Sir handler

Chapter 10

THREE DIMENSIONAL GEOMETRY I (LINES AND PLANES)

10.1.1. Introduction:

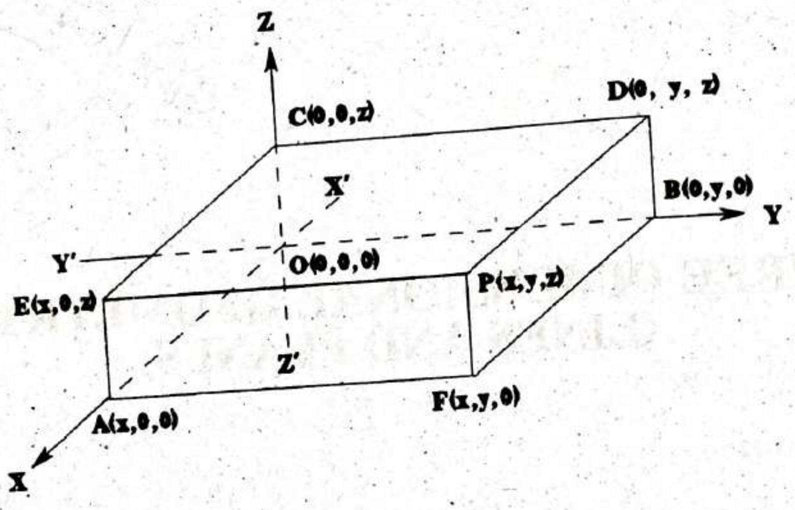
In plane the position of a point is determined by two numbers x, y obtained with reference to two straight lines in the plane generally at right angles. The position of a point in space is determined by three numbers x, y, z. The plane is regarded as R x R or R^2 space and the three dimensional space as R x R x R or simply R^3 space.

In plane we see that there is a one — one correspondence between the points in plane and the ordered pair (x, y). Now we will see that there is a one — one correspondence between the points in 3-space and the ordered triple (x, y, z). Two ordered triples are regarded as equal if and only if the corresponding components are equal. Thus (x, y, z) = (a, b, c) if and only if x = a, y = b, and z = c

An equation in two variables represents a curve in plane, while an equation in three variables represents a surface in three dimensional space or simply 3-space. The simplest equation in two variables is the linear equation $ax + by + c = 0$ which represents a straight line, while the simplest equation in three variables $ax + by + cz + d = 0$ represents a plane. Thus equation can take several forms depending upon the information given and the information sought.

10.1.2. Rectangular Coordinate System:

In order to locate points in three dimensional space we must have some fixed reference frame. We obtain such a frame by selecting a fixed point O and selecting at O three mutually perpendicular lines as indicated in the figure below. On each of these lines a positive direction is assigned and is shown by an arrowhead. These three lines are called the x-axis, y-axis and z-axis.



The x-axis and y-axis together determine a horizontal plane called the xy-plane. Similarly the xz-plane is the vertical plane containing the x-axis and z-axis, and the yz-plane is the plane determined by the y-axis and z-axis

If P is any point in space it has three coordinates with respect to this fixed frame of reference, and these coordinates are indicated by writing P(x, y, z). These coordinates can be defined thus:

x is the directed distance of P from the yz-plane,
y is the directed distance of P from the xz-plane,
z is the directed distance of P from the xy-plane.

In the figure these are the directed distances DP, EP and FP respectively. These line segments form the edges of a box, with each face perpendicular to one of the coordinate axes. With the lettering of the figure, A is the projection of P on the x-axis, B is the projection of P on the y-axis, and C is the projection of P on the z-axis. Clearly an alternate definition for the coordinates of P is:

x is the directed distance OA,
y is the directed distance OB,
z is the directed distance OC.

The symbol P(x, y, z) denotes that the coordinates of the point P are x, y, z.

Conversely, given any three numbers, x, y, z we can find a point P whose coordinates are x, y, z. To effect this we measure off OA = x units along the x-axis, AN parallel to the y-axis and equal to y units and NP parallel to the z-axis and equal to z units. Then P is the required point.

Thus the position of a point in space is uniquely determined by three coordinates. Hence we see that there is a one — one correspondence between the points in space and the ordered triples (x, y, z).

It is clear that if x is negative, the point (x, y, z) lies in back of the yz-plane. If y is negative the point lies to the left of the xz-plane, and if z is negative the point lies below the xy-plane. These three coordinate planes divide space into 8 separate pieces called *octants*. The octant in which all three coordinates are positive is called the *first octant*. The other octants could be named, but there is no real reason for doing so.

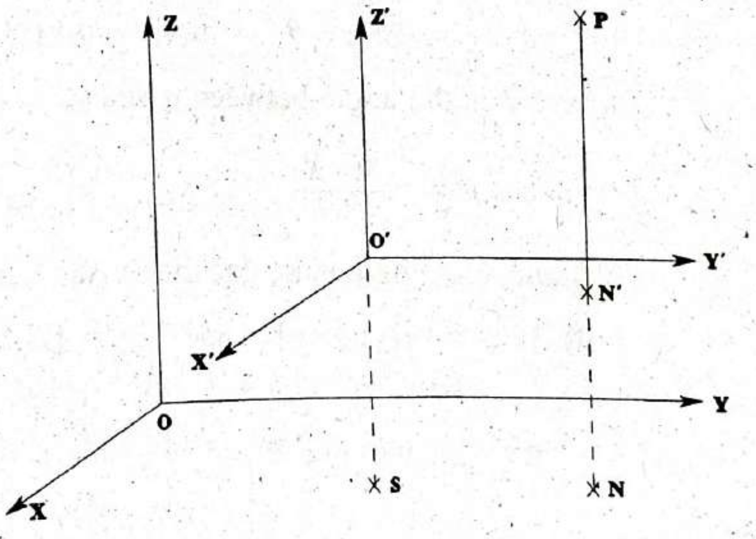
10.1.3. Convention for Signs:

When the directions of the x and y-axis have been fixed, the positive direction of the z-axis is chosen along the direction in which a right-handed screw will move when rotated from x to y. Such a system of axes is called a right-handed system.

10.1.4. Translation of Axes:

The technique of introducing a new system of coordinate axes through a point other than the origin O and parallel to the x, y, z axes is called translation of axes. It is also known as shifting of origin or change of origin.

Let OX, OY, OZ, and O′X′, O′Y′, O′Z′, be two sets of parallel axes through O and O′ respectively.



Let (a, b, c) be the coordinates of O′ referred to OX, OY, OZ. Let the coordinates of a point P in space be (x, y, z) referred to the X, Y, Z coordinate system and (x′, y′, z′) be its coordinates referred to X′, Y′, Z′ coordinate system.

Let PN′, perpendicular from P to X′Y′-plane meet the XY-plane in N. Since X′Y′-plane is parallel to XY-plane, PN is also perpendicular to XY-plane. Let O′S be perpendicular to XY-plane. Then $NN' = SO' = c$

$$\therefore \quad z = NP = NN' + N'P = c + z'$$

Similarly $x = a + x'$, $y = b + y'$

Hence
$$x' = x - a$$
$$y' = y - b$$
$$z' = z - c$$

10.1.5. Some Results of Vector Algebra:

Analytic geometry of three dimensions or solid geometry can be presented without the use of vectors. However when vectors are used the presentation can be simplified. Here we give some useful results from vector algebra.

(i) If O(0, 0, 0) is origin and P(x, y, z) is any point then

$\overrightarrow{OP} = x\underline{i} + y\underline{j} + z\underline{k} = [x, y, z]$ and it is called position vector of the point P.

(ii) The length or norm of the vector $\underline{v} = [v_1, v_2, v_3]$ denoted by $|\underline{v}|$ is the distance from origin to the point

(v_1, v_2, v_3). Thus $|\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

(iii) A vector with unit magnitude or norm is called a unit vector. Thus if $\underline{v}$ is a vector with magnitude v the $\underline{v}/v$ is a unit vector along $\underline{v}$.

(iv) If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points then

$$\overrightarrow{PQ} = (x_2 - x_1)\underline{i} + (y_2 - y_1)\underline{j} + (z_2 - z_1)\underline{k}$$

(v) If two vectors $\underline{u}$ and $\underline{v}$ are collinear or parallel we can write $\underline{u} = t\,\underline{v}$ where t is a scalar.

(vi) If $\underline{u} = [u_1, u_2, u_3]$ and $\underline{v} = [v_1, v_2, v_3]$ are two vectors then

$$\underline{u} \cdot \underline{v} = uv\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$$

where θ is the angle between $\underline{u}$ and $\underline{v}$

$$\therefore \quad \cos\theta = \frac{\underline{u} \cdot \underline{v}}{uv}$$

If $\underline{u}$ and $\underline{v}$ are perpendicular then $u_1v_1 + u_2v_2 + u_3v_3 = 0$

(vii) If $\underline{u} = [u_1, u_2, u_3]$ and $\underline{v} = [v_1, v_2, v_3]$ are two vectors then

$$\underline{u} \times \underline{v} = uv\sin\theta\,\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where θ is the angle between $\underline{u}$ and $\underline{v}$ and $\underline{n}$ is a unit vector perpendicular to both $\underline{u}$ and $\underline{v}$.

(viii) If $\underline{p}$, $\underline{q}$ are position vectors of two points P and Q respectively, then the position vector of the point R, which divides the line segment PQ in the ratio

m:n is $\dfrac{n\underline{p} + m\underline{q}}{m + n}$

(ix) Scaler Triple Product $[\underline{u}\ \underline{v}\ \underline{w}] = \underline{u} \cdot (\underline{v} \times \underline{w})$

$$= \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v}) = (\underline{u} \times \underline{v}) \cdot \underline{w} = (\underline{v} \times \underline{w}) \cdot \underline{u} = (\underline{w} \times \underline{u}) \cdot \underline{v}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

where $\underline{u} = [u_1, u_2, u_3]$ etc.

(x) $[\underline{u}\ \underline{v}\ \underline{w}]$ = volume of the parallelopiped with $\underline{u}$, $\underline{v}$, $\underline{w}$ as coterminus edges.

(xi) The dot product $\underline{u} \cdot \underline{v}$ of $\underline{u}$ and $\underline{v}$ is equal to the product of the length of either of them and the projection of the other upon it.

In particular if one of the vectors say $\underline{u}$ is of unit length then

$\underline{u} \cdot \underline{v} = |\underline{v}| \cos\theta$, which is just the projection, or component of $\underline{v}$ in the direction of the unit vector $\underline{u}$.

(xii) $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$

$$= \begin{vmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{vmatrix}$$

(xiii) $(\underline{a} \times \underline{b}) \times (\underline{c} \times \underline{d}) = [\underline{a}\,\underline{b}\,\underline{d}]\,\underline{c} - [\underline{a}\,\underline{b}\,\underline{c}]\,\underline{d}$

10.1.6. Distance Between Two Points:

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Then

$\overrightarrow{PQ} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$

and its magnitude is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Hence the distance between P and Q, i.e., magnitude of the vector $\overrightarrow{PQ}$ is

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

10.1.7. Division of a Line Segment in a Given Ratio:

Let $P(x_1, y_1, z_1)$, and $Q(x_2, y_2, z_2)$ be two given points, and $R(x, y, z)$ divides the line segment PQ in the ratio m:n.

We know that if position vectors of P and Q are $\underline{p}$ and $\underline{q}$ respectively then the position vector $\underline{r}$ of the point R is given by

$$\underline{r} = \frac{n\underline{p} + m\underline{q}}{m + n}$$

Now $\underline{p} = [x_1, y_1, z_1]$ and $\underline{q} = [x_2, y_2, z_2]$

$$\text{Hence } [x, y, z] = \frac{n[x_1, y_1, z_1] + m[x_2, y_2, z_2]}{m + n}$$

$$= \frac{[nx_1 + mx_2, ny_1 + my_2, nz_1 + mz_2]}{m + n}$$

i.e., R is $\left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}, \frac{nz_1 + mz_2}{m + n}\right)$

Cor 1: The mid point of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Cor 2: If R divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $\lambda:1$, then R is $\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda}\right)$

10.1.8. Direction Cosines of a Vector:

Let $\underline{u} = [u_1, u_2, u_3] = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$

then u_1, u_2, u_3 are called the *direction numbers* or *direction ratios* of the vector $\underline{u}$.

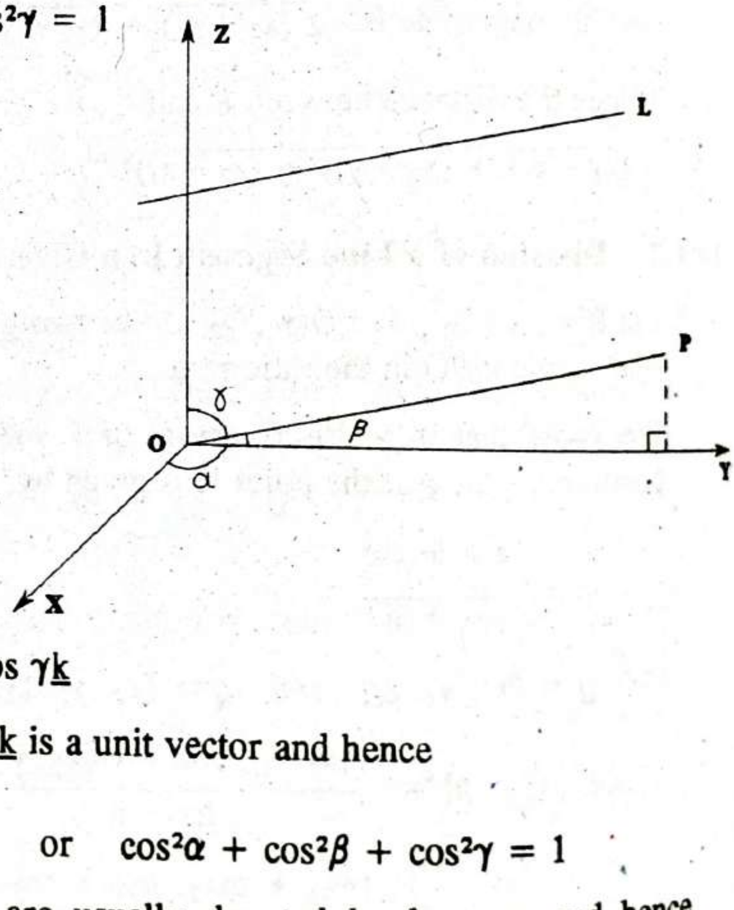
If u is magnitude of $\underline{u}$ then $\dfrac{u_1}{u}, \dfrac{u_2}{u}, \dfrac{u_3}{u}$ are the *direction cosines* of the vector u

It is obvious that $\left(\dfrac{u_1}{u}\right)^2 + \left(\dfrac{u_2}{u}\right)^2 + \left(\dfrac{u_3}{u}\right)^2 = 1$

10.1.9. Direction Cosines of a Line:

If a line makes angles α, β, γ, with the positive directions of x, y and z-axes respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called its direction cosines and α, β, γ are called its direction angles.

To prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



Let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of the line L.

and $\overrightarrow{OP} = \underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be a vector parallel to the line L, which obviously makes angles α, β, γ with the axes.

From the figure we see that

$x = r\cos\alpha$, $y = r\cos\beta$ and $z = r\cos\gamma$

where $r = |\underline{r}|$

Now $\underline{r} = r\cos\alpha\underline{i} + r\cos\beta\underline{j} + r\cos\gamma\underline{k}$

$\therefore \quad \dfrac{\underline{r}}{r} = \cos\alpha\underline{i} + \cos\beta\underline{j} + \cos\gamma\underline{k}$ is a unit vector and hence

$$\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1 \quad \text{or} \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Note: Direction cosines of a line are usually denoted by l, m, n and hence $l^2 + m^2 + n^2 = 1$.

10.1.10. Direction Ratios:

The three numbers proportional to the direction cosines of a line are called the direction ratios of that line.

i.e., If l, m, n are direction cosines of a line and $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c}$ then, a, b, c are called direction ratios of the line.

Note 1: Direction ratios of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

Note 2: If a, b, c are direction ratios of a line then direction cosines are given by

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

i.e., $l = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$, $m = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$, $n = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$

Note 3: One should always make a distinction between direction cosines and direction ratios. It is only when l, m, n are direction cosines, that we have the relation $l^2 + m^2 + n^2 = 1$.

Example: Find the direction cosines of the line joining the points P(2, 3, 4) and Q(4, 7, −2).

Solution: Direction ratios of PQ are $4 - 2, 7 - 3, -2 - 4$

i.e., 2, 4, −6 or 1, 2, −3

∴ direction cosines are $\dfrac{1}{\sqrt{1 + 4 + 9}}, \dfrac{2}{\sqrt{1 + 4 + 9}}, \dfrac{-3}{\sqrt{1 + 4 + 9}}$

i.e., $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}$

10.1.11 Angle Between Two Straight Lines:

If l_1, m_1, n_1 are direction cosines of a line L and l_2, m_2, n_2 are direction cosines of another straight line M, then $\theta \in [0, \pi]$, the angle between the two lines is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

Definition: The angle between two non-intersecting (skew) lines is the angle between two lines drawn parallel to them through any point in space.

10.1.12. Tetrahedron:

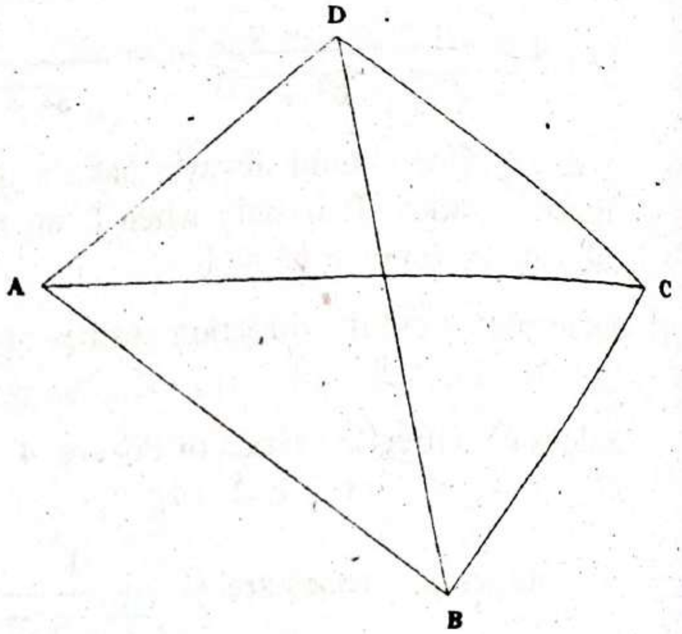
A tetrahedron is a three dimensional figure bounded by four planes. It has four vertices, each vertex arising as a point of intersection of three of the four planes. It has six edges each edge arising as the line of intersection of two of the four planes. ($^4C_2 = 6$).

To construct a tetrahedron, we start with three points A, B, C and any point D, not lying on the plane determined by the points A, B, C. Then the four faces of the tetrahedron are the four triangles

ABC, BCD, CAD, ABD; the four vertices are the points A, B, C, D and the six edges are the lines

AB, CD; BC, AD; CA, BD.

The two edges AB, CD joining separately the points A, B and C, D are called a pair of *opposite* edges. Similarly BC, AD and CA, BD are the two other pairs of opposite edges.



10.1.13. Volume of a Tetrahedron:

Let $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ be the position vectors of the vertices of the tetrahedron ABCD.

Volume of tetrahedron

$= \frac{1}{3}$ (area of Δ ABC)

(h, the length of the altitude from D to the plane ABC)

$= \frac{1}{6}$ (2 . area of Δ ABC) . h

$= \frac{1}{6}$ (area of the parallelogram with AB and AC as adjacent sides) . h

$= \frac{1}{6}$ (volume of the parallelopiped with $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ as coterminus edges)

$= \frac{1}{6}[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \frac{1}{6}[\underline{b}-\underline{a} \;\; \underline{c}-\underline{a} \;\; \underline{d}-\underline{a}]$

Cor: If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ are the vertices of the tetrahedron.

$$\overrightarrow{AB} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

$$\overrightarrow{AC} = [x_3 - x_1, y_3 - y_1, z_3 - z_1]$$

and $\overrightarrow{AD} = [x_4 - x_1, y_4 - y_1, z_4 - z_1]$

and thus volume of tetrahedron is

$$\frac{1}{6}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Solved Examples:

Example 1: The points A(3, 2, −4), B(−1, 1, −2), C(−2, 3, 3), D(−3, −2, 1) are corners of a tetrahedron. Find volume of the tetrahedron. (P.U. 1990)

Solution: Volume of the tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 3 & 2 & -4 & 1 \\ -1 & 1 & -2 & 1 \\ -2 & 3 & 3 & 1 \\ -3 & -2 & 1 & 1 \end{vmatrix}$$

Substracting first row from others

$$= \frac{1}{6}\begin{vmatrix} 3 & 2 & -4 & 1 \\ -4 & -1 & 2 & 0 \\ -5 & 1 & 7 & 0 \\ -6 & -4 & 5 & 0 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} -4 & -1 & 2 \\ -5 & 1 & 7 \\ -6 & -4 & 5 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} 4 & -1 & 2 \\ 5 & 1 & 7 \\ 6 & -4 & 5 \end{vmatrix}$$

$$= \frac{1}{6}\{4(5 + 28) + 1(25 - 42) + 2(-20 - 6)\} = \frac{1}{6}[132 - 17 - 52]$$

$$= \frac{63}{6} = 10\tfrac{1}{2} = 10.5 \text{ cubic units.}$$

Example 2: Show that the points (1, 6, 1), (1, 3, 4), (4, 3, 1) and (0, 2, 0) are the vertices of a regular tetrahedron.

Solution: Let the given points be A(1, 6, 1), B(1, 3, 4), C(4, 3, 1) and D(0, 2, 0)

Now $AB = \sqrt{(1-1)^2 + (6-3)^2 + (1-4)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$AC = \sqrt{(1-4)^2 + (6-3)^2 + (1-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$AD = \sqrt{(1-0)^2 + (6-2)^2 + (1-0)^2} = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

$BC = \sqrt{(1-4)^2 + (3-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

$BD = \sqrt{(1-0)^2 + (3-2)^2 + (4-0)^2} = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$

and $CD = \sqrt{(4-0)^2 + (3-2)^2 + (1-0)^2} = \sqrt{16+1+1} = \sqrt{18} = 3\sqrt{2}$

$\therefore$ AB = AC = AD = BC = BD = CD

Hence the given points form the vertices of a regular tetrahedron.

Example 3: Find the coordinates of the point dividing the join of (−3, 1, 4) and (5, −1, 6) in the ration 3 : 5.

Solution: Here ratio m : n = 3 : 5

$\therefore$ if (x, y, z) is the required point then

$$x = \frac{5(-3) + 3(5)}{3+5} = 0, \quad y = \frac{5(1) + 3(-1)}{5+3} = \frac{5-3}{8} = \frac{1}{4}$$

$$z = \frac{5(4) + 3(6)}{5 + 3} = \frac{20 + 18}{8} = \frac{19}{4}$$

Hence the required point is $\left(0, \frac{1}{4}, \frac{19}{4}\right)$

Example 4: Find the locus of the point which is equidistant from the points (−1, 2, 3) and (3, 2, 1).

Solution: The given points are A(−1, 2, 3) and B(2, 3, 1).

If P(x, y, z) is a point on the locus PA = PB

$\therefore \quad \sqrt{(x+1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2}$

i.e., $2x - 4y - 6z + 1 + 4 + 9 = -4x - 6y - 2z + 4 + 9 + 1$

or $6x + 2y - 4z = 0$ or $3x + y - 2z = 0$

Example 5: Find the direction ratios, direction cosines and measures of direction angles of the straight line through the points (1, −2, 0) and (5, −10, 1).

Solution: Direction ratios of the line are

$5 - 1, -10 + 2, 1 - 0$ i.e., 4, −8, 1

and $\sqrt{4^2 + (-8)^2 + 1^2} = \sqrt{16 + 64 + 1} = \sqrt{81} = 9$

$\therefore$ Direction cosines of the line are $\frac{4}{9}, \frac{-8}{9}, \frac{1}{9}$

and the measures of the direction angles are

$$\cos^{-1}\frac{4}{9}, \cos^{-1}\left(\frac{-8}{9}\right), \cos^{-1}\frac{1}{9}$$

Example 6: Find the direction cosines of a straight line that has all the direction angles congruent.

Solution: Here $\alpha = \beta = \gamma$

Therefore $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \quad 3\cos^2\alpha = 1, \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Hence the direction cosines of the straight line are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \text{ or } -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$$

Example 7: The direction cosines of two straight lines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$. Find measure of the angle between them. (P.U. 1985)

Solution: $l + m + n = 0$ I

$l^2 + m^2 - n^2 = 0$ II

From I $n = -(l + m)$

Substituting the value in equation II, we get

$$l^2 + m^2 - (l + m)^2 = 0 \text{ i.e., } -2lm = 0$$

$\therefore$ either $l = 0$ or $m = 0$

If $l = 0$, $n = -m$ or $\frac{m}{-1} = \frac{n}{1}$

$\therefore \frac{l}{0} = \frac{m}{-1} = \frac{n}{1}$ i.e., l, m, n are proportional

to $0, -1, 1$ respectively and their actual values are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

If $m = 0$, $n = -l$ or $\frac{l}{1} = \frac{n}{-1}$ or $\frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$

$\therefore$ l, m, n are $\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$

Thus the direction cosines of the two lines are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$

If θ is the measure of the angle between them then

$$\cos\theta = 0 . \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) . 0 + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

EXERCISE 10.1

1. Show that the points (3, −1, 3), (1, −1, 2), (2, 1, 0) and (4, 1, 1) are the vertices of a rectangle.
2. A(3, 2, 0), B(5, 3, 2) and C(−9, 6, −3) are the vertices of a triangle ABC. Find the coordinates of the point of intersection of the internal bisector of the angle A with the side BC.
3. Find the ratio in which the yz-plane divides the line segment joining the points (2, 4, 7) and (−3, −5, 8). (P.U. 1990)
4. What are the direction cosines of the coordinate axes.
5. A line makes angles of 30° and 60° with the x and y-axes respectively. What angle does it make with the z-axis?

6. Prove that if measures of the direction angles of a straight line are α, β and γ then

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

7. If the edges of a rectangular parallelopiped are a, b, c. Show that measures of the angles between the four diagonals are given by arc cos $\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right)$

(P.U. 1989)

8. Prove that the acute angle between any two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

(P.U. 1989)

9. A straight line makes angles of measures α, β, γ, δ with the four diagonals of a cube, prove that

$$\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma + \text{Cos}^2\delta = \frac{4}{3}$$ (P.U. 1988, 91)

10. Find the angle between the two straight lines whose direction cosines are given by the following equations.

(i) $l - 2m - 2n = 0$
$lm + mn + nl = 0$

(P.U. 1987)

(ii) $l + m + n = 0$
$2lm + 2ln - mn = 0$

(iii) $2l + 2m - n = 0$
$lm + mn + nl = 0$

11. Find the direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to 3, −2, 3 and 1, −2, −1.

12. If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$; $m_1 + m_2 + m_3$; $n_1 + n_2 + n_3$ makes congruent angles with them.

(P.U. 1986)

13. A variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l$, $m + \delta m$, $n + \delta n$. Show that the measure of the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$

(P.U. 1987)

14. If A, B are the points (3, 4, 5), (−1, 3, −7), find the locus of a point P such that $|PA|^2 - |PB|^2 = \text{constant} = k(\text{say})$.

15. Find the angle between the two straight lines if their direction cosines are given by the equations $l + 2m + 3n = 0$, $3lm + mn - 4ln = 0$.

(P.U. 1984)

Exercise 10.1

1. Solution:

Let the points be A(3, −1, 3), B(1, −1, 2), C(2, 1, 0) and D(4, 1, 1)

$$AB = \sqrt{(1-3)^2 + (-1+1)^2 + (2-3)^2} = \sqrt{4+0+1} = \sqrt{5}$$

$$CD = \sqrt{(4-2)^2 + (1-1)^2 + (1-0)^2} = \sqrt{4+0+1} = \sqrt{5}$$

$\therefore \quad AB = CD$

And $\quad AC = \sqrt{(2-3)^2 + (1+1)^2 + (0-3)^2} = \sqrt{1+4+9} = \sqrt{14}$

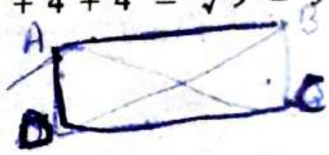
$$BD = \sqrt{(4-1)^2 + (1+1)^2 + (1-2)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$\therefore \quad AC = BD$

$$BC = \sqrt{(2-1)^2 + (1+1)^2 + (0-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$AD = \sqrt{(4-3)^2 + (1+1)^2 + (1-3)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

$\therefore \quad BC = AD$. Hence ABCD is a rectangle.



2. Solution:

The internal bisector AD of $\hat{A}$ divides BC in the ratio of the sides AB and AC.

i.e. $\quad BD : DC = AB : AC$

$$AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2}$$

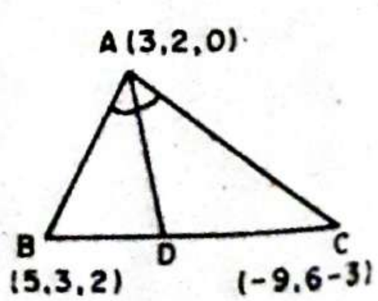
$$= \sqrt{144+16+9} = \sqrt{169} = 13$$

$\therefore \quad BD : DC = 3 : 13$

Hence for D $\quad x = \dfrac{3(-9) + 13(5)}{3+13} = \dfrac{38}{16}$

$$y = \frac{3(6) + 13(3)}{3+13} = \frac{57}{16}$$

$$z = \frac{3(-3) + 13(2)}{3+13} = \frac{17}{16}$$



Hence required point D is $\left(\dfrac{38}{16}, \dfrac{57}{16}, \dfrac{17}{16}\right)$

3. Solution:

Suppose that the yz-plane divides the join of the given points in the ratio m : n. Then x-coordinate of the point which divides the join of the given points in the ratio m : n is $\dfrac{m(-3) + n(2)}{m+n}$ which must be zero as it lies in the yz-plane.

$\therefore \quad \dfrac{-3m+2n}{m+n} = 0$ i.e. $3m - 2n = 0$ or $\dfrac{m}{n} = \dfrac{2}{3}$ i.e. $m : n = 2 : 3$

$3m = 2n$

$\dfrac{m}{n} = \dfrac{2}{3}$

4. **Solution:**

For x-axis the directional angles are 0, 90°, 90°.

Hence direction csines of x-axis are 1, 0, 0.

Similarly we can see that the direction cosines of y-axis are 0, 1, 0 and those of z-axis are 0, 0, 1.

5. **Solution:**

Here $\alpha = 30^\circ$, $\beta = 60^\circ$. We know $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore \quad \cos^2 30^\circ + \cos^2 60^\circ + \cos^2\gamma = 1$

$\therefore \quad \frac{3}{4} + \frac{1}{4} + \cos^2\gamma = 1 \quad \therefore \cos^2\gamma = 0 \quad \therefore \gamma = 90^\circ$

Hence the line makes an angle of 90° with z-axis.

6. **Solution:**

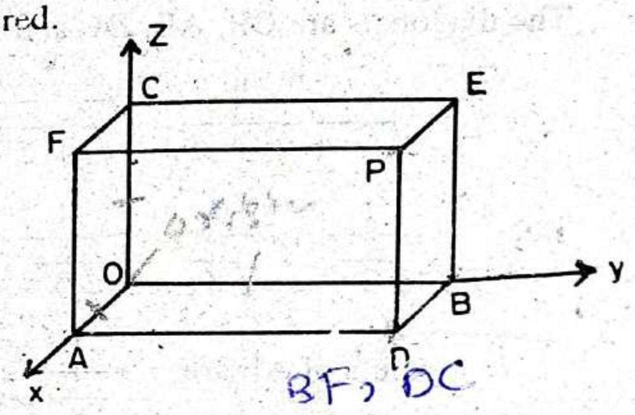
We know that if α, β, γ are the direction angles of a straight line then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \text{ i.e. } 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

i.e. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ as required.

7. **Solution:**

The edges OA, OB and OC are a, b, c respectively. We take O as the origin, OA as x-axis OB as y-axis and OC as z-axis.



Therefore the coordinates of the vertices of the rectangular parallelopiped are

O(0, 0, 0), A(a, 0, 0), B(0, b, 0), C(0, 0, c), D(a, b, 0), E(0, b, c), F(a, 0, c) and P(a, b, c)

The four diagonals are OP, AE, DC and BF.

Direction ratios of OP are a, b, c and direction ratios of AE are −a, b, c.

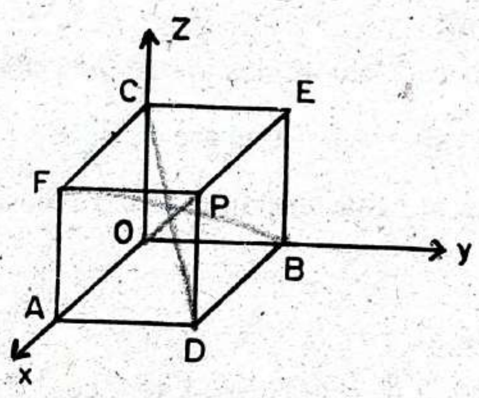
Therefore if α be the angle between OP and AE then

$$\cos\alpha = \frac{-a(a) + b \cdot b + c \cdot c}{\sqrt{a^2+b^2+c^2} \cdot \sqrt{a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

i.e. $\quad \alpha = \arccos \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$

Similarly we can find the angles between the other diagonals. These angles will be one of $\arccos \frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}$.

8. Solution:



Let a side of the cube be a. So as in last question the vertics are

O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), D(a, a, 0), E(0, a, a), F(a, 0, a), P(a, a, a)

Direction ratios of OP are a, a, a and those of AE are −a, a, a.

∴ If α is the angle between OP and AE then

$$\cos\alpha = \frac{-a^2 + a^2 + a^2}{a^2 + a^2 + a^2} = \frac{1}{3} \quad \therefore \alpha = \cos^{-1}\frac{1}{3}.$$

9. Solution:

See figure of question 8. Then as in that question the points are

O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), D(a, a, 0), E(0, a, a), F(a, 0, a), P(a, a, a)

The diagonals are OP, AE, DC and BF. The d.c's of OP are

$$\frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{a}{\sqrt{a^2 + a^2 + a^2}}$$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Similarly we see that

d.c's of AE are $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

d.c's of DC are $-\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

and d.c's of BF are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Let the d.c's of the line which makes angles α, β, γ and δ with the diagonals be l, m, n.

$$\therefore \quad \cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}, \quad \cos\beta = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

$$\cos\gamma = \frac{-l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

and $$\cos\delta = \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}}$$

Squaring and adding we get

$$\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma + \text{Cos}^2\delta = {}^4/_3\,(\lambda^2 + \mu^2 + \nu^2) = {}^4/_3$$

10.(i) Solution:

The direction cosines of the two lines are givne by

$$l - 2m - 2n = 0 \quad \text{......(i)}$$

$$lm + mn + nl = 0 \quad \text{......(ii)}$$

From (i) $l = 2m + 2n$ ∴ (ii) becomes $m(2m + 2n) + mn + n(2m + 2n) = 0$

i.e. $2m^2 + 5mn + 2n^2 = 0$ or $(m + 2n)(2m + n) = 0$

∴ Either $m + 2n = 0$ ......(iii)

or $2m + n = 0$ ......(iv)

From (iii) $m = -2n$, ∴ $l = -4n + 2n = -2n$

$$\therefore \quad \frac{l}{2} = \frac{m}{2} = \frac{n}{-1} \quad \text{......(A)}$$

From (iv) $n = -2m$, ∴ $l = 2m - 4m = -2m$

$$\therefore \quad \frac{l}{2} = \frac{m}{-1} = \frac{n}{2} \quad \text{......(B)}$$

The equations (A) and (B) give the direction ratios of the two lines.

If α be the angle between the two lines then

$$\cos\alpha = \frac{2.2 + (-1)(2) + 2(-1)}{\sqrt{4 + 1 + 4}\,.\,\sqrt{4 + 4 + 1}} = 0$$

$$\therefore \quad \alpha = 90° = \frac{\pi}{2}$$

(ii) The direction cosines of the two lines are given by

$$l + m + n = 0 \quad \text{......(i)}$$

$$2lm + 2ln - mn = 0 \quad \text{......(ii)}$$

From (i) $n = -(l + m)$. Substituting in (ii) we get

$2lm - 2l(l + m) + m(l + m) = 0$ i.e. $2lm - 2l^2 - 2lm + lm + m^2 = 0$

or $-2l^2 + lm + m^2 = 0$ or $2l^2 - lm - m^2 = 0$ or $(l - m)(2l + m) = 0$

So the direction cosines of two line are given by

$$\left.\begin{array}{l} l + m + n = 0 \\ l - m = 0 \end{array}\right\} \quad \text{......(A)}$$

and

$$\left.\begin{array}{l} l + m + n = 0 \\ 2l + m = 0 \end{array}\right\} \quad \text{......(B)}$$

From (A) $\dfrac{l}{1} = \dfrac{m}{1} = \dfrac{n}{-2} = \dfrac{1}{\sqrt{6}}$

From (B) $\dfrac{l}{-1} = \dfrac{m}{2} = \dfrac{n}{-1} = \dfrac{1}{\sqrt{6}}$

If θ is the measure of the angle between the two lines

$$\cos\theta = \frac{1}{\sqrt{6}}\left(\frac{-1}{\sqrt{6}}\right) + \frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{-1}{\sqrt{6}}\right)$$

$$= \frac{-1}{6} + \frac{2}{6} + \frac{2}{6} = \frac{3}{6} = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3}$$

(iii) The direction cosines of the two liens are given by

$2l + 2m - n = 0$ ...... (i)

$lm + mn + nl = 0$ ...... (ii)

From (i) $n = 2l + 2m$, substituting in (ii)

$lm + m(2l + 2m) + l(2l + 2m) = 0$ i.e. $2l^2 + 5lm + 2m^2 = 0$

or $(2l+m)(l+2m) = 0$. So the direction cosines of the two lines are given by

$$\left.\begin{aligned} 2l + 2m - n &= 0 \\ 2l + m &= 0 \end{aligned}\right\} \quad \text{...... (A)}$$

and

$$\left.\begin{aligned} 2l + 2m - n &= 0 \\ l + 2m &= 0 \end{aligned}\right\} \quad \text{...... (B)}$$

From (A) $\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2} = \frac{1}{3}$

From (B) $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2} = \frac{1}{3}$

If θ is the angle between the two lines

$$\cos\theta = \frac{1}{3}\cdot\frac{2}{3} + \left(\frac{-2}{3}\right)\left(\frac{-1}{3}\right) + \left(\frac{-2}{3}\right)\cdot\frac{2}{3}$$

$$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0, \quad \theta = \frac{\pi}{2}$$

11. Solution:

Let the required d.c's be l, m, n then

$3l - 2m + 3n = 0$ and $l - 2m - n = 0$

which give $\frac{l}{8} = \frac{m}{6} = \frac{n}{-4}$ or $\frac{l}{4} = \frac{m}{3} = \frac{n}{-2}$

$\therefore$ l, m, n are $\frac{4}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-2}{\sqrt{29}}$

12. Solution:

If θ is the angle between the line whose d.c's are l_1, m_1, n_1 and the line whose d.c's are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$.

$$\cos\theta = \frac{l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)}{\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}}$$

$$= \frac{l_1^2 + m_1^2 + n_1^2 + l_1l_2 + m_1m_2 + n_1n_2 + l_1l_3 + m_1m_3 + n_1n_3}{\sqrt{\begin{pmatrix} l_1^2 + m_1^2 + n_1^2 + l_2^2 + m_2^2 + n_2^2 + l_3^2 + m_3^2 + n_3^2 \\ + 2\Sigma l_1l_2 + 2\Sigma l_2l_3 + \Sigma l_3l_1 \end{pmatrix}}}$$

As the lines are perpendicular $l_1l_2 + m_1m_2 + n_1n_2 = 0$

$l_2l_3 + m_2m_3 + n_2n_3 = 0$

$l_1l_3 + m_1m_3 + n_1n_3 = 0 \quad \therefore \cos\theta = \dfrac{1 + o + o}{\sqrt{1 + 1 + 1 + 0 + 0 + 0}} = \dfrac{1}{\sqrt{3}}$

This is independent of l's, m's, n's. Hence the result.

13. Solution:

As l, m, n and $l + \delta l$, $m + \delta m$, $n + \delta n$ are d.c's we have

$$l^2 + m^2 + n^2 = 1 \qquad \ldots\ldots (1)$$

$$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$$

or $\quad l^2 + m^2 + n^2 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2 + 2l\,\delta l + 2m\,\delta m + 2n\,\delta n = 1$

or $\quad (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2\,(l\,\delta l + m\,\delta m + n\,\delta n) \qquad \ldots\ldots (2)$

$$\cos\delta\theta = l\,(l + \delta l) + m\,(m + \delta m) + n\,(n + \delta n)$$
$$= l^2 + l\,\delta l + m^2 + m\,\delta m + n^2 + n\,\delta n$$
$$= 1 - \frac{1}{2}\,\{(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$$

i.e. $\quad 1 - 2\sin^2\dfrac{\delta\theta}{2} = 1 - \dfrac{1}{2}\,[(\delta l)^2 + (\delta m)^2 + (\delta n)^2\}$

$4\sin^2\dfrac{\delta\theta}{2} = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ As $\delta\theta$ is small

$4\left(\dfrac{1}{2}\delta\theta\right)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ i.e. $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$2\sin^2\theta/2 = 1-\cos 2\theta$

$\cos 2\theta = 1-2\sin^2\theta$

$\cos\delta\theta = 1-2\sin^2\dfrac{\delta\theta}{2}$

14. Solution:

Let P(x, y, z) be a point on the locus.

$\therefore$ The equation $(PA)^2 - (PB)^2 = K$ becomes

$[\sqrt{(x-3)^2 + (y-4)^2 + (z-5)^2}\,]^2 - [\sqrt{(x+1)^2 + (y-3)^2 + (z+7)^2}\,]^2 = K$

i.e. $\quad -6x - 8y - 10z + 50 - 2x + 6y - 14z - 59 = K$

or $\quad -8x - 2y - 24z - 9 = K$ or $8x + 2y + 24z + 9 + K = 0$

15. Solution:

The direction cosines of the two lines are given by

$$l + 2m + 3n = 0 \qquad \ldots\ldots (i)$$

$$3\,lm + mn - 4\,ln = 0 \qquad \ldots\ldots (ii)$$

From (i) $l = -2m - 3n$, substituting in (ii)

$3m\,(-2m - 3n) + mn - 4n\,(-2m - 3n) = 0$

i.e. $\quad -6m^2 - 9mn + mn + 8mn + 12n^2 = 0$ or $-6m^2 + 12n^2 = 0$ or $m^2 - 2n^2 = 0$

i.e. $\quad (m + \sqrt{2}\,n)\,(m - \sqrt{2}\,n) = 0$ Either $m + \sqrt{2}\,n = 0$

or $\quad m - \sqrt{2}\,n = 0$, If $m + \sqrt{2}\,n = 0$

$m = -\sqrt{2}\,n$. $\dfrac{m}{\sqrt{2}} = \dfrac{n}{-1} = K$ (say) $\therefore 1 = -2\sqrt{2}K + 3K$

$\dfrac{l}{3 + 2\sqrt{2}} = K$, $\therefore$ d.c's of one line are given by

$$\frac{l}{3 - 2\sqrt{2}} = \frac{m}{\sqrt{2}} = \frac{n}{-1}$$

Now if $m - \sqrt{2}\,n = 0$, $m = \sqrt{2}\,n$ or $\dfrac{m}{\sqrt{2}} = \dfrac{n}{1}$

As before we can get d.c's of the second line as

$$\frac{l}{-(2\sqrt{2} + 3)} = \frac{m}{\sqrt{2}} = \frac{n}{1}$$

If θ is the angle between the two lines then

$$\cos\theta = \frac{-(3 - 2\sqrt{2})(2\sqrt{2} + 3) + \sqrt{2}.\sqrt{2} + (-1).1}{\sqrt{(3 - 2\sqrt{2})^2 + 2 + 1}\,.\,\sqrt{(2\sqrt{2} + 3)^2 + 2 + 1}} = 0$$

$\therefore \quad \theta = \dfrac{\pi}{2}$

THE PLANE

10.2.1. Definition:

A plane surface (or briefly a plane) is a surface such that every point on the straight line joining any two points of the surface lies on the surface.

A plane in 3-space is uniquely determined by specifying a point in the plane and a vector perpendicular to the plane. A vector perpendicular to a plane is called a *normal* to the plane.

10.2.2. Equation of Plane:

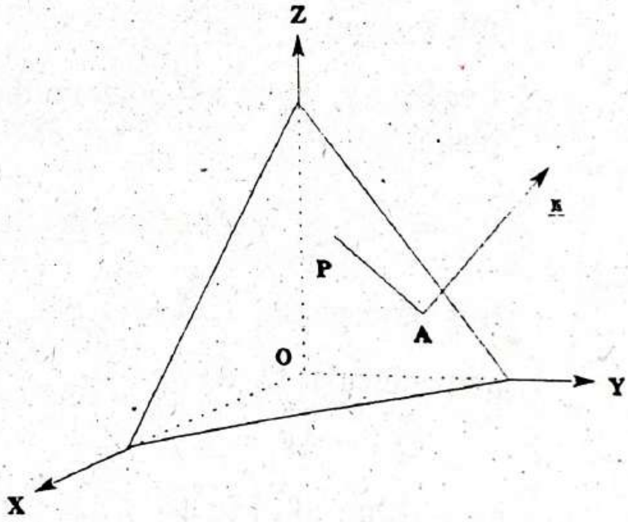
To find the equation of the plane through a given point $A(x_1, y_1, z_1)$ and with the non-zero normal vector $\underline{n} = [a, b, c]$.

Let $P(x, y, z)$ be any point in the plane. It is obvious that the vector $\overrightarrow{AP}$ is perpendicular to $\underline{n}$.

$\overrightarrow{AP} = [x - x_1, y - y_1, z - z_1]$

and $n = [a, b, c]$

As $\overrightarrow{AP}$ is perpendicular to $\underline{n}$.



$$\underline{n} \cdot \overrightarrow{AP} = 0$$

i.e., $[a, b, c] \cdot [x - x_1, y - y_1, z - z_1] = 0$

i.e., $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

is the required equation of the plane. This is known as *point normal* form of the equation of the plane.

10.2.3. Theorem:

If a, b, c and d are constants and a, b and c not all zero, then the equation $ax + by + cz + d = 0$ represents a plane having vector, $\underline{n} = [a, b, c]$ as a normal.

Proof: By hypothesis, a, b and c are not all zero. For the moment, we assume that $a \neq 0$. The equation $ax + by + cz + d = 0$ can be re-written as

$$a\left(x + \frac{d}{a}\right) + by + cz = 0$$

i.e., $a\left[x - \left(-\frac{d}{a}\right)\right] + b[y - 0] + c[z - 0] = 0$

But this is a point normal form of the equation of the plane passing through the point $\left(-\frac{d}{a}, 0, 0\right)$ and having $\underline{n} = [a, b, c]$ as normal.

If $a = 0$, then either $b \neq 0$ or $c \neq 0$. These cases can be handled in a similar way.

The equation $ax + by + cz + d = 0$ is called the *general form* of the equation of a plane.

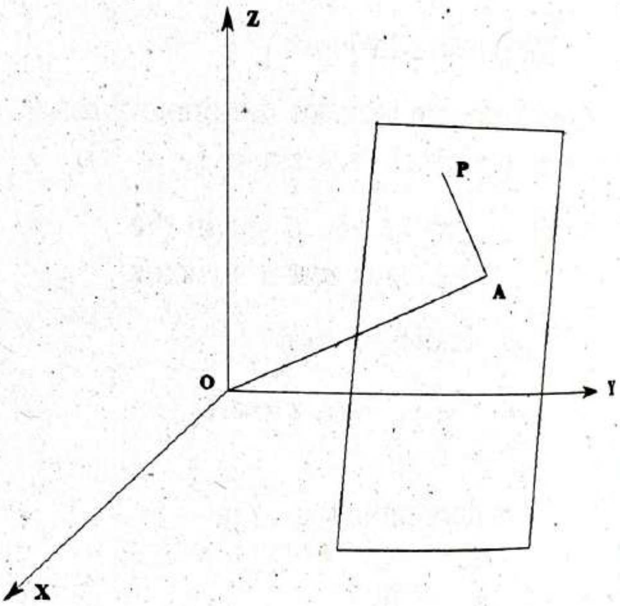
10.2.4. Normal Form of the Equation of a Plane:

To find the equation of a plane in terms of p, the lenght of the normal from the origin to it and l, m, n, the direction cosines of the normal. (p is to be always regarded positive).

Let OA be the normal from O to the given plane; A being the foot of perpendicular.

Then $|OA| = p$ and l, m, n are direction cosines of OA. Hence the coordinates of A are (pl, pm, pn).

Let P(x, y, z) be any point on the plane.



$$\overrightarrow{PA} = [x - pl, y - pm, z - pn]$$

It is obvious that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{AP}$.

$$\therefore \quad \overrightarrow{OA} \cdot \overrightarrow{AP} = 0$$

i.e., $[l, m, n] \cdot [x - pl, y - pm, z - pn] = 0$

i.e., $l(x - pl) + m(y - pm) + n(z - pn) = 0$

i.e., $lx + my + nz = p(l^2 + m^2 + n^2)$

or $lx + my + nz = p$ because $l^2 + m^2 + n^2 = 1$

10.2.5. To reduce the general equation $ax + by + cz + d = 0$ of a plane to the normal form $lx + my + nz = p$.

As the two equations represent the same plane, comparing coefficients of the two equations we get

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

Thus the direction cosines l, m, n of a normal to the plane are proportional to a, b, c and

$$p = \frac{-d}{\pm\sqrt{a^2 + b^2 + c^2}} = \frac{-d}{\pm\sqrt{\Sigma a^2}}$$

But p is always regarded positive, therefore, we take positive or negative sign before the expression $\sqrt{\Sigma a^2}$ according as d is negative or positive.

Thus if d be positve

$$l = -\frac{a}{\sqrt{\Sigma a^2}},\ m = -\frac{b}{\sqrt{\Sigma a^2}},\ n = -\frac{c}{\sqrt{\Sigma a^2}} \text{ and } p = \frac{d}{\sqrt{\Sigma a^2}}$$

If, d, be negative, we have only to change the signs of all these. Thus the normal form is obtained by dividing the given equation throughout by $\sqrt{\Sigma a^2}$ or $-\sqrt{\Sigma a^2}$ according as d is negative or positive.

Example: Reduce the equation $2x - y + 2z + 1 = 0$ to the normal form.

$\therefore$ the direction cosines of a normal to the plane are $-\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$ and the length of the perpendicular from the origin to the plane is $\frac{1}{3}$.

10.2.6. Intercept Form of the Equation of a Plane:

To find the equation of a plane in terms of intercepts a, b, c which it makes on the axes.

Let the equation of the plane be

$$Ax + By + Cz + D = 0 \qquad \ldots\ldots \text{(i)}$$

The plane meets the axes in the points (a, 0, 0), (0, b, 0), (0, 0, c).

As these points lie on plane (i), we get

$Aa + D = 0$ i.e., $A = -\frac{D}{a}$,

$Bb + D = 0$ i.e., $B = -\frac{D}{b}$,

$Cc + D = 0$ i.e., $C = -\frac{D}{c}$,

$\therefore$ (i) becomes $-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$

or $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which is the required equation.

10.2.7. Plane Through Three Points:

To find the equation of the plane passing through the three non-collinear points (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3)

Let the required equation of the plane be

$$ax + by + cz + d = 0 \quad \text{...... (i)}$$

As the given points lie on the plane, we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{(ii)}$$
$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{(iii)}$$
$$ax_3 + by_3 + cz_3 + d = 0 \quad \text{(iv)}$$

Eliminating, a, b, c,d from (i) — (iv), we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

which is the required equation of the plane.

Note: 1. Through three collinear points, infinite number of planes pass.

2. In actual numerical exercises it is more convinient to follow the method used in the example below.

Example: Find the equation of the plane through the three points A(2, 1, 1), B(6, 3, 1) and C(−2, 1, 2).

Example: The general equation of a plane through (2, 1, 1) is

$$a(x-2) + b(y-1) + c(z-1) = 0 \quad \text{...... (i)}$$

It will pass through B and C if

$$4a + 2b + 0c = 0$$

and $-4a + 0b + c = 0$

These give $\frac{a}{2} = \frac{b}{-4} = \frac{c}{8}$ or $\frac{a}{1} = \frac{b}{-2} = \frac{c}{4}$

Substituting these values in (i) we have

$$1(x-2) - 2(y-1) + 4(z-1) = 0$$

i.e., $x - 2y + 4z - 4 = 0$ as the required equation.

10.2.8. Angle Between Two Planes:

The angle between two planes is defined as the angle between their normal vectors. Thus the angle between the two planes;

$$a_1x + b_1y + c_1z + d_1 = 0, \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

is equal to the angle between the lines with direction ratios

$a_1, b_1, c_1 \quad ; \quad a_2, b_2, c_2$

and is, therefore, $= \cos^{-1}\left(\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(\Sigma a_1^2)}\sqrt{(\Sigma a_2^2)}}\right)$

Therefore, two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular. *Thus the two planes.*

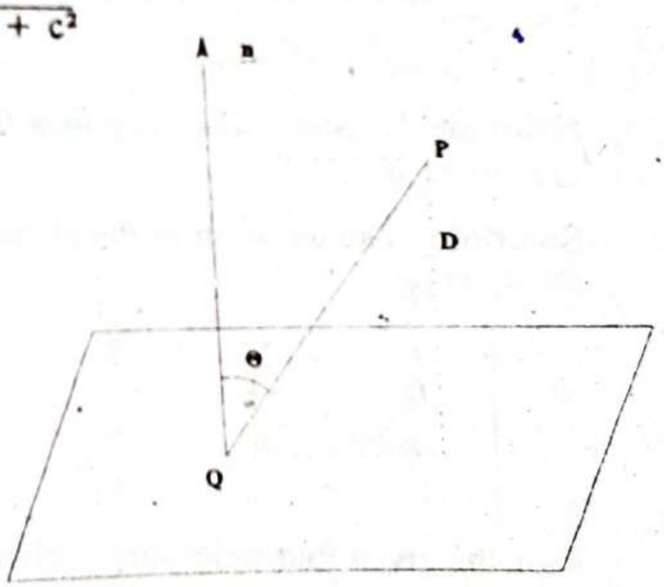
$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

will be parallel, if $a_1/a_2 = b_1/b_2 = c_1/c_2$

and will be perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

10.2.9. Distance of a Point From a Plane:

The perpendicular distance of a point $P(x_1, y_1, z_1)$ from the plane $ax + by + cz + d = 0$ is $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$



Proof: Let $Q(x_0, y_0, z_0)$ be any point in the plane, and position of the normal. $\underline{n} = [a, b, c]$ be so that its intitial point is at Q.

As in the figure, the distance D is equal to PQ Cosθ

$$D = \frac{QP \cos\theta\ |\underline{n}|}{|\underline{n}|} = \frac{|\overline{QP} \cdot \underline{n}|}{|\underline{n}|}$$

But $\overrightarrow{QP} = [x_1 - x_0, y_1 - y_0, z_1 - z_0]$

$$\overrightarrow{QP} \cdot \underline{n} = [x_1 - x_0, y_1 - y_0, z_1 - z_0] \cdot [a, b, c]$$
$$= a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)$$

and $|\underline{n}| = \sqrt{a^2 + b^2 + c^2}$

Thus distance $D = \dfrac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since the point $Q(x_0, y_0, z_0)$ lies in the plane, its coordinates satisfy the equation of the plane, so that

$$ax_0 + by_0 + cz_0 + d = 0 \text{ or } ax_0 + by_0 + cz_0 = -d$$

So we get distance $= D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Note: By distance between two points we mean the straight distance between the two points and by distance of a point from a line or plane we mean the perpendicular distance of the point from the line or plane.

10.2.10. Solved Examples:

Example 1: Find the distance between the parallel planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

Solution: To find the distance between the planes, we may select an arbitrary point in one of the planes and compute its distance to the other plane. By setting $y = z = 0$ in the equation $x + 2y - 2z = 3$, we obtain the point $P(3, 0, 0)$ in this plane. Thus the required distance is the distance from P to the plane $2x + 4y - 4z - 7 = 0$

i.e., $\dfrac{|2(3) + 4(0) - 4(0) - 7|}{\sqrt{2^2 + 4^2 + 4^2}} = \dfrac{1}{6}$

Example 2: Show that the points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar.

Solution: The equation of the plane through the points $(0, -1, -1)$, $(4, 5, 1)$ and $(3, 9, 4)$ is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix} = 0$$

Thus the given four points are coplanar if

$$\begin{vmatrix} -4 & 4 & 4 & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix} = 0$$

$$\text{Now L.H.S} = \begin{vmatrix} -4 & 4 & 4 & 1 \\ 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \end{vmatrix}$$

Adding 4th column in 2nd and 3rd column

$$= \begin{vmatrix} -4 & 5 & 5 & 1 \\ 0 & 0 & 0 & 1 \\ 4 & 6 & 2 & 1 \\ 3 & 10 & 5 & 1 \end{vmatrix} = \begin{vmatrix} -4 & 5 & 5 \\ 4 & 6 & 2 \\ 3 & 10 & 5 \end{vmatrix}$$

$= -4(30 - 20) - 5(20 - 6) + 5(40 - 18)$

$= -40 - 70 + 110 = 0$. Hence the four points are coplanar.

Example 3: Find the equation of the plane through (5, −1, 4) and perpendicular to each of the planes, $x + y - 2z - 3 = 0$ and $2x - 3y + z = 0$.

Solution: A plane through (5, −1, 4) is

$$a(x - 5) + b(y + 1) + c(z - 4) = 0$$

As it is perpendicular to the two given planes, we have

$$a + b - 2c = 0$$

$$2a - 3b + c = 0$$

$$\therefore \quad \frac{a}{1 - 6} = \frac{b}{-4 - 1} = \frac{c}{-3 - 2} \quad \text{or} \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

Hence the equation of the required plane is

$$x - 5 + y + 1 + z - 4 = 0 \text{ i.e., } x + y + z - 8 = 0$$

Example 4: Find the equation of the plane through the points (1, 0, 1) and (2, 2, 1) and perpendicular to the plane $x - y - z + 4 = 0$. (P.U. 1988)

Solution: A plane through (1, 0, 1) is

$$a(x - 1) + b(y - 0) + c(z - 1) = 0 \qquad \text{...... I}$$

As it passes through the point (2, 2, 1)

$$a + 2b + 0c = 0 \qquad \text{...... II}$$

Also plane I is perpendicular to the plane $x - y - z + 4 = 0$, therefore

$$a - b - c = 0 \qquad \text{...... III}$$

From II & III we get.

$$\frac{a}{-2 - 0} = \frac{b}{0 + 1} = \frac{c}{-1 - 2} \quad \text{or} \quad \frac{a}{2} = \frac{b}{-1} = \frac{c}{3}$$

∴ Required equation of the plane is

$$2(x - 1) - y + 3(z - 1) = 0 \quad \text{i.e.,} \quad 2x - y + 3z - 5 = 0$$

Example 5: Find the equation of the plane through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and the point (1, 1, 1).

Solution: Tthe plane

$$x + y + z - 6 + k(2x + 3y + 4z + 5 = 0 \qquad \text{...... (i)}$$

passes through the intersection of the given planes for all values of k.

It will pass through (1, 1, 1) if

$$-3 + 14k = 0 \text{ or } k = \frac{3}{14}$$

Putting $k = \frac{3}{14}$ in (i), we get

$20x + 23y + 26z - 69 = 0$ which is the required equation of the plane.

Example 6: Find the measure of the acute angle between the planes
$2x + y - z = 7, \quad x + 2y + z = 6$ (P.U. 1991)

Solution: The direction ratios of the normals to the two planes are 2, 1, −1 and 1, 2, 1.

The angle between the two planes is the same as the angle between the normals to the two planes. Therefore, if θ is the angle between them

$$\text{Cos}\theta = \frac{2 \cdot 1 + 1 \cdot 2 + (-1) \cdot 1}{\sqrt{4+1+1} \cdot \sqrt{1+4+1}} = \frac{3}{6} = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

EXERCISE 10.2

1. Find the equation of the plane through the three given points
   (i) (2, 2, −1), (3, 4, 2), (7, 0, 6)
   (ii) (4, −1, 2), (−3, −2, −1), (7, −1, 3)
2. Convert the equations of the planes
   $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ to normal forms. Also find the measure of the angle between them.
3. Find the equation of the plane through the point (2, −3, 1) and which is normal to the straight line joining the points (3, 4, −1) and (2, −1, 5).
4. Find the measure of the acute angle between the planes.
   (i) $2x - y + z - 6 = 0, \quad x + y + 2z - 3 = 0$
   (ii) $2x + y - z - 5 = 0, \quad x - y - 2z + 7 = 0$
5. Find the equation of the plane which is perpendicular bisector of the line segment joining the points (3, 4, −1) and (5, 2, 7). (P.U. 1990)
6. Find the equation of the plane passing through the point (−1, 3, 2) and perpendicular to the planes $x + 2y - 2z = 5$ and $3x + 3y + 2z = 8$. (P.U. 1991)
7. Find the equation of the plane through the points (2, 2, 1), (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 9$.
8. Write the equation of the family of all planes whose distance from the origin is seven. Find those members which are parallel to the plane $x + y + z + 5 = 0$. (P.U. 1990)

9. Find the equation of the plane which passes through the point (3, 4, 5), has an x-intercept equal to −5 and is perpendicular to the plane $2x + 3y - z = 8$.

(P.U. 1985)

10. Find the equation of the plane which passes through the intersection of the planes $2x + y - 4 = 0$, $y + 2z = 0$ and which

(i) is perpendicular to the plane $3x + 2y - 3z = 7$.

(ii) passes through the point (2, −1, 1).

11. Write the equation of the family of planes having x-intercept 5, y-intercept 2 and a non-zero, z-intercept. Find the member of the family which is perpendicular to the plane $3x - 2y + z - 4 = 0$.

12. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

(P.U. 1991)

13. The vertices of a tetrahedron are (0, 0, 0), (3, 0, 0), (0, −4, 0) and (0, 0, 5). Find the equations of the planes that form the sides.

14. Find the equation of the plane through the intersection of the planes

$2x - y + 3z = 0$ and $x + 2y - 2z - 3 = 0$

and (i) at a unit distance from the origin.

(ii) perpendicular to the plane $3x - 2y + 4z - 5 = 0$ (P.U. 1988)

(iii) having x-intercept 6.

15. Find the locus of the point whose distance from the origin is three times its distance from the plane $2x - y + 2z = 3$.

Exercise 10.2

1.(i) Solution:

The three points are P (2, 2, −1), Q (3, 4, 2) and R (7, 0, 6)

The general equation of a plane through P (2, 2, −1) is

$$a(x-2)+b(y-2)+c(z+1)=0$$

It will pass through Q and R, if

$$a+2b+3c=0. \text{ and}$$

$$5a-2b+7c=0,$$

These give $\frac{a}{20}=\frac{b}{8}=\frac{c}{-12}$ or $\frac{a}{5}=\frac{b}{2}=\frac{b}{-3}$

Subsituting these values in (i), we ahve

$$5(x-2)+2(y-2)-3(z+1)=0,$$

i.e. $5x+2y-3z-17=0$ as the required equation.

(ii) The three points are P (4, −1, 2), Q (−3, −2, −1) and R (7, −1, 3)

Equation of a plane through P (4, −1, 2) is $a(x-4)+b(y+1)+c(z-2)=0$

Since the other two points lie on this plane

$$-7a-b-3c=0 \quad \text{and} \quad 3a+0b+c=0$$

which give $\frac{a}{-1}=\frac{b}{-2}=\frac{c}{3}$

Hence the equation of the plane is

$$4+2(y+1)-3(z-2)=0 \text{ i.e. } x+2y-3z+4=0$$

2. Solution:

$3x - 4y + 5z = 0$ ...... (i)

Dividing by $\sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$

We get the normal form of the equation

i.e. $\frac{3}{5\sqrt{2}}x - \frac{4}{5\sqrt{2}}y + \frac{5}{5\sqrt{2}}z = 0$

And $2x - y - 2z = 5$ ...... (ii)

Dividing by $\sqrt{4 + 1 + 4} = 3$ we get $\frac{2}{3}x - \frac{1}{3}y - \frac{2}{3}z = \frac{5}{3}$

Measure of the angle between the planes is given by

$$\cos\theta = \frac{3}{5\sqrt{2}} \cdot \frac{2}{3} + \left(\frac{-4}{5\sqrt{2}}\right)\left(-\frac{1}{3}\right) + \frac{5}{5\sqrt{2}}\left(-\frac{2}{3}\right)$$

$$= \frac{2}{5\sqrt{2}} + \frac{4}{15\sqrt{2}} - \frac{2}{3\sqrt{2}}$$

$$= \frac{6 + 4 - 10}{15\sqrt{2}} = 0. \text{ Hence } \theta = \frac{\pi}{2}.$$

3. Solution:

The direction ratios of the given line are

$[2 - 3, -1 - 4, 5 - (-1)]$ or $[-1, -5, 6]$,

Hence the required plane through the point $(2, -3, 1)$ is

$(-1)(x - 2) + (-5)(y + 3) + 6(z - 1) = 0$

or $-x - 5y + 6z - 19 = 0$ or $x + 5y - 6z + 19 = 0$

4. Solution:

(i) The direction ratios of the normals to the planes are $2, -1, 1$ and $1, 1, 2$.

$\therefore$ Measure of the angle between them is given by

$$\cos\theta = \frac{(2)(1) + (-1)(1) + (1)(2)}{\sqrt{2^2 + (-1)^2 + 1^2} \cdot \sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{6} = \frac{1}{2}, \therefore \theta = \frac{\pi}{3}$$

(ii) The direction ratios of the normals to the planes are $2, 1, -1$ and $1, -1, -2$.

Let θ be measure of the angle between the planes. Then

$$\cos\theta = \frac{2 - 1 + 2}{\sqrt{6} \cdot \sqrt{6}} = \frac{3}{6} = \frac{1}{2}. \text{ Hence } \theta = \frac{\pi}{3}.$$

5. Solution:

The direction ratios of the line joining the given points are

$5 - 3, 2 - 4, 7 - (-1)$ i.e. $2, -2, 8$

Also the mid-point of the segment joining the given points is

$\left(\frac{5 + 3}{2}, \frac{2 + 4}{2}, \frac{7 - 1}{2}\right) \equiv (4, 3, 3)$. Hence the required plane is

$$2(x-4)+(-2)(y-3)+8(z-3)=0$$

or $\quad 2x-2y+8z=26 \quad$ or $\quad x-y+4z=13$

6. Solution:

Equation of a plane through the point $(-1, 3, 2)$ is

$$a(x+1)+b(y-3)+c(z-2)=0 \qquad \text{.....(i)}$$

It will be perpendicular to the two given planes if

$$a+2b-2c=0 \quad \text{and} \quad 3a+3b+2c=0$$

which give $\dfrac{a}{4+6}=\dfrac{b}{-6-2}=\dfrac{c}{3-6}$ i.e. $\dfrac{a}{10}=\dfrac{b}{-8}=\dfrac{c}{-3}$

∴ The required equation of the the plane is

$$10(x+1)-8(y-3)-3(z-2)=0 \text{ i.e. } 10x-8y-3z+40=0$$

7. Solution:

Any plane through $(2, 2, 1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0 \qquad \text{......(i)}$$

It will pass through $(9, 3, 6)$ if

$$a(9-2)+b(3-2)+c(6-1)=0$$

i.e. $\quad 7a+b+5c=0 \qquad$ ......(ii)

The plane (1) will be perpendicular to the given plane if

$$2a+6b+6c=0 \qquad \text{......(iii)}$$

From (ii) and (iii), we have

$$\frac{a}{-24}=\frac{b}{-32}=\frac{c}{40} \quad \text{or} \quad \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}$$

Substituting in (i), we see that the equation of the required plane is

$$3(x-2)+4(y-2)-5(z-1)=0 \quad \text{or} \quad 3x+4y-5z=9$$

8. Solution:

The equation of the family in normal form is $lx+my+nz=7$ where l, m, n are direction cosines of a normal to the plane. The equation of the plane $x+y+z+5=0$ in the normal form is

$$\frac{x}{-\sqrt{3}}+\frac{y}{-\sqrt{3}}+\frac{z}{-\sqrt{3}}+\frac{5}{-\sqrt{3}}=0 \qquad \text{......(1)}$$

A plane parallel to (1) has normal vector with direction cosines

$$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ or } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus, there are two members of the family parallel to (1). They are

$$-\frac{1}{\sqrt{3}}x-\frac{1}{\sqrt{3}}y-\frac{1}{\sqrt{3}}z-7=0 \text{ and } \frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z-7=0$$

9. Solution:

Let the equation of the plane be

$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$, Here $A = -5$.

Thus $\boxed{\frac{x}{-5} + \frac{y}{B} + \frac{z}{C} = 1}$ ...... (1)

As this plane is perpendicular to $2x + 3y - z = 8$ we have

$$2\left(-\frac{1}{5}\right) + 3\left(\frac{1}{B}\right) - \frac{1}{C} = 0 \quad \therefore \quad -\frac{2}{5} + \frac{3}{B} - \frac{1}{C} = 0$$

$$\frac{3}{B} - \frac{1}{C} = \frac{2}{5} \qquad ..... (2)$$

Also the plane (1) passes through (3, 4, 5),

$$\therefore \quad \frac{3}{-5} + \frac{4}{B} + \frac{5}{C} = 1 \text{ or } \frac{4}{B} + \frac{5}{C} = 1 + \frac{3}{5} = \frac{8}{5} \qquad ...... (3)$$

$\therefore$ From (2) and (3)

$\therefore$ $B = \frac{95}{18}$, $C = \frac{95}{16}$

$\therefore$ Equation of the plane is

$$\frac{x}{-5} + \frac{18y}{95} + \frac{16z}{95} = 1 \implies 19x - 18y - 16z + 95 = 0$$

10. Solution:

Let the equation of the plane be $2x + y - 4 + k(y + 2z) = 0$

or $2x + (k + 1)y + 2kz - 4 = 0$ ..... (i)

(i) It will be perpendicualr to $3x + 2y - 3z = 7$

if $2.3 + (k + 1).2 + 2k.(-3) = 0 \implies k = 2$

$\therefore$ The required plane is $2x + 3y + 4z - 4 = 0$

(ii) The plane (i) will pass through the point (2, −1, 1) if

$2.2 + (k + 1)(-1) + 2k.1 - 4 = 0$ i.e. $4-k-1+2k-4=0 \implies k=1$

The required equation of the plane is

$2x + 2y + 2z - 4 = 0$ or $x + y + z = 2$

11. Solution:

The given plane is

$3x - 2y + z - 4 = 0$ ..... (i)

Let non-zero z-intercept be c. Equation of the required family of planes is

$$\frac{x}{5} + \frac{y}{2} + \frac{z}{c} = 1, \text{ where c is a parameter.}$$

If a member of this family is perpendicular to (1), then we have

$$\frac{3}{5} - \frac{2}{2} + \frac{1}{c} = 0 \text{ or } \frac{1}{c} = \frac{2}{5} \text{ i.e. } c = \frac{5}{2}$$

The required plane is $\frac{x}{5} + \frac{y}{2} + \frac{z}{5/2} = 1$ i.e. $\frac{x}{5} + \frac{y}{2} + \frac{2z}{5} = 1$.

12. Solution:

Let the equation of the plane be

$lx + my + nz = p, \quad l^2 + m^2 + n^2 = 1$ ...... (1)

$$\Rightarrow \quad \frac{x}{p/l} + \frac{y}{p/m} + \frac{z}{p/n} = 1$$

Thus coordinates of A, B, C are respectivley $(p/l, 0, 0)$, $(0, p/m, 0)$, $(0, 0, p/n)$

Equation of the plane through A $\left(\frac{p}{l}, 0, 0\right)$ and parallel to yz plane is $x = \frac{p}{l}$

Similarly plane through $B\left(0, \frac{p}{m}, 0\right)$ and parallel to zx plane is $y = \frac{p}{m}$

and plane through $C\left(0, 0, \frac{p}{n}\right)$ and parallel to xy-plane is $z = \frac{p}{n}$

Thus $l = \frac{p}{x}, m = \frac{p}{y}, n = \frac{p}{z}$. Since $l^2 + m^2 + n^2 = 1$

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 1 \; ; \quad \therefore x^{-2} + y^{-2} + z^{-2} = p^{-2}$$

13. Solution:

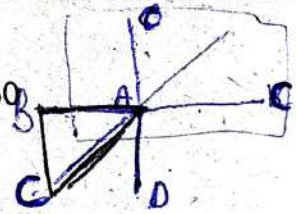
Let the vertices be denoted by $A = (0, 0, 0)$, $B = (3, 0, 0)$, $C = (0, -4, 0)$ and $D = (0, 0, 5)$.

Equation of any plane through the point A is $ax + by + cz = 0$ → (1)

If it passes through, B and C, then $a = 0$ and $b = 0$. Thus the plane through A, B, C is $cz = 0$, or $z = 0$.

Similarly, the plane through A, B and D is $y = 0$ and the plane through A, C and D is $x = 0$, Now we fined the plane through B, C and D. This can be written in the intercept form as

$$\frac{x}{3} + \frac{y}{-4} + \frac{z}{5} = 1 \quad \text{or} \quad 20x - 15y + 12z = 60$$



14. Solution:

Any plane through the intersection of the given planes is

$$2x - y + 3z + k(x + 2y - 2z - 3) = 0$$

or $\quad (2 + k)x + (2k - 1)y + (3 - 2k)z = 3k$ ...... (1)

(i) Now the perpendicular distance of this plane from the origin is

$$\frac{3k}{\pm\sqrt{(2+k)^2 + (2k-1)^2 + (3-2k)^2}} = 1 \quad (\because \text{ of given})$$

Squaring and cross multiplying we get

$$9k^2 = 9k^2 - 12k + 14 \; ; \quad \therefore k = \frac{7}{6}$$

Putting this value of k in (1) we get $19x + 8y + 4z = 21$

which is the required equaiton of the plane at a unit distance from the origin.

(ii) Since the plane (1) is perpendicular to the plane $3x - 2y + 4z - 6 = 0$, therefore,

$$3(2 + k) - 2(2k - 1) + 4(3 - 2k) = 0 \text{ or } 9k = 20 \text{ i.e. } k = \frac{20}{9}$$

Putting this value of k in (1) we have

$$38x + 31y - 13z - 60 = 0$$ which is the required plane.

(iii) Here x-intercept $= \dfrac{3k}{k+2} = 6$, $3k = 6k + 12$ or $k = -4$

Equation of the required plane is

$$-2x - 9y + 11z + 12 = 0 \quad \text{i.e.} \quad 2x + 9y - 11z - 12 = 0$$

15. Solution:

Given plane is $2x - y + 2z - 3 = 0$ ...... (i)

Let P (x, y, z) be any point on the locus

PO = 3. distance of P from (i) where O is the origin.

$$\therefore \quad \sqrt{x^2 + y^2 + z^2} = 3 \cdot \frac{2x - y + 2z - 3}{\sqrt{4 + 1 + 4}}$$

or $\quad x^2 + y^2 + z^2 = (2x - y + 2z - 3)^2$

or $\quad x^2 + y^2 + z^2 = 4x^2 + y^2 + 4z^2 + 9 - 4xy + 8xz - 12x - 4yz + 6y - 12z$

i.e. $\quad 3x^2 + 3z^2 - 4xy + 8xz - 4yz - 12x + 6y - 12z + 9 = 0$ locus:

9. Find the equation of the plane which passes through the point (3, 4, 5), has an x-intercept equal to −5 and is perpendicular to the plane $2x + 3y - z = 8$. (P.U. 1985)

10. Find the equation of the plane which passes through the intersection of the planes
$2x + y - 4 = 0, \quad y + 2z = 0$ and which
(i) is perpendicular to the plane $3x + 2y - 3z = 7$.
(ii) passes through the point (2, −1, 1).

11. Write the equation of the family of planes having x-intercept 5, y-intercept 2 and a non-zero, z-intercept. Find the member of the family which is perpendicular to the plane $3x - 2y + z - 4 = 0$.

12. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$. (P.U. 1991)

13. The vertices of a tetrahedron are (0, 0, 0), (3, 0, 0), (0, −4, 0) and (0, 0, 5). Find the equations of the planes that form the sides.

14. Find the equation of the plane through the intersection of the planes
$2x - y + 3z = 0$ and $x + 2y - 2z - 3 = 0$
and (i) at a unit distance from the origin.
(ii) perpendicular to the plane $3x - 2y + 4z - 5 = 0$ (P.U. 1988)
(iii) having x-intercept 6.

15. Find the locus of the point whose distance from the origin is three times its distance from the plane $2x - y + 2z = 3$.

THE STRAIGHT LINE

10.3.1. Definition:

If A and B are two distinct but fixed points in 3-space, we define the straight line AB as the set of all points P such that the vector $\overrightarrow{AP}$ is collinear with the vector $\overrightarrow{AB}$. In plane analytic geometry a line is represented by a single linear equation in x, y. In 3-space a line is, represented (as we will see) by two linear equations in x, y, z.

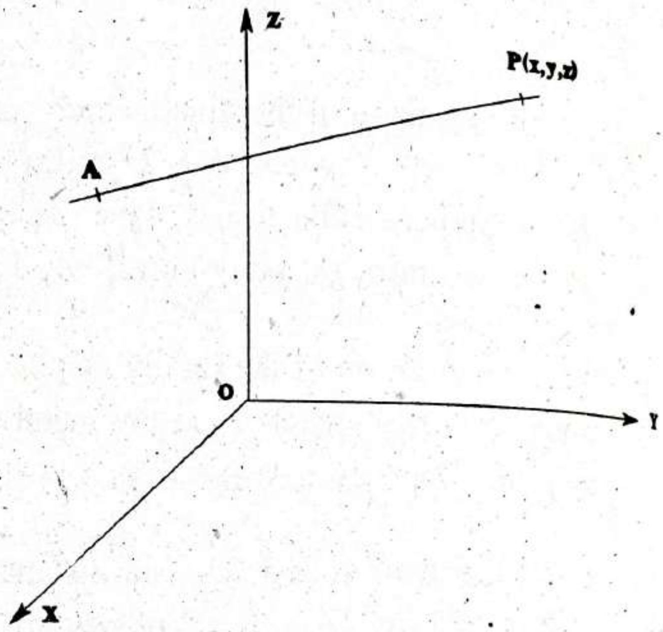
10.3.2. Symmetric Equations of a Line:

Let L be a line passing through the point $A(x_1, y_1, z_1)$ and parallel to a non-zero vector

$\underline{a} = l\underline{i} + m\underline{j} + n\underline{k}$

where l, m, n are the direction cosines of the vector $\underline{a}$, as well as, those of the line L.

Then L is the set of points $P(x, y, z)$ such that the vector $\overrightarrow{AP}$ is parallel to the given vector $\underline{a}$.



Thus there is a scalar $t \in R$ such that

$$\overrightarrow{AP} = t\,\underline{a}$$

or $\quad [x - x_1, y - y_1, z - z_1] = t[l, m, n]$

which gives $x - x_1 = tl$, $y - y_1 = tm$, $z - z_1 = tn$

Eliminating t we get $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ ...... (A)

as the equations of the line in the *symmetric form.*

The equations (A) written as

$$x = x_1 + tl$$
$$y = y_1 + tm$$
$$z = z_1 + tn$$

are called the *parametric equations* of the line L.

Note 1: The equation (A) may be written as the equations of three planes — but they are not independent, since any two may be combined to give the equation of the third plane. Because the coordinates of every point on the line must satisfy each of these three equations; each plane contains the line and is perpednicular to one of the coordinate axes. These planes are called the *projecting planes* of the line.

In particular, as the x-axis is the intersection of the XZ and XY planes, its equations are $y = 0$, $z = 0$ taken together. Similarly the equations of the y-axis are $x = 0$, $z = 0$ and of the z-axis are $x = 0$, $y = 0$.

Note 2: If l, m, n are proportional to l_1, m_1, n_1, equations (A) become

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$

These may also be taken as the equations of the line L.

Example: Find the equations to the line passing through the point (0, −3, 2) and parallel to the line joining the points (3, 4, 7) and (2, 7, 5).

Solution: The direction ratios of the line joining the points (3, 4, 7) and (2, 7, 5) are 2 − 3, 7 − 4, 5 − 7 or −1, 3, −2 or 1, −3, 2.

Hence the equations of the required line through the point (0, −3, 2) are

$$\frac{x}{1} = \frac{y+3}{-3} = \frac{z-2}{2}$$

10.3.3. Two Point Form of a Line:

The line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) has $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ as its direction ratios. Thus the required equations of the line are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Example: Find the equations of the line through the points (3, 4, 5) and (5, −2, 3).

Solution: The direction ratios of the line are 2, −6, −2 or −2, 6, 2. The equations of the line are

$$\frac{x-3}{2} = \frac{y-4}{-6} = \frac{z-5}{-2} \quad \text{or} \quad \frac{x-5}{-2} = \frac{y+2}{6} = \frac{z-3}{-2}$$

depending upon which point we select as (x_1, y_1, z_1). The alternate answers give the same projecting planes.

10.3.4. General Equations of a Line:

A straight line is determined when two planes intersect in space.

If $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

are the equations of two intersecting planes, any point whose coordinates satisfy both equations is a point on the line determined by the two planes. Conversely, the coordinates of any point on the line will satisfy both the equations.

Thus the equations of two intersecting planes may be considered as the equations of a line.

The general equations of a line may be reduced to the symmetric form by eliminating first one variable, say x, and so obtaining a projecting plane in the other two variables; and then by eliminating a second variable, say y, and so obtaining another projecting plane in x and z. If the two resulting equations are solved for z, the symmetric form may be obtained by equating these values of z.

Example: Reduce the equations of the line $3x + 2y + 4z + 5 = 0$ and $x - y + 2z - 4 = 0$ to the symmetric form.

Solution: Solving the pair of equations simultaneously — first eliminating y, then eliminating x, we get

$$z = \frac{3 - 5x}{8} \quad \text{and} \quad z = \frac{5y + 17}{2}$$

Equating these values of z, and rearranging terms, we get

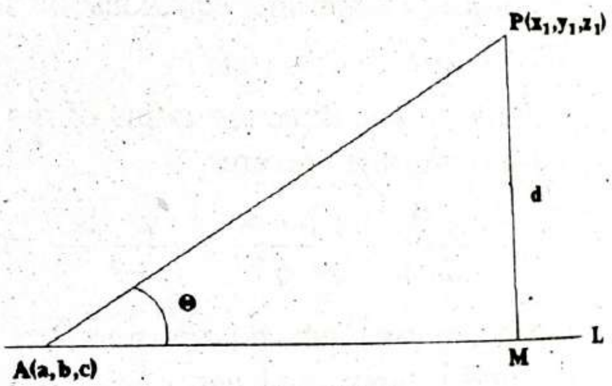
$$\frac{-5x + 3}{8} = \frac{5y + 17}{2} = z \text{ or } \frac{x - 3/5}{-8/5} = \frac{y + 17/5}{2/5} = \frac{z - 0}{1}$$

i.e. $$\frac{x - 3/5}{-8} = \frac{y + 17/5}{-2} = \frac{z - 0}{5}$$

10.3.5. Distance of a Point From a Line:

To find the perpendicular distance of the point $P(x_1, y_1, z_1)$ from the line.

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n}$$



Let $d = |PM|$ be the perpendicular distance of the point $P(x_1, y_1, z_1)$ from the given line. Then $d = |PM| = |\overrightarrow{AP}| \sin\theta$ where A(a, b, c) is a point on the line and θ is the measure of the angle between $\overrightarrow{AP}$ and the line.

Let $\underline{b} = [l, m, n]$ be the direction vector of the line then

$$d = |\overrightarrow{AP}| \sin\theta = \frac{|\overrightarrow{AP}|\,|\underline{b}| \sin\theta}{|\underline{b}|} = \frac{|\overrightarrow{AP} \times \underline{b}|}{|\underline{b}|}$$

Since $|b| \neq 0$ for all lines, we have $d = \dfrac{|\overrightarrow{AP} \times \underline{b}|}{|b|}$ as the formula.

10.3.6. Solved Examples:

Example 1: Show that the straight lines

$$\frac{x - 1}{1} = \frac{y}{2} = \frac{z - 3}{1} \quad \text{and} \quad \frac{x - 2}{1} = \frac{y - 2}{-1} = \frac{z - 4}{1}$$

are perpendicular

Solution: The direction ratios of the two lines are 1, 2, 1 and 1, −1, 1. And (1) (1) + (2) (−1) + (1) (1) = 0. The two lines are perpendicular.

Example 2: Find the equations of the straight line passing through the point [illegible] −2) and perpendicular to each of the straight lines

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \text{ and } \frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}$$ (P.U. 1985)

Solution: The direction ratios of the given lines are 2, 2, 2 and 3, −1, 2. Let l_1, m_1, n_1, be the direction ratios of the required line, then by the condition of perpendicularity, we have

$$2l_1 + 2m_1 + 2n_1 = 0$$
$$3l_1 - m_1 + 2n_1 = 0$$

$$\therefore \quad \frac{l_1}{4+2} = \frac{m_1}{6-4} = \frac{n_1}{-2-6}$$

$$\text{or} \quad \frac{l_1}{6} = \frac{m_1}{2} = \frac{n_1}{-8} \text{ or } \frac{l_1}{3} = \frac{m_1}{1} = \frac{n_1}{-4}$$

∴ the equations of the required line through (2, 0, −2) are

$$\frac{x-2}{3} = \frac{y}{1} = \frac{z+2}{-4}$$

Example 3: Prove that the planes $4x + 4y - 5z - 12 = 0$ and $8x + 12y - 13z - 32 = 0$ intersect and the equations of their line of intersection can be written as $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$ (P.U. 1988)

Solution: The direction ratios of the normals to the two planes are 4, 4, −5 and 8,12, −13. As these direction ratios are not proportional, the two planes intersect.

To find the equations of the line in the symmetric form we choose $z = 0$. Then we have $4x + 4y = 12$ and $8x + 12y = 32$, which give $x = 1, y = 2$.

Hence (1, 2, 0) is a point on the line. Similarly taking $x = 0$, we have $4y - 5z = 12$ and $12y - 13z = 32$ which give $y = \frac{1}{2}, z = -2$.

Hence $(0, \frac{1}{2}, -2)$ is an other point on the line.

Thus the equations of the line in symmertric form are

$$\frac{x-1}{0-1} = \frac{y-2}{\frac{1}{2}-2} = \frac{z-0}{-2-0} \quad \text{or} \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$$

Example 4: Find the equations of the line through the point (1, 2, 3) and parallel to the line $x - y + 2z - 5 = 0 = 3x + y + z + 6$.

Solution: If l_1, m_1, n_1 are the direction ratios of the required line, we have

$$l_1 - m_1 + 2n_1 = 0 \text{ and } 3l_1 + m_1 + n_1 = 0$$

which give $\frac{l_1}{-3} = \frac{m_1}{5} = \frac{n_1}{4}$

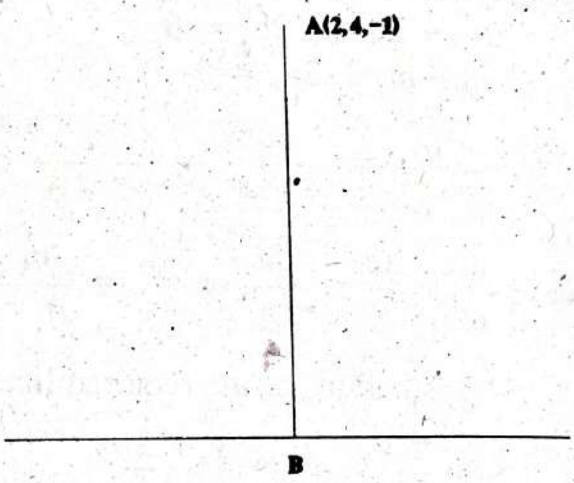
Hence the equations of the required line through the point (1, 2, 3) are

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

Example 5: Find the equations of the perpendicular from the point (2, 4, −1) to the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ (P.U. 1989)

Solution: The line is

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = t(\text{say})$$



Let $B(-5 + t, -3 + 4t, 6 - 9t)$ be a point on the line.

$$\overrightarrow{AB} = [-7 + t, -7 + 4t, 7 - 9t]$$

If B is foot of the perpendicular from the point A(2, 4, −1) to the line

$$1(-7+t)+4(-7+4t)-9(7-9t) = 0$$

which gives $t = 1$.

∴ point B is (−4, 1, −3) and the equations of the perpendicular from the point (2, 4, −1) to the line are

$$\frac{x-2}{-4-2} = \frac{y-4}{1-4} = \frac{z+1}{-3+1} \text{ or } \frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$$

Example 6: Determine whether the following given pair of lines intersect or not, and find the common point if they do;

$$\frac{x+3}{2} = \frac{y}{-2} = \frac{z-7}{6} \text{ and } \frac{x+6}{1} = \frac{y+5}{-3} = \frac{z-1}{2} \quad \text{(P.U. 1990)}$$

Solution: The parametric form of the equations are

$$x = 2t - 3, \; y = -2t, \; z = 6t + 7$$

and $x = t' - 6, \; y = -3t' - 5, \; z = 2t' + 1$

If the two lines intersect,

$$2t - 3 = t' - 6$$
$$-2t = -3t' - 5$$

and $6t + 7 = 2t' + 1$ should be consistant.

Solving the first two equations we get $t = -\frac{7}{2}$ and $t' = -4$.

But these values of t and t′ do not satisfy the third equation. Hence the two lines do not intersect.

Example 7: Show that for the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) to be collinear

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

is a necessary condition.

By taking three suitable points, show that the above condition is not sufficient.
(P.U. 1987)

Solution: Let the three points be collinear.

Then $[x_1, y_1, z_1] = m[x_2, y_2, z_2] + n[x_3, y_3, z_3]$

$$\therefore \quad x_1 = mx_2 + nx_3$$
$$y_1 = my_2 + ny_3$$
and $$z_1 = mz_2 + nz_3$$

or $$x_1 - mx_2 - nx_3 = 0$$
$$y_1 - my_2 - ny_3 = 0$$
and $$z_1 - mz_2 - nz_3 = 0$$

Eliminating m, n we get

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Now consider the three points (0, 0, 0), (0, 0, 1) and (0, 1, 0). They are

obviously not collinear, but $\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 0$

Hence the condition $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

is necessary but not sufficient.

EXERCISE 10.3

1. Find the equations of the line passing through the points (−3, 1, 4) and (5, −1, 6).

2. Determine whether the given pair of lines in each case intersect or not, and find the common point if they do.
   (i) $x = p - 1,\ y = 2 + p,\ z = -1 - 2p$ and
   $x = 1 + 2q,\ y = -6 + q,\ z = 5 - 3q$

   (ii) $\dfrac{x-3}{-1} = \dfrac{y-2}{-3} = \dfrac{z}{2}$ and
   $x = -1 + 2p,\ y = 2 + 2p,\ z = 5 - 3p.$ (P.U. 1990)

3. Find the coordinates of the foot of the perpendicular from the point (−3, 0, −2) to the straight line $\dfrac{x-2}{2} = \dfrac{y-2}{-2} = \dfrac{z-1}{1}$
   Also find the length and the equations of the perpendicular. (P.U. 1989)

4. Find the equations of the straight line passing through the point (0, −3, 2) and

parallel to the straight line joining the points (3, 4, 7) and (2, 7, 5).

5. Find the equations of the perpendicular from the point (1, 6, 3) to the straight line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Also obtain its length and coordinates of the foot of the perpendicular.

6. Find in symmetric form, the equations of the line

$$x + y - z + 1 = 0 = 4x + y - 2z + 2$$

and find its direction cosines. (P.U. 1990)

7. Find the equations of the straight line perpendicular to both the lines

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}, \quad \frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$$

and passing through their point of intersection. (P.U. 1986, 87)

8. Show that the lines $x + 2y - z - 7 = 0 = y + z - 2x - 6$ and $3x + 6y - 3z - 8 = 0 = 2x - y - z$ are parallel.

9. Show that the lines $x + 2y - 1 = 0 = 2y - z - 1$ and $x - y - 1 = 0 = x - 2z - 3$ are perpendicular.

10. Find the equation of the perpendicular from the origin to the line

$$x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$$

Also find the coordinates of the foot of the perpendicular.

11. Determine the direction cosines of the line given by the equations $z + 2y - 3 = 0$ and $x + 3y - z + 5 = 0$.
Also find the equations of the straight line passing through the point (5, −3, 2) and parallel to the line given above. (P.U. 1991)

12. Find the equations of the straight line passing through the point (3, 4, 5) and intersecting the z-axis at right angles.

13. Find the angle between the lines

$$3x + 2y + z - 5 = 0 = x + y - 2z - 3$$

and $2x - y - z = 0 = 7x + 10y - 8z$

14. Prove that the symmetric equations of the line formed by the intersection of the two planes $2x - 5y + z - [illegible] = 0$ and $x + y - 2z + 3 = 0$ are

$$x = \frac{y - 1/9}{5/9} = \frac{z - 14/9}{7/9}$$

Putting this value of k in (1) we have

$$38x + 31y - 13z - 60 = 0$$ which is the required plane.

(iii) Here x-intercept $= \dfrac{3k}{k+2} = 6$, $3k = 6k + 12$ or $k = -4$

Equation of the required plane is

$$-2x - 9y + 11z + 12 = 0 \quad \text{i.e.} \quad 2x + 9y - 11z - 12 = 0$$

15. Solution:

Given plane is $2x - y + 2z - 3 = 0$ ...... (i)

Let P (x, y, z) be any point on the locus

PO = 3. distance of P from (i) where O is the origin.

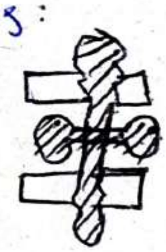
$\therefore$ $$\sqrt{x^2 + y^2 + z^2} = 3 \cdot \frac{2x - y + 2z - 3}{\sqrt{4 + 1 + 4}}$$

or $x^2 + y^2 + z^2 = (2x - y + 2z - 3)^2$

or $x^2 + y^2 + z^2 = 4x^2 + y^2 + 4z^2 + 9 - 4xy + 8xz - 12x - 4yz + 6y - 12z$

i.e. $3x^2 + 3z^2 - 4xy + 8xz - 4yz - 12x + 6y - 12z + 9 = 0$ locus:

Exercise 10.3



1. Solution:

The direction ratios of the line are 5 + 3, −1 − 1, 6 − 4 or 8, −2, 2

$\therefore$ The equations of the required line are

$$\frac{x+3}{8} = \frac{y-1}{-2} = \frac{z-4}{2} \quad \text{or} \quad \frac{x+3}{4} = \frac{y-1}{-1} = \frac{z-4}{1}$$

2.(i) Solution:

If the two lines intersect, the equations

$x = p - 1 = 1 + 2q$ i.e. $p = 2 + 2q$

$y = 2 + p = -6 + q$ i.e. $p = -8 + q$

and $z = -1 - 2p = 5 - 3q$ i.e. $2p = -6 + 3q$ should be consistant.

Solving the first two equations we get $q = -10$ and $p = -18$

These values satisfy the third equation, therefore the two lines intersect and the point of intersection is (−19, −16, 35)

(ii) Let $\dfrac{x-3}{-1} = \dfrac{y-2}{-3} = \dfrac{z}{2} = t$

$\therefore$ $x = 3 - t$, $y = 2 - 3t$, $z = 2t$

If the two lines intersect then the equations

$3 - t = -1 + 2p$, $2 - 3t = 2 + 2p$, $2t = 5 - 3p$ should be consistant.

Solving the first two equations we get

$t = -2$, $p = 3$, these values satisfy the third equation.

Thus, the two lines intersect and the point of intersection is (5, 8, −4)

3. Solution:

A point on the line

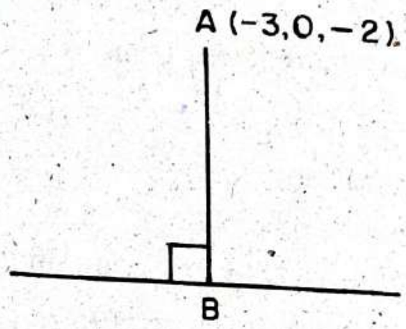
$$\frac{x-2}{2}=\frac{y-2}{-2}=\frac{z-1}{1}=t \text{ is } B\,(2+2p,\ 2-2p,\ 1+p)$$

Direction ratios of AB are $2+2p+3,\ 2-2p-0,\ 1+p+2$

i.e. $2p+5,\ -2p+2,\ p+3$

If B is the foot of the perpendicular from A to the given line, then

$$2(2p+5)-2(-2p+2)+1(p+3)=0 \text{ i.e. } 9p+9=0 \Longrightarrow p=-1$$



∴ The foot of the perpendicular is (0, 4, 0)

∴ Length of the perpendicular is

$$|AB|=\sqrt{(-3-0)^2+(0-4)^2+(-2-0)^2}=\sqrt{9+16+4}=\sqrt{29}$$

Equations of the perpendicular AB are

$$\frac{x+3}{0+3}=\frac{y-0}{4-0}=\frac{z+2}{0+2} \text{ or } \frac{x+3}{3}=\frac{y}{4}=\frac{z+2}{2}$$

4. Solution:

The line joining the points (3, 4, 7) and (2, 7, 5) has direction ratios:

$$[2-3,\ 7-4,\ 5-7]=[-1,\ 3,\ -2]$$

As the required line is parallel to the join of (3, 4, 7) and (2, 7, 5) it must have the same direction ratios i.e., [−1, 3, −2].

Thus the line through the point (0, −3, 2) with direction ratios −1, 3, −2 is

$$\frac{x-0}{-1}=\frac{y+3}{3}=\frac{z-2}{-2} \text{ or } \frac{x}{1}=\frac{y+3}{-3}=\frac{z-2}{2}$$

5. Solution:

Let A = (1, 6, 3) and B be the foot of the perpendicular on

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}=t,$$

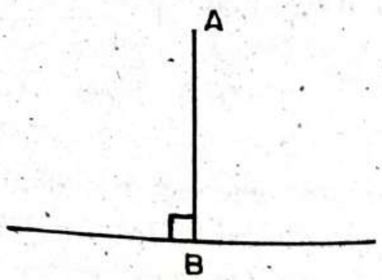
If $x=t,\ y=1+2t,\ z=3t+2$ be the coordinates of B, so that direction ratios of AB are

$$t-1,\ 1+2t-6,\ 3t+2-3 \text{ i.e. } t-1,\ 2t-5,\ 3t-1$$

Also the direction ratios of the given line are 1, 2, 3. By the condition of perpendicularity we have

$$(t-1)+2(2t-5)+3(3t-1)=0 \text{ or } 14t=14$$

i.e. $t=1$



Hence the coordinates of the foot of the perpendicular B are (1, 3, 5)

Length of the perpendicular

$$|AB| = \sqrt{(1-1)^2 + (3-6)^2 + (5-3)^2} = \sqrt{0+9+4} = \sqrt{13}$$

Equations of the perpendicular AB are

$$\frac{x-1}{1-1} = \frac{y-6}{3-6} = \frac{z-3}{5-3} \text{ or } \frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

6. Solution:

The line is $x + y - z + 1 = 0$; $4x + y - 2z + 2 = 0$

If we take $z = 0$, we get $x + y + 1 = 0$ and $4x + y + 2 = 0$.

which give $x = \frac{-1}{3}$, $y = \frac{-2}{3}$

$\therefore$ A point on the line is $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$.

If l, m, n are direction ratios of the line $l + m - n = 0$; $4l + m - 2n = 0$

which give $\frac{l}{1} = \frac{m}{2} = \frac{n}{3}$. $\therefore$ The equations of the line in symmetric form are

$$\frac{x + \frac{1}{3}}{1} = \frac{y + \frac{2}{3}}{2} = \frac{z}{3}$$

And direction cosines of line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

7. Solution:

The two lines are

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3} = p \text{ or } \frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2} = q$$

If the two lines intersect $x = 1 + p = -2 + 2q$ i.e. $p - 2q = -3$

$y = 1 + 2p = 5 - q$ i.e. $2p + q = 4$

$z = -2 + 3p = -3 + 2q$ i.e. $3p - 2q = -1$ are consistant.

Solving the first two equations we get $p = 1$, $q = 2$, which satisfy the third equation. Thus the two lines intersect and the point of intersection is (2, 3, 1).

If l, m, n are the direction ratio of the line perpendicular to the two given lines

$l + 2m + 3n = 0$ and $2l - m + 2n = 0$

which give $\frac{l}{7} = \frac{m}{4} = \frac{n}{-5}$

$\therefore$ The equations of the required line are $\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}$

8. Solution:

If l_1, m_1, n_1 are the direction ratios of the 1st line, then we have

$1 \cdot l_1 + 2m_1 - n_1 = 0$ and $-2l_1 + m_1 + n_1 = 0$

$\therefore \frac{l_1}{3} = \frac{m_1}{1} = \frac{n_1}{5}$ ...... (i)

Again if l_2, m_2, n_2 are the direction ratios of the 2nd line, we have

$$3l_2 + 6m_2 - 3n_2 = 0 \quad \text{and} \quad 2l_2 - m_2 - n_2 = 0$$

From (i) and (ii) we see that $\therefore \quad \frac{l_2}{-9} = \frac{m_2}{-3} = \frac{n_2}{-15}$ ...... (ii)

From (i) and (ii) we find that $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Therefore, the given two lines are parallel.

9. Solution:

Let the direction ratios of the 1st line be l_1, m_1, n_1. Then we have

$$1 \cdot l_1 + 2m_1 + 0n_1 = 0 \quad \text{and} \quad 0\, l_1 + 2m_1 - n_1 = 0$$

$$\therefore \quad \frac{l_1}{-2} = \frac{m_1}{1} = \frac{n_1}{2} \quad \text{...... (i)}$$

Now, if l_2, m_2, n_2 are the direction ratios of the 2nd line then we have

$$1 \cdot l_2 - 1 \cdot m_2 + 0n_2 = 0 \quad \text{and} \quad 1 \cdot l_2 - 0m_2 - 2n_2 = 0$$

$$\therefore \quad \frac{l_2}{2} = \frac{m_2}{2} = \frac{n_2}{1} \quad \text{...... (ii)}$$

$\therefore$ from (i) and (ii) we have $l_1l_2 + m_1m_2 + n_1n_2 = 0$;

therefore, the given lines are perpendicular to each other.

10. Solution:

The line is $x + 2y + 3z + 4 = 0$; $2x + 3y + 4z + 5 = 0$

If $z = 0$, $x + 2y + 4 = 0$ and $2x + 3y + 5 = 0$ which give $x=2$, $y = -3$

$\therefore$ A point on the given line is $(2, -3, 0)$. If l, m, n are direction ratios of the line.

$$l + 2m + 3n = 0 \quad ; \quad 2l + 3m + 4n = 0$$

which give $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$ or $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$

Therefore the equations of the given line in symmetric form are

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = p \text{ (say)}$$

A point on this line is $P(2 + p, -3 - 2p, p)$

If O is origin, the direction ratios of OP are $2 + p, -3 - 2p, p$.

If OP is perpendicular to the given line $1(2 + p) - 2(-3 - 2p) + p = 0$

i.e. $6p + 8 = 0 \Rightarrow p = \frac{-4}{3}$, $\therefore$ P the foot of the perpendicular is

$$\left(2 - \frac{4}{3}, -3 + \frac{8}{3}, \frac{-4}{3}\right) \text{ i.e. } \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$$

$\therefore$ The equations of the perpendicular from origin to the given line are

$$\frac{x}{\frac{2}{3} - 0} = \frac{y}{\frac{-1}{3} - 0} = \frac{z}{\frac{-4}{3} - 0} \quad \text{or} \quad \frac{x}{-2} = \frac{y}{1} = \frac{z}{4}$$

11. Solution:

The equations of the given line are

$$x + 3y - z + 5 = 0 \quad \text{...... (i)}$$

$$2y + z - 3 = 0 \quad \text{...... (ii)}$$

Adding (i) and (ii) we get $x + 5y + 2 = 0$ or $x + 2 = -5y$

i.e. $$\frac{x+2}{5} = \frac{y}{-1} \quad \text{...... (iii)}$$

Also from (ii) we have $-2y = z - 3$

i.e. $$\frac{y}{-1} = \frac{z-3}{2} \quad \text{...... (iv)}$$

Combining (iii) and (iv) we get $\frac{x+2}{5} = \frac{y}{-1} = \frac{z-3}{2}$

$\therefore$ d.c's fo the line are $\frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}, \frac{2}{\sqrt{30}}$

Equations of the line through (5, −3, 2) and parallel to the given line are

$$\frac{x-5}{5} = \frac{y+3}{-1} = \frac{z-2}{2}$$

12. Solution:

Direction cosines of z-axis are 0, 0, 1.

Let direction ratios of the required line be l, m, n.

Since the line is perpendicular to z-axis, we have $l.0 + m.0 + n.1 = 0$ i.e. $n=0$

$\therefore$ The equations of line through (3, 4, 5) are

$$\frac{x-3}{l} = \frac{y-4}{m} = \frac{z-5}{0} \; ; \quad \therefore z = 5$$

Thus the line passes through the points (3, 4, 5) and (0, 0, 5).

Hence the required line is :-

$$\frac{x-3}{0-3} = \frac{y-4}{0-4} = \frac{z-5}{5-5} \quad \text{or} \quad \frac{x-3}{3} = \frac{y-5}{4} = \frac{z-5}{0}$$

13. Solution:

The given lines are

$$3x + 2y + z - 5 = 0 = x + y - 2z - 3 \quad \text{...... (i)}$$

$$2x - y - z = 0 = 7x + 10y - 8z \quad \text{...... (ii)}$$

If l_1, m_1, n_1, are direction ratios of line (i)

$$3l_1 + 2m_1 + n_1 = 0 \text{ and } l_1 + m_1 - 2n_1 = 0$$

which give $\frac{l_1}{-5} = \frac{m_1}{7} = \frac{n_1}{1}$

If l_2, m_2, n_2 are direction ratios of line (ii) $2l_2 - m_2 - n_2 = 0$

$$7l_2 + 10m_2 - 8n_2 = 0$$

which give $\frac{l_2}{18} = \frac{m_2}{9} = \frac{n_2}{27}$ or $\frac{l_2}{2} = \frac{m_2}{1} = \frac{n_2}{3}$

If θ is the angle between the two lines

$$\cos\theta = \frac{-5.2 + 7.1 + 1.3}{\sqrt{25 + 49 + 1}\,.\,\sqrt{4 + 1 + 9}} = 0 \; ; \; \therefore \; \theta = 90^{\circ}$$

14. **Solution:**

The equations of the given line are $2x - 5y + z - 1 = 0 \; ; \; x + y - 2z + 3 = 0$

If $x = 0$, we have $-5y + z - 1 = 0$ and $y - 2z + 3 = 0$

which give $y = \dfrac{1}{9}$, $z = \dfrac{14}{9}$.

$\therefore$ a point on the given line is $\left(0, \dfrac{1}{9}, \dfrac{14}{9}\right)$

If l, m, n are the direction ratios of the line $2l - 5m + n = 0$ and $l + m - 2n = 0$

which give $\dfrac{l}{10 - 1} = \dfrac{m}{1 + 4} = \dfrac{n}{2 + 5}$ or $\dfrac{l}{9} = \dfrac{m}{5} = \dfrac{n}{7}$

Hence the equations of the line in symmetric form are

$$\frac{x - 0}{9} = \frac{y - \frac{1}{9}}{5} = \frac{z - \frac{14}{9}}{7} \quad \text{or} \quad \frac{x}{1} = \frac{y - \frac{1}{9}}{\frac{5}{9}} = \frac{z - \frac{14}{9}}{\frac{7}{9}} \text{ as requ.}$$

parallel to the straight line joining the points (3, 4, 7) and (2, 7, 5).

5. Find the equations of the perpendicular from the point (1, 6, 3) to the straight line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Also obtain its length and coordinates of the foot of the perpendicular.

6. Find in symmetric form, the equations of the line

$$x + y - z + 1 = 0 = 4x + y - 2z + 2$$

and find its direction cosines. (P.U. 1990)

7. Find the equations of the straight line perpendicular to both the lines

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z+2}{3}, \quad \frac{x+2}{2} = \frac{y-5}{-1} = \frac{z+3}{2}$$

and passing through their point of intersection. (P.U. 1986, 87)

8. Show that the lines $x + 2y - z - 7 = 0 = y + z - 2x - 6$ and $3x + 6y - 3z - 8 = 0 = 2x - y - z$ are parallel.

9. Show that the lines $x + 2y - 1 = 0 = 2y - z - 1$ and $x - y - 1 = 0 = x - 2z - 3$ are perpendicular.

10. Find the equation of the perpendicular from the origin to the line

$$x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$$

Also find the coordinates of the foot of the perpendicular.

11. Determine the direction cosines of the line given by the equations $z + 2y - 3 = 0$ and $x + 3y - z + 5 = 0$.

Also find the equations of the straight line passing through the point (5, −3, 2) and parallel to the line given above. (P.U. 1991)

12. Find the equations of the straight line passing through the point (3, 4, 5) and intersecting the z-axis at right angles.

13. Find the angle between the lines

$$3x + 2y + z - 5 = 0 = x + y - 2z - 3$$

and $2x - y - z = 0 = 7x + 10y - 8z$

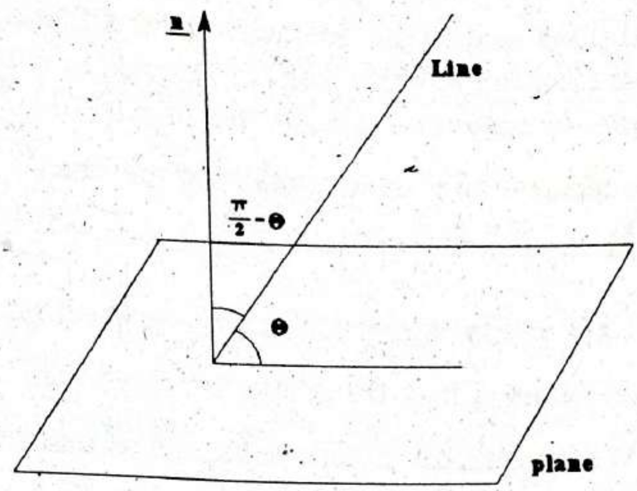
14. Prove that the symmetric equations of the line formed by the intersection of the two planes $2x - 5y + z - 3 = 0$ and $x + y - 2z + 3 = 0$ are

$$x = \frac{y - 1/9}{5/9} = \frac{z - 14/9}{7/9}$$

A LINE AND A PLANE

10.4.1: Angle Between a Line and a Plane:

The angle between a line and a plane is the angle between the line and its projection on the plane. It is clearly the complement of the angle between the line and the normal to the plane.



10.4.2. To Find the Angle Between the Line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, and the plane $ax+by+cz+d=0$.

Since the direction cosines of the normal to the given plane and of the given line are proportional to a, b, c and l, m, n respectively, we have.

$$\sin\theta = \frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(l^2+m^2+n^2)}},$$

where θ is the required angle.

The straight line is parallel to the plane, if $\theta = 0$

i.e., $al+bm+cn=0$

which is also evident from the fact *that if a line be parallel to a plane, it is perpendicular to the normal to it.*

Note: The line will intersect the plane in a single point, if $al+bm+cn \neq 0$.

10.4.3. Conditions for a Line to Lie in a Plane:

To find the conditions that the line

L: $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

may be in the plane α : $ax+by+cz+d=0$

The line would lie in the given plane, if and only if, the coordinates

$$lr+x_1,\quad mr+y_1,\quad nr+z_1$$

of any point on the line satisfy the equation of the plane for all values of r so that

$$r(al+bm+cn)+(ax_1+by_1+cz_1+d)=0$$ is an identity.

This gives

$$al+bm+cn=0$$
$$ax_1+by_1+cz_1+d=0,$$

which are the required *two* conditions.

These conditions lead to the geometrical facts that a line will lie in a given plane, if (i) *the normal to the plane is perpendicular to the line,*
(ii) *any one point on the line lies in the plane.*

or: The equation of a plane containing the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

where $Al+Bm+Cn=0$

Remark: We summarize the above results as under:

1. The straight line L and the plane α intersect in one point if and only if
$al+bm+cn\neq 0$.

2. The straight line L lies in the plane α if and only if
$al+bm+cn=0$ and $ax_1+by_1+cz_1+d=0$

3. The straight line L is parallel to the plane α if and only if
$al+bm+cn=0$ and $ax_1+by_1+cz_1+d\neq 0$

10.4.4. Coplanar Lines:

Two coplanar lines either intersect in a finite point or they are parallel. In the latter case they are said to intersect at infinity so that two coplanar lines always intersect.

Two non-coplanar or skew lines are such that they neither intersect nor are they parallel.

10.4.5. Condition for Coplanarity of Lines:

To find the condition that the two straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{...... (i)}$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad \text{...... (ii)}$$

should intersect i.e., be coplanar.

If the lines intersect, they must lie in a plane. Equation of a plane containing the line (i) is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \quad \text{...... (1)}$$

where $al_1+bm_1+cn_1=0$ ...... (2)

The plane (1) will contain the line (ii) if the point (x_2, y_2, z_2) lies on it and the line is parallel to it.

This requires

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0 \quad \text{...... (3)}$$

and $al_2 + bm_2 + cn_2 = 0$ ...... (4)

Eliminating a, b, c from (2), (3) and (4) we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \qquad \text{...... (A)}$$

which is the reqdied condition for the lines to intersect.

If the condition (A) is satisfied, the equation of the plane containing the two straight lines is obtained by eliminating a, b, c between (1) (2) and (4) i.e.,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

10.4.6. Solved Examples:

Example 1: Show that the line

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ is parallel to the plane}$$

$2x + y - 2z = 3.$

Solution: The direction ratios of the given line are 3, 4, 5 and the direction ratios of the normal to the given plane are 2, 1, –2.

And since $3(2) + 4(1) + 5(-2) = 0$

The normal to the plane is perpendicular to the given line which shows that the line is parallel to the given plane.

Example 2: Show that the line

$$\frac{x+10}{1} = \frac{8-y}{2} = \frac{z}{1} \text{ lies in the plane } x + 2y + 3z - 6 = 0.$$

Solution: Equations of the line are

$$\frac{x+10}{1} = \frac{8-y}{2} = \frac{z}{1} = t(\text{say})$$

A point on the line is $(-10 + t,\ 8 - 2t,\ t)$.

As $1(-10 + t) + 2(8 - 2t) + 3t - 6 = 0$ for every value of t.

The line lies in the plane $x + 2y + 3z - 6 = 0$.

Example 3: Prove that the straight lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \text{ and } \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

intersect. Also find their point of intersection and the plane through them.

(P.U. 1988)

Solution: The two lines are

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = p \text{ (say)}$$

and $$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = q \text{ (say)}$$

A point on each line is

$(1 + 2p, -1 - 3p, -10 + 8p)$ and $(4 + q, -3 - 4q, -1 + 7q)$

If the two lines intersect, the equations

$1 + 2p = 4 + q$

$-1 - 3p = -3 - 4q$

and $-10 + 8p = -1 + 7q$ should be consistant.

Solving the first two equations we get $p = 2$, $q = 1$, which satisfy the third equation.

Hence the two lines intersect and their point of intersection is $(5, -7, 6)$.

Now the plane through the two lines contains the point $(5, -7, 6)$ and its normal is perpendicular to the two lines.

Hence the equation of the plane through the given lines is

$$a(x - 5) + b(y + 7) + c(z - 6) = 0$$

where $2a - 3b + 8c = 0$

and $a - 4b + 7c = 0$.

Eliminating a, b, c from these equations we get the equation of the plane as

$$\begin{vmatrix} x-5 & y+7 & z-6 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

i.e., $(x - 5)(-21 + 32) - (y + 7)(14 - 8) + (z - 6)(-8 + 3) = 0$

or $11x - 6y - 5z - 67 = 0$

Example 4: Find the equation of the plane through the point $(3, -2, 5)$ and perpendicular to the line $x = 2 + 3t$, $y = 1 - 6t$, $z = -2 + 2t$.

Solution: Direction ratios of the given line are $3, -6, 2$.

As the plane is perpendicular to the line, the directin ratios of a normal to the plane are $3, -6, 2$.

$\therefore$ Equation of a plane perpendicular to the line is

$3x - 6y + 2z + d = 0$

Since it passes through the point $(3, -2, 5)$ we have $9 + 12 + 10 + d = 0$ i.e., $d = -31$

Hence the equation of the required plane is $3x - 6y + 2z - 31 = 0$.

Example 5: Find the equation of the plane passing through the point (2, –3, 1) and containing the line $x - 3 = 2y = 3z - 1$.

Solution: The equations of the line are

$$x - 2y - 3 = 0 = 2y - 3z + 1.$$

A plane through this line is

$$x - 2y - 3 + k(2y - 3z + 1) = 0$$

As it passes through the point (2, –3, 1)

$$2 + 6 - 3 + k(-6 - 3 + 1) = 0 \text{ i.e., } k = \frac{5}{8}$$

Hence the equation of the required plane is

$$x - 2y - 3 + \frac{5}{8}(2y - 3z + 1) = 0 \text{ i.e., } 8x - 6y - 15z - 19 = 0$$

Example 6: Find the equation of the plane that passes through the points (2, –1, 1) and (1, 2, –1) and is parallel to the straight line $2x = -3y = 6z$.

(P.U. 1988)

Solution: A plane through the point (2, –1, 1) is

$$a(x - 2) + b(y + 1) + c(z - 1) = 0 \quad \text{...... (1)}$$

As it passes through the point (1, 2, –1),

$$-a + 3b - 2c = 0 \quad \text{...... (2)}$$

Since the plane is parallel to the line

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{1}$$

$$3a - 2b + c = 0 \quad \text{...... (3)}$$

Eliminating a, b c from (1), (2) and (3), we get

$$\begin{vmatrix} x - 2 & y + 1 & z - 1 \\ -1 & 3 & -2 \\ 3 & -2 & 1 \end{vmatrix} = 0$$

i.e., $(x - 2)(3 - 4) - (y + 1)(-1 + 6) + (z - 1)(2 - 9) = 0$

or $-x + 2 - 5y - 5 - 7z + 7 = 0$

or $x + 5y + 7z - 4 = 0$ which is the required equation.

Example 7: Find the condition that the straight lines $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ and $a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$ are coplanar.

Solution: If the lines are coplanar, they intersect in some point say (x_1, y_1, z_1), which will lie on the four given planes.

$\therefore \quad a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$
$\quad a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$
$\quad a_3x_1 + b_3y_1 + c_3z_1 + d_3 = 0$
and $a_4x_1 + b_4y_1 + c_4z_1 + d_4 = 0$

Eliminating x_1, y_1, z_1 between these equations we get

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0 \text{ which is the required condition.}$$

In case, this condition is satisfied, the coordinates of the point of intersection can be obtained by solving any three of the four equations simultaneously.

EXERCSIE 10.4

1. Show that the straight line $\frac{x-2}{1} = \frac{y-5}{2} = \frac{z}{3}$ is perpendicular to the plane $4x + 8y + 12z + 19 = 0$.

2. Prove that the straight lines
$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$
are coplanar.

3. Prove that the straight lines $\frac{x-3}{1} = \frac{y}{-2} = \frac{z}{1}$
and $x = 1 + t$, $y = 1$, $z = 1 - t$ lie in the same plane and find the equation of this plane.

4. Find the equation of the plane passing through the line $x + 2z = 4$, $y - z = 8$ and parallel to the line $\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}$.

5. Find the equations of the planes through the points $(4, -5, 3)$, $(2, 3, 1)$ and parallel to the coordinate axes.

6. Show that there is no plane which passes through the straight line $x = 2t - 3$, $y = 4t - 2$, $z = t - 3$ and is parallel to the plane $2x - y + z = 0$. (P.U. 1989)

7. Prove that the straight lines
$4x + 4y - 5z - 12 = 0 = 8x + 12y - 13z - 32$
and $3x - 2y + 2 = 0 = 2x - z - 4$ are parallel. Find the equation of the plane passing through them. (P.U. 1989)

8. Find the equations of the planes through the straight line
$x + y - z = 0 = 2x - y + 3z - 5$ which are perpendicular to the coordinate planes.

9. Find the equation of the plane containing the line $x = 2t$, $y = 3t$, $z = 4t$ and the

10. Show that the straight line $\frac{x}{-1} = \frac{y+1}{2} = \frac{z-2}{-5}$

and the plane $3x + 4y - 2z = 22$ have a unique point of intersection. Find the point of intersection.

11. Determine the point, if any common to the straight line

$\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-1}{-1}$ and the plane $x + y + z = 3$.

12. Show that the straight lines

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5;$$

and $2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$ are coplanar.

13. Find the equations of the straight line passing through the point (5, −3, 2) and perpendicular to the yx-plane. (P.U. 1990)

14. A line with direction cosines proportional to 2, 7, −5 is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$; $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$

Find the coordinates of the points of intersection and the length intercepted upon it. (P.U. 1986)

15. Find the equation of the plane containing the line $x - 3 = 2y = 3z - 1$ and passing through the point (2, −3, 1). (P.U. 1991)

16. Find the distance of the point (−1, −5, −10) from the point of intersectin of the line

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$

If θ is the angle between the two lines

$$\cos\theta = \frac{-5.2 + 7.1 + 1.3}{\sqrt{25 + 49 + 1}\,.\,\sqrt{4 + 1 + 9}} = 0 \; ; \; \therefore \; \theta = 90^\circ$$

14. Solution:

The equations of the given line are $2x - 5y + z - 1 = 0$; $x + y - 2z + 3 = 0$

If $x = 0$, we have $-5y + z - 1 = 0$ and $y - 2z + 3 = 0$

which give $y = \frac{1}{9}$, $z = \frac{14}{9}$.

$\therefore$ a point on the given line is $\left(0, \frac{1}{9}, \frac{14}{9}\right)$

If l, m, n are the direction ratios of the line $2l - 5m + n = 0$ and $l + m - 2n = 0$

which give $\frac{l}{10 - 1} = \frac{m}{1 + 4} = \frac{n}{2 + 5}$ or $\frac{l}{9} = \frac{m}{5} = \frac{n}{7}$

Hence the equations of the line in symmetric form are

$$\frac{x - 0}{9} = \frac{y - \frac{1}{9}}{5} = \frac{z - \frac{14}{9}}{7} \text{ or } \frac{x}{1} = \frac{y - \frac{1}{9}}{\frac{5}{9}} = \frac{z - \frac{14}{9}}{\frac{7}{9}} \text{ as requ.}$$

Exercise 10.4

1. Solution:

The direction ratios l_1, m_1, n_1 of the given line are 1, 2, 3.

And the direction ratios l_2, m_2, n_2 of the normal to the plane are 4, 8, 12

As $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ the given line is parallel to the normal to the given plane. i.e. the given line is perpendicular to the plane.

2. Solution:

We know that if the straight lines

$$\frac{x - \alpha_1}{l_1} = \frac{y - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1} \text{ and } \frac{x - \alpha_2}{l_2} = \frac{y - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2}$$

are coplanar, then

$$\begin{vmatrix} \alpha_2 - \alpha_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

so the given straight lines are coplanar if

$$\begin{vmatrix} 2 - 1 & 3 - 2 & 4 - 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

which is, of course, zero since the substruction of the 2nd row from the third row makes 1st and 3rd rows identical. Hence the given straight lines are identical.

3. Solution:

The given lines are

$$\frac{x-3}{1} = \frac{y}{-2} = \frac{z}{1} \quad \text{and} \quad \frac{x-1}{1} = \frac{y-1}{0} = \frac{z-1}{-1}$$

The lines will be coplanar if

$$\begin{vmatrix} 1-3 & 1-0 & 1-0 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 0$$

$$\text{L.H.S.} = (-2)(2) - (1)(-2) + 1(2) = 0 = \text{R.H.S.}$$

Therefore the given lines are coplanar, Also the equation of the required plane (through the given lines) is

$$\begin{vmatrix} x-3 & y & z \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 0$$

i.e. $(x-3)(2) - y(-2) + z(2) = 0$ i.e. $x + y + z = 3$

4. Solution:

A plane through the line

$$x + 2z - 4 = 0 = y - z - 8 \text{ is } x + 2z - 4 + k(y - z - 8) = 0$$

i.e. $x + ky + (2-k)z - 4 - 8k = 0$ ...... (i)

direction ratios of the normal to this plane are $1, k, 2-k$.

If plane (i) is parallel to the line $\dfrac{x-3}{2} = \dfrac{y+4}{3} = \dfrac{z-7}{4}$

$$2.1 + 3.k + 4(2-k) = 0 \implies k = 10$$

Putting $k = 10$ in (i) we get the equation of the required plane

i.e. $x + 10y - 8z - 84 = 0$

5. Solution:

Let the plane parallel to x-axis be $ax + by + cz = 1$ where a, b, c are the direction ratios of a normal to the plane. The direction cosines of x-axis are (1, 0, 0)

$\therefore \quad a.1 + b.0 + c.0 = 0 \quad \text{or} \quad a = 0$

Hence the equation of the plane parallel to x-axis reduces to $by + cz = 1$

Since the points (4, −5, 3) and (2, 3, 1) lie in this plane.

$$-5b + 3c = 1 \quad \text{and} \quad 3b + 1c = 1$$

These equation give $b = \dfrac{1}{7}$, $c = \dfrac{4}{7}$

$\therefore$ The plane parallel to x-axis is $\dfrac{1}{7}y + \dfrac{4}{7}z = 1$ or $y + 4z = 7$

Similarly the plane parallel to y-axis is $ax + cz = 1$. On making the coordinates of the given points satisfy this equation we have

$$4a + 3c = 1 \text{ and } 2a + c = 1 \text{ which give } a \Rightarrow 1, c = -1.$$

$\therefore$ The plane parallel to y-axis is $x - z = 1$

Similarly the plane parallel to z-axis is $ax + by = 1$. As it passes through the given points $4a - 5b = 1$ and $2a + 3b = 1$ which give $a = \frac{4}{11}$, $b = \frac{1}{11}$

$\therefore$ The plane parallel to z-axis is $4x + y = 11$.

6. Solution:

The given line is $\frac{x+3}{2} = \frac{y+2}{4} = \frac{z+3}{1} = t$ ...... (1)

Direction ratios of this line are 2, 4, 1.

A plane through line (1) will be parallel to the plane

$$2x - y + z = 0 \quad \text{......(2)}$$

if the line is perpendicular to the normal to the plane

But $2.2 + 4.(-1) + 1.1 \neq 0$

$\therefore$ The line is not perpendicular to the normal to plane (2).

Hence there is no plane which passes through the line (1) and is parallel to the plane (2).

7. Solution:

If l_1, m_1, n_1 are the direction ratios of the first line, then

$$4l_1 + 4m_1 - 5n_1 = 0 \quad \text{and} \quad 8l_1 + 12m_1 - 13n_1 = 0$$

which give $\frac{l_1}{8} = \frac{m_1}{12} = \frac{n_1}{16}$ or $\frac{l_1}{2} = \frac{m_1}{3} = \frac{n_1}{4}$

If l_2, m_2, n_2 are the direction ratios of the second line then

$$3l_2 - 2m_2 + 0n_2 = 0 \quad \text{and} \quad 2l_2 + 0m_2 - n_2 = 0$$

which give $\frac{l_2}{2} = \frac{m_2}{3} = \frac{n_2}{4}$ As $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$, the two lines are parallel.

Now we find a point on the first line. Let $z = 0$, then

$4x + 4y - 12 = 0$ and $8x + 12y - 32 = 0$ which give $x = 1$, $y = 2$

$\therefore$ A point on the line is $(1, 2, 0)$

A plane through the second line is $3x - 2y + 2 + k(2x - z - 4) = 0$

If the point $(1, 2, 0)$ lies on it $3 - 4 + 2 + k(2 - 4) = 0$ gives $k = \frac{1}{2}$.

Hence the equation of the plane through the two lines is

$$3x - 2y + 2 + \frac{1}{2}(2x - z - 4) = 0 \text{ i.e. } 8x - 4y - z = 0$$

8. Solution:

Any plane through the given line is ...... (A)

$$x + y - z + k(2x - y + 3z - 5) = 0$$

or $\quad (1 + 2k)x + (1 - k)y + (3k - 1)z - 5k = 0$

The normal vector to this plane is $[1 + 2k, 1 - k, 3k - 1]$. (i)

For planes perpendicular to yz-plane, zx-plane, and xy-plane (i) is perpendicular to $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$ respectively.

Hence for plane perpendicular to yz-plane we have

$$1(1+2k)=0 \text{ which gives } k=-\frac{1}{2}$$

Putting this value of k in (A) we have

$$\frac{3}{2}y-\frac{5}{2}z+\frac{5}{2}=0 \quad \text{or} \quad 3y-5z+5=0$$

as the equation of plane perpendicular to yz–plane.

Similarly $1-k=0$ and $3k-1=0$ give the values of k for planes perpendicular to the zx–plane and xy–plane respectively.

$\therefore$ For $k=1$ the plane is $3x+2z-5=0$ and for $k=\frac{1}{3}$ the equation of the plane is $5x+2y-5=0$

9. Solution:

A plane through the intersection of $x+y+z=0$ and $2y-z=0$ is

$$x+y+z+k(2y-z)=0$$

i.e. $\quad x+(1+2k)y+(1-k)z=0 \quad$ ...... (1)

Since this plane contains the line $x=2t$, $y=3t$, $z=4t$, normal of the plane is perpendicular to the line. Therefore

$$2+3(1+2k)+4(1-k)=0 \text{ (since 2, 3, 4 are direction ratios of the line).}$$

or $\quad 2+3+6k+4-4k=0$ or $k=-\frac{9}{2}$

Substituting for k in (1), the required plane is

$$x+(1-9)y+\left(1+\frac{9}{2}\right)z=0 \quad \text{or} \quad 2x-16y+11z=0$$

10. Solution:

Line is $\frac{x}{-1}=\frac{y+1}{2}=\frac{z-2}{-5}$ ...... (i)

Any point on the straight line (i) has coordinates $(-t, -1+2t, 2-5t)$.

If this point lies on the plane

$$3x+4y-2z=22\ ;\quad 3(-t)+4(-1+2t)-2(2-5t)=22$$

or $\quad -3t-4+8t-4+10t=22$ i.e. $15t=30 \Rightarrow t=2$

Hence the point of intersection is $(-2, 3, -8)$.

11. Solution:

A point on the straight line $\frac{x-3}{1}=\frac{y-2}{0}=\frac{z-1}{-1}$ is $(3+p, 2, 1-p)$

If it lies on the plane $x+y+z=3$ we have $3+p+2+1-p=3$ which does not determine p. Hence there is no point common to the given line and the plane.

12. Solution:

The given straight lines will be coplanar if

$$\begin{vmatrix} 1 & 2 & -5 & 9 \\ 3 & -1 & 2 & -5 \\ 2 & 3 & -1 & -3 \\ 4 & -5 & 1 & 3 \end{vmatrix} = 0$$

By the operations $R_2 - 3R_1$, $R_3 - 2R_1$ and $R_4 - 4R_1$ we have

$$\text{L.H.S.} = \begin{vmatrix} 1 & 2 & -5 & 9 \\ 0 & -7 & 17 & -32 \\ 0 & -1 & 9 & -21 \\ 0 & -13 & 21 & -33 \end{vmatrix}$$

$$= \begin{vmatrix} -7 & 17 & -32 \\ -1 & 9 & -21 \\ -13 & 21 & -33 \end{vmatrix} = \begin{vmatrix} 7 & 17 & 32 \\ 1 & 9 & 21 \\ 13 & 21 & 33 \end{vmatrix}$$

$$= 7\,(297 - 441) - 17\,(33 - 273) + 32\,(21 - 117)$$

$$= 7\,(-144) - 17\,(-240) + 32\,(-96) = -1008 + 4080 - 3072 = 0$$

Hence the given lines are coplanar.

13. Solution:

Direction cosines of the normal to the yx-plane are 0, 0, 1.

A line perpendicular to yx-plane will have 0, 0, 1 as it direction cosines.

Hence the equations of the line through (5,−3,2) and perpendicular to yx-plane are

$$\frac{x-5}{0} = \frac{y+3}{0} = \frac{z-2}{1}$$

14. Solution:

The two straight lines are

$$\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} \quad \text{and} \quad \frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$$

A point on the first line is $P(3p + 5, -p + 7, p - 2)$ and a point on the second line is $Q(-3q - 3, 2q + 3, 4q + 6)$.

P and Q will be the points of intersection of the given lines with the intersecting line, if line PQ has direction cosines proportional to 2, 7, −5.

Now the direction cosines of PQ are $3p + 3q + 8, -p - 2q + 4, p - 4q - 8$

$$\therefore \quad \frac{3p + 3q + 8}{2} = \frac{-p - 2q + 4}{7} = \frac{p - 4q - 8}{-5} \quad \text{which give}$$

$$23p + 25q = -48\ ,\quad 17p + 7q = -24 \text{ which give } p = -1, q = -1.$$

Hence the points of intersection are P (2, 8, −3) and Q (0, 1, 2).

Length intercepted $= PQ = \sqrt{(2-0)^2 + (8-1)^2 + (-3-2)^2} = \sqrt{4+49+25} = \sqrt{78}$

15. Solution:

The given line is $x - 3 = 2y = 3z - 1$.

These equations can be written as $x - 2y - 3 = 0 = 2y - 3z + 1$

A plane containing this line is $x - 2y - 3 + k\,(2y - 3z + 1) = 0$

If it passes through (2, −3, 1) we have $2 + 6 - 3 + k\,(-6 - 3 + 1) = 0 \Rightarrow k = \frac{5}{8}$.

Hence the equation of the required plane is

$$x - 2y - 3 + \frac{5}{8}(2y - 3z + 1) = 0 \text{ i.e. } 8x - 6y - 15z - 19 = 0$$

16. Solution:

Line is $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = p$

A point on this line is $(3p + 2, 4p - 1, 12p + 2)$

If it lies on the plane $x - y + z = 5$, $3p + 2 - 4p + 1 + 12p + 2 = 5 \Rightarrow p = 0$

∴ The point of intersection of the line and plane is $(2, -1, 2)$

∴ Required distance is the distance between the points $(2, -1, 2)$ and $(-1, -5, -10)$

∴ The distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$

10. Show that the straight line $\frac{x}{-1} = \frac{y+1}{2} = \frac{z-2}{-5}$

and the plane $3x + 4y - 2z = 22$ have a unique point of intersection. Find the point of intersection.

11. Determine the point, if any common to the straight line

$$\frac{x-3}{1} = \frac{y-2}{0} = \frac{z-1}{-1} \text{ and the plane } x + y + z = 3.$$

12. Show that the straight lines

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5;$$

and $2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$ are coplanar.

13. Find the equations of the straight line passing through the point $(5, -3, 2)$ and perpendicular to the yx-plane. (P.U. 1990)

14. A line with direction cosines proportional to $2, 7, -5$ is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$; $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$

Find the coördinates of the points of intersection and the length intercepted upon it. (P.U. 1986)

15. Find the equation of the plane containing the line $x - 3 = 2y = 3z - 1$ and passing through the point $(2, -3, 1)$. (P.U. 1991)

16. Find the distance of the point $(-1, -5, -10)$ from the point of intersectin of the line

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} \text{ and the plane } x - y + z = 5$$

SHORTEST DISTANCE BETWEEN TWO STRAIGHT LINES

10.5.1. Definition:

Two straight lines which neither intersect nor are parallel are called non-coplanar or skew lines.

Two coplanar lines either intersect in a finite point or they are parallel. If the lines intersect the shortest distance between them is zero. If they are parallel, the shortest distance between them is the distance of any point on one straight line to the other straight line.

Before discussing the shortest distance between two non-coplanar lines, we state the following simple results from pure solid Geometry.

If two straight lines are skew, then

(i) they lie in parallel planes;

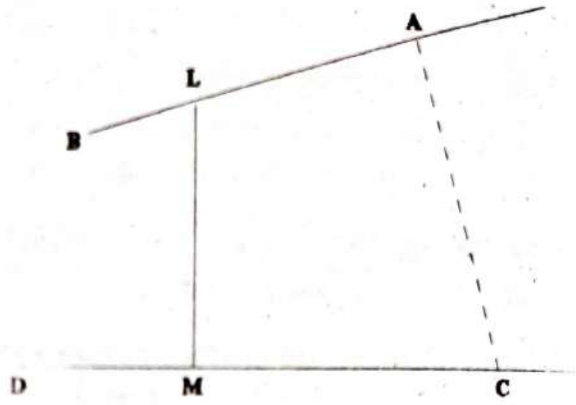
(ii) there is one and only one straight line perpendicular to both of them;

(iii) the intercept of this common perpendicular on the lines is the shortest distance betweeen them;

of shortest distance of the line joining any two points, one on each of the given lines.

10.5.2. To find the magnitude and equation of the line of shortest distance between two straight lines.

If AB, CD are the two given straight lines and LM is the line which meets them both at right angles at L and M, then LM is the line of shortest distance between the given lines and the length LM is the magnitude.



Let the equations of the given lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \ldots\ldots (i)$$

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \quad \ldots\ldots (ii)$$

and let the shortest distance LM lie along the line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \ldots\ldots (iii)$$

Line (iii) is perpendicular to both the lines (i) and (ii). Therefore, we have

$$ll_1 + mm_1 + nn_1 = 0,$$
$$ll_2 + mm_2 + nn_2 = 0,$$

or $$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \frac{1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

$$\therefore \quad l = \frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \; m = \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}, \; n = \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}} \quad \ldots\ldots (iv)$$

The line of shortest distance is perpendicular to both the lines. Therefore the magnitude of the shortest distance is the projection on the line of shortest distance of the line joining any two points, one on each of the given lines (i) and (ii).

Taking the projection of the join of (x_1, y_1, z_1), (x_2, y_2, z_2) on the line with direction cosines l, m, n, we see that the shortest distance

$$= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n,$$

where l, m, n have the values as given in (iv).

To find the equations of the line of shortest distance, we observe that it is coplanar with both the given lines.

The equation of the plane containing the coplanar lines (i) and (iii) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad \text{...... (v)}$$

and that of the plane containing the coplanar lines (ii) and (iii) is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \qquad \text{...... (vi)}$$

Thus (v) and (vi) are the two equations of the line of shortest distance, where l, m, n are given in (iv)

Note: Other methods of determining the shortest distance are explained in the solved examples.

10.5.3. Solved Examples:

Example 1: Find the shortest distance between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$.

Solution: The equations of the two straight lines can be written as

$$\frac{x + a}{12} = \frac{y}{6} = \frac{z}{-1} \qquad \text{...... (1)}$$

and $x - y - 2a = 0 = x - 6z + 6a$ ...... (2)

A plane through line (2) is

$$x - y - 2a + k(x - 6z + 6a) = 0$$

or $(1 + k)x - y - 6kz - 2a + 6ak = 0$

If it is parallel to line (1)

$$12(1 + k) + 6(-1) + (-1)(-6k) = 0 \text{ i.e., } k = -\frac{1}{3}$$

∴ Equation of a plane through line (2) and parallel to line (1) is

$$x - y - 2a - \frac{1}{3}(x - 6z + 6a) = 0$$

i.e., $2x - 3y + 6z - 12a = 0$ ...... (3)

A point on straight line (1) is $(-a, 0, 0)$.

∴ the shortest distance between the two lines = distance of $(-a, 0, 0)$ from the plane (3)

$$= \frac{|\,2(-a) - 3 \cdot 0 + 6 \cdot 0 - 12a\,|}{\sqrt{4 + 9 + 36}} = \frac{14a}{7} = 2a$$

Example 2: Obtain the coordinates of the points where the line of shortest distance between the lines

$$\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3} \text{ and}$$

$$\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2} \text{ meets them.}$$

Solution: Let P, Q be the points where the line of shortest distance meets the two lines. P, Q are on the two lines. Let P be $(23-6p, 19-4p, 25+3p)$ and Q be $(12-9q, 1+4q, 5+2q)$, one on each line.

$$\vec{PQ} = [-9q+6p-11, 4q+4p-18, 2q-3p-20]$$

As $\vec{PQ}$ is perpendicular to the two lines

$$-6(-9q+6p-11) - 4(4q+4p-18) + 3(2q-3p-20) = 0$$

and $\quad -9(-9q+6p-11) + 4(4q+4p-18) + 2(2q-3p-20) = 0$

or $\quad 44q - 61p + 78 = 0 \quad$ and $\quad 101q - 44p - 13 = 0$

Solving the two equations, we get $p = 2, \; q = 1$.

Hence the two points are $P(11, 11, 31)$ and $Q(3, 5, 7)$.

Example 3: Given the two lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

Find the coordinates of the points where the common perpendicular meets these lines. Also find the length and equations of the common perpendicular. (P.U. [illegible])

Solution: Let $P(1+2p, 2+3p, 3+4p)$ be a point on the first line and $Q(2+3q, 3+4q, 4+5q)$ be a point on the second line. The direction ratios of line PQ are $3q-2p+1, \; 4q-3p+1, \; 5q-4p+1$.

As $\vec{PQ}$ is perpendicular to the two lines

$$2(3q-2p+1) + 3(4q-3p+1) + 4(5q-4p+1) = 0$$

and $\quad 3(3q-2p+1) + 4(4q-3p+1) + 5(5q-4p+1) = 0$

or $\quad 38q - 29p + 9 = 0 \quad$ and $\quad 50q - 38p + 12 = 0$

Solving the two equations we get $p = -1, \; q = -1$.

For these values of p and q, P is $(-1, -1, -1)$ and Q is $(-1, -1, -1)$.

Hence the two lines intersect at the point $(-1, -1, -1)$.

Thus the shortest distance i.e., the length of the common perpendicular is zero. And there is no common perpendicular in the plane of the lines.

Any how we can find the equations of a line prependicular to the two lines through the common point $(-1, -1, -1)$.

If l, m, n are the direction ratios of this line, then

$$2l + 3m + 4n = 0 \quad \text{and} \quad 3l + 4m + 5n = 0$$

$$\therefore \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

Hence the equations of the line are $\frac{x+1}{1} = \frac{y+1}{-2} = \frac{z+1}{1}$

Example 4: Find the length and the equations of the line of shortest distance between $5x - y - z = 0 = x - 2y + z + 3$

and $7x - 4y - 2z = 0 = x - y + z - 3$

Solution: First we transform the equations to the symmetric form. So we have the equations of the two lines as

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{3} \quad \text{and} \quad \frac{x}{2} = \frac{y+1}{3} = \frac{z-2}{1}$$

If l, m, n are the direction ratios of the line of shortest distance, we have

$$l + 2m + 3n = 0 \quad \text{and} \quad 2l + 3m + n = 0$$

i.e., $\frac{l}{7} = \frac{m}{-5} = \frac{n}{1} = \frac{1}{\sqrt{75}} \therefore l = \frac{7}{\sqrt{75}},\ m = \frac{-5}{\sqrt{75}},\ n = \frac{1}{\sqrt{75}}$

Hence the length of the shortest distance is $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= \frac{7}{\sqrt{75}}(0-0) - \frac{5}{\sqrt{75}}(-1-1) + \frac{1}{\sqrt{75}}(2+1) = \frac{10}{\sqrt{75}} + \frac{3}{\sqrt{75}} = \frac{13}{\sqrt{75}}.$$

The equations of the line of shortest distance are

$$\begin{vmatrix} x & y-1 & z+1 \\ 1 & 2 & 3 \\ 7 & -5 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x & y+1 & z-2 \\ 2 & 3 & 1 \\ 7 & -5 & 1 \end{vmatrix}$$

i.e., $17x + 20y - 2z - 39 = 0 = 8x + 5y - 31z + 67$

Example 5: Find the shortest distance between the straight line joining the points A(3, 2, −4) and B(1, 6, −6) and the straight line joining the points C(−1, 1, −2) and D(−3, 1, −6). Find also the equations of the line of the shortest distance and the coordinates of the points where it meets AB and CD. (P.U. :98C)

Solution: Equations of the line through A and B are

$$\frac{x-3}{1-3} = \frac{y-2}{6-2} = \frac{z+4}{-6+4} \quad \text{or} \quad \frac{x-3}{-2} = \frac{y-2}{4} = \frac{z+4}{-2} = p \quad \text{...... (1)}$$

and the equations of the line joining C and D are

$$\frac{x+1}{-3+1} = \frac{y-1}{1-1} = \frac{z+2}{-5+2} \quad \text{or} \quad \frac{x+1}{-2} = \frac{y-1}{0} = \frac{z+2}{-4} = q$$

Let P and Q be the feet of the common perpendicular. The coordinates of P are $(3 - 2p, 2 + 4p, -4 - 2p)$ and those of Q are $(-1 - 2q, 1, -2 - 4q)$.

$$\overrightarrow{PQ} = [-2q + 2p - 4, -4p - 1, -4q + 2p + 2]$$

Since $\overrightarrow{PQ}$ is perpendicular to both (1) and (2)

$$-2(-2q + 2p - 4) + 4(-4q - 1) - 2(-4q + 2p + 2) = 0$$

and $$-2(-2q + 2p - 4) + 0(-4p - 1) - 4(-4q + 2p + 2) = 0$$

or $12q - 24p = 0$ and $20q - 12p = 0$

Solving these equations we get $p = 0, q = 0$.

$\therefore$ the point P is $(3, 2, -4)$ and Q is $(-1, 1, -2)$ and the shortest distance $= |PQ|$

$$= \sqrt{(-1-3)^2 + (1-2)^2 + (-2+4)^2} = \sqrt{21}$$

and equations of the line PQ are

$$\frac{x-3}{-1-3} = \frac{y-2}{1-2} = \frac{z+4}{-2+4} \quad \text{or} \quad \frac{x-3}{-4} = \frac{y-2}{-1} = \frac{z+4}{2}$$

EXERCISE 10.5

1. Find the magnitude and the equations of the line of shortest distance between the straight lines
$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

2. Find the shortest distance between the axis of z and the line
$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$

3. Find the shortest distance between the straight lines
$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$
Also find the equations of the straight line which is perpendicular to both the given straight lines. (P.U. 1985)

4. Find the magnitude and the equations of the line of shortest distance between the straight lines
$$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5} \quad \text{and} \quad \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$$

5. Find the shortest distance between the straight lines
$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \text{and} \quad \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$
Find the equations of the straight line which is perpendicular to both the given straight lines and also its point of intersection with the given straight lines. (P.U. 1987)

6. Find the length and equations of the line of shortest distance between the straight lines $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$ and $\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}$

7. Find the coordinates of the point on the join of (–3, 7, –13) and (–6, 1, –10) which is nearest to the intersection of the planes

$$2x - y - 3z + 32 = 0 \text{ and } 3x + 2y - 15z - 8 = 0$$

8. Find the length and equations of the common perpendicular of the straight lines.

$$6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$$
$$3x - 9y + 5z = 0 = x + y - z$$

MISCELLANEOUS EXERCISE 10

1. Find the ratios in which the join of the points (3, 2, 1), (1, 3, 2) is divided by the surface represented by the equation $3x^2 - 72y^2 + 128z^2 = 3$.

2. Prove that if two pairs of opposite edges of a tetrahedron are perpendicular, then so is the third pair.

3. Find the equation of the locus of a point such that its distance from the plane $2x - y + 3z + 5 = 0$ is always twice its distance from the plane $x + 2y - 3z - 4 = 0$.
(P.U. 1990)

4. Show that the necessary and sufficient condition for the points P_1, P_2, P_3 to be collinear is that there should exist scalars k_1, k_2, k_3, $(k_1 + k_2 + k_3 = 0)$ such that $k_1 \overline{V_1} + k_2 \overline{V_2} + k_3 \overline{V_3} = 0$ where $\overline{V_1}, \overline{V_2}, \overline{V_3}$ are the vectors $\overline{OP_1}, \overline{OP_2}, \overline{OP_3}$ respectively.
(P.U. 1991)

5. Show that the distance of point (3, –4, 5) from the plane $2x + 5y - 6z = 16$ measured parallel to the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ is $\frac{60}{7}$

6. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C, show that
   (i) the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = 9\,p^{-2}$
   (ii) the locus of the centroid of the tetrahedron OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$

7. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = a$ is $\frac{2a}{\sqrt{6}}$, and the three straight lines of shortest distance intersect at the point (–a, –a, –a).

8. Prove that the tetrahedron formed by the coordinate planes and a tangent plane to the surface $xyz = c^3$ is of constant volume.
(P.U. 1986)

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Exercise 10.5

1. Solution:

Let l, m, n be the direction cosines of the lie of shortest distance.

As it is perpendicular to the two lines, we have

$$3l - 16m + 7n = 0 \text{ and } 3l + 8m - 5n = 0, \therefore \frac{l}{24} = \frac{m}{36} = \frac{n}{72}$$

or $\frac{l}{2} = \frac{m}{3} = \frac{n}{6}$. Hence $l = \frac{2}{7}$, $m = \frac{3}{7}$, $n = \frac{6}{7}$

The magnitude of shortest distance is the projection of the join of the points (8, −9, 10). and (15, 29, 5), on the line of he shortest distane and is therefore

$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$$= 7 \cdot \frac{2}{7} + 38 \cdot \frac{3}{7} - 5 \cdot \frac{6}{7} = 14$$

Again, the equation of the plane containing the first of the two given lines and the line of shortest distance is

$$\begin{vmatrix} x - 8, & y + 9, & z - 10 \\ 3, & -16, & 7 \\ 2, & 3, & 6 \end{vmatrix} = 0, \text{ or } 117x + 4y - 41z - 490 = 0$$

Also the equation of the plane containing the second line and the shortest distance line is

$$\begin{vmatrix} x - 15, & y - 29, & z - 5 \\ 3, & 8, & -5 \\ 2, & 3, & 6 \end{vmatrix} = 0, \text{ or } 9x - 4y - z = 14$$

Hence the equations of the shortest distance line are

$$117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14$$

2. Solution:

Any plane through the second given line is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0$$

i.e. $(a + ka')x + (b + kb')y + (c + kc')z + (d + kd') = 0$ ...... (i)

It will be parallel to z-axis whose direction cosines are 0, 0, 1, if the normal to the plane is ⊥ z-axis, i.e. if,

$$0 \cdot (a + ka') + 0 \cdot (b + kb') + 1 \cdot (c + kc') = 0, \text{ i.e. } k = \frac{-c}{c'}$$

Substituting this value of k in (i), we see that the equation of the plane through the second line parallel to the first is

$$(ac' - a'c)\,x + (bc' + b'c)\,y + (dc' - d'c) = 0 \qquad \text{......(ii)}$$

The required S.D. is the distance of any point on z-axis from the plane (ii).

$\therefore$ S.D. = perpendicular from (0, 0, 0), (a point on z-axis)

$$= \pm \frac{dc' - d'c}{\sqrt{[(ac' - a'c)^2 + (bc' - b'c)^2]}}$$

3. Solution:

If l, m, n are the direction cosines of the shortest distance line, then it being perpendicular to the given lines, we have

$$2l + 3m + 4n = 0 \quad \text{and} \quad 3l + 4m + 5n = 0$$

$$\therefore \quad \frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9} \quad \text{or} \quad \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$$\therefore \quad l = \frac{-1}{\sqrt{6}}, \; m = \frac{2}{\sqrt{6}}, \; n = \frac{-1}{\sqrt{6}}.$$ Length of the shortest distance

$$= l\,(x_2 - x_1) + m\,(y_2 - y_1) + n\,(z_2 - z_1)$$

$$= \frac{-1}{\sqrt{6}}(2-1) + \frac{2}{\sqrt{6}}(4-2) + \left(\frac{-1}{\sqrt{6}}\right)(5-3) = \frac{1}{\sqrt{6}}$$

Equations of the shortest distance are, by the standard formula

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}$$

which reduce to $11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7$ as required.

4. Solution:

A point on the first line is P $(2p + 3, -7p - 15, 5p + 9)$ and a point on the second line is Q $(2q - 1, q + 1, -3q + 9)$.

The direction cosines of PQ are proportional to

$$2p - 2q + 4, \; -7p - q - 16, \; 5p + 3q$$

If PQ is perpendicular to the two lines we have

$$2(2p - 2q + 4) - 7(-7p - q - 16) + 5(5p + 3q) = 0$$

and $\quad 2(2p - 2q + 4) + 1(-7p - q - 16) - 3(5p + 3q) = 0$

or $\quad 39p + 9q = -60$ and $9p + 7q = -4$ which give $p = -2, q = 2$

$\therefore$ P is $(-1, -1, -1)$ and Q is $(3, 3, 3)$

The shortest distance = PQ = $\sqrt{(3+1)^2 + (3+1)^2 + (3+1)^2} = \sqrt{48} = 4\sqrt{3}$

And the equations of the line of shortest distance are

$$\frac{x+1}{3+1} = \frac{y+1}{3+1} = \frac{z+1}{3+1} \text{ or } x+1 = y+1 = z+1, \text{or } x = y = z$$

5. **Solution:**

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = p \qquad \text{...... (i)}$$

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} = q \qquad \text{...... (ii)}$$

A point on line (i) is P $(3 + p, 5 - 2p, 7 + p)$ and a point on line (ii) is Q $(-1 + 7q, -1 - 6q, -1 + q)$.

Direction ratios of the line PQ are $7q - p - 4, -6q + 2p - 6, q - p - 8$

If PQ is perpendicular to the two lines

$$1(7q - p - 4) - 2(-6q + 2p - 6) + 1(q - p - 8) = 0$$

and $\quad 7(7q - p - 4) - 6(-6q + 2p - 6) + 1(q - p - 8) = 0$

i.e. $\quad 20q - 6p = 0$ and $86q - 20p = 0$ which give $p = 0, q = 0$,

Hence P is $(3, 5, 7)$ and Q is $(-1, -1, -1)$

$\therefore$ The shortest distance between the two lines

$= PQ = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16 + 32 + 64} = \sqrt{116} = 2\sqrt{29}$

The equations of the line PQ which is perpendicular to the two lines are

$$\frac{x+1}{3+1} = \frac{y+1}{5+1} = \frac{z+1}{7+1} \text{ i.e. } \frac{x+1}{4} = \frac{y+1}{6} = \frac{z+1}{4}$$

or $$\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}$$

and the points of intersection are P $(3, 5, 7)$ and Q $(-1, -1, -1)$.

6. **Solution:**

Let l, m, n be the direction cosines of the line of shortest distance.

As it is perpendicular to the two lines

$$-4l + 3m + 2n = 0 \; ; \; -4l + m + n = 0, \therefore \frac{l}{1} = \frac{m}{-4} = \frac{n}{8}$$

Hence $l = \frac{1}{9}$, $m = \frac{-4}{9}$, $n = \frac{8}{9}$

The magnitude of shortest distance is the projection of the join of $(-3, 6, 0)$ and $(-2, 0, 7)$ on the line of shortest distance and is therefore,

$$\frac{1}{9}(-2 + 3) - \frac{4}{9}(0 - 6) + \frac{8}{9}(7 - 0) = \frac{1}{9} + \frac{24}{9} + \frac{56}{9} = 9$$

The equations of the line of shortest distance are

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ 1 & -4 & 8 \end{vmatrix} = 0 = \begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -4 & 8 \end{vmatrix}$$

i.e. $\quad 32x + 34y + 13z - 108 = 0 = 4x + 11y + 5z - 27.$

7. Solution:

Equations of the line through the points (−3, 7, −13) and (−6, 1, −10) are

$$\frac{x+3}{1} = \frac{y-7}{2} = \frac{z+13}{-1} = t \qquad \text{...... (1)}$$

A point on (1) is $P(-3+t, 7+2t, -13-t)$.

We transform the equations

$$2x - y - 3z + 32 = 0 \text{ and } 3x + 2y - 15z - 8 = 0 \qquad \text{...... (2)}$$

of the second straight line into symmetric form. Let [l, m, n] be a direction vector of (2). Then

$$2l - m - 3n = 0 \quad , \quad 3l + 2m - 15n = 0$$

$$\frac{l}{21} = \frac{m}{21} = \frac{n}{7} \text{ or } \frac{l}{3} = \frac{m}{3} = \frac{n}{1}$$

Taking $z = 0$, equations (2) become $2x - y + 32 = 0$ and $3x + 2y - 8 = 0$

or $\dfrac{x}{-56} = \dfrac{y}{112} = \dfrac{1}{7}$ or $x = -8, y = 16, z = 0$ is a point on (2).

Hence symmetric equations of (2) are

$$\frac{x+8}{3} = \frac{y-16}{3} = \frac{z}{1} = s \qquad \text{...... (3)}$$

A point on (3) is $Q(-8+3s, 16+3s, s)$.

$$\overrightarrow{PQ} = [3s - t - 5, 3s - 2t + 9, s + t + 13]$$

Let $\overrightarrow{PQ}$ be normal to both (1) and (3). Then $3s-t+5+6s-4t+18-s-t-1=0$

and $9s - 3t - 15 + 9s - 6t + 27 + s + t + 13 = 0$ or $8s - 6t = 0$

and $19s - 8t + 25 = 0$. Solving for s and t we have $s = -3, t = -4$

Substituting $t=-4$ in the coordinates of P, we get $P(-7,-1,-9)$ as the required point

8. Solution:

The lines are L: $6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$

M. $3x - 9y + 5z = 0 = x + y - z$

We first transform the equations into the symmetric form.

Putting $z = 0$ in the equations for L we have

$$6x + 8y - 13 = 0 \text{ and } x + 2y - 3 = 0, \therefore \frac{x}{2} = \frac{y}{5} = \frac{1}{4}$$

or $x = \dfrac{1}{2}, y = \dfrac{5}{4}, z = 0$ is a point on L. Let [a, b, c] be a direction vector of L. Since L is perpendicular to normal of each plane constituting it, we have

$$6a + 8b + 3c = 0 \quad \text{and} \quad a + 2b + c = 0$$

or $\dfrac{a}{2} = \dfrac{b}{-3} = \dfrac{c}{4}$. Hence, [a, b, c] = [2, −3, 4]. Equation of L are

$$\frac{x - 1/2}{3} = \frac{y - 5/4}{2} = \frac{z}{4} \qquad \text{...... (1)}$$

Similarly, we can written M as $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ ...... (2)

Let $\underline{n} = [l, m, n]$ be a vector perpendicular to both (1) and (2).
Then $2l - 3m + 4n = 0$ and $l + 2m + 3n = 0$

or $\quad \dfrac{l}{-17} = \dfrac{m}{-2} = \dfrac{n}{7}$. Hence, $\underline{n}\,[l, m, n] = [-17, -2, 7]$.

The points P $\left(\dfrac{1}{2}, \dfrac{5}{4}, 0\right)$ and Q (0, 0, 0) lie on (1) and (2) respectively.

$$\vec{QP} = \left[\frac{1}{2}, \frac{5}{4}, 0\right]$$

Length of the common perpendicular to the two lines is the orthogonal projection of QP on $\underline{n}$. Therefore length of the common perpendicular

$$= \frac{|\vec{QP} \cdot \underline{n}|}{|\underline{n}|} = \frac{\frac{17}{2} + \frac{5}{2}}{\sqrt{342}} = \frac{11}{\sqrt{342}}$$

The equtions of the common perpendicular are

$$\begin{vmatrix} x + 1/2 & y - 5/4 & z \\ 2 & -3 & 4 \\ -17 & -2 & 7 \end{vmatrix} = 0 = \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ -17 & -2 & 7 \end{vmatrix}$$

i.e. $\quad \left(x - \dfrac{1}{2}\right)(-21 + 8) - \left(y - \dfrac{5}{4}\right)(14 + 68) + z(-4 - 5) =$

$\quad 0 = x(14 + 6) - y(7 + 51) + z(-2 + 34)$

or $\quad -13x - 82y - 55z + 109 = 0 = 20x - 58y + 32z$

i.e. $\quad 13x + 82y + 55z - 109 = 0 = 10x - 29y + 16z$

Miscellaneous Exercise 10

1. Solution:

Let the ratio be r:1. ∴ a point on the join of (3, 2, 1) and (1, 3, 2) is

$$\left(\frac{3 + r}{1 + r}, \frac{2 + 3r}{1 + r}, \frac{1 + 2r}{1 + r}\right)$$

if it lies on the surface $3x^2 - 72y^2 + 128z^2 = 3$

$$3\left(\frac{3 + r}{1 + r}\right)^2 - 72\left(\frac{2 + 3r}{1 + r}\right)^2 + 128\left(\frac{1 + 2r}{1 + r}\right)^2 = 3$$

which on simplifcation reduces to $2r^2 + 5r + 2 = 0$, $\therefore\ r = -2, -\dfrac{1}{2}$.

∴ The required ratio is $\quad -1:2$ or $-2:1$

2. Solution:

Let $\underline{a}, \underline{b}, \underline{c}, \underline{d}$ be the position vectors of the vertices A, B, C, D of the tetrahedron referred to an origin O.

If AB is perpendicular to CD $(\underline{b} - \underline{a}) \cdot (\underline{d} - \underline{c}) = 0$

i.e. $\quad \underline{b} \cdot \underline{d} - \underline{b} \cdot \underline{c} - \underline{a} \cdot \underline{d} + \underline{a} \cdot \underline{c} = 0$ ..... (1)

If AC is perpendicular to BD $(\underline{c} - \underline{a}).(\underline{d} - \underline{b}) = 0$
i.e. $\underline{c}.\underline{d} - \underline{c}.\underline{b} - \underline{a}.\underline{d} + \underline{a}.\underline{b} = 0$ ...... (2)
Subtracting (2) from (1) we get $\underline{b}.\underline{d} - \underline{c}.\underline{d} + \underline{a}.\underline{c} - \underline{a}.\underline{b} = 0$
or $\underline{d}.(\underline{b} - \underline{c}) - \underline{a}.(\underline{b} - \underline{c}) = 0$ i.e. $(\underline{b} - \underline{c}).(\underline{d} - \underline{a}) = 0$
which shows that BC is perpendicular to AD.

3. Solution:

Two planes are $2x - y + 3z + 5 = 0$ ...... (1)
$x + 2y - 3z - 4 = 0$ ...... (2)

Let P (x, y, z) be any point on the locus. Therefore
distance of P from (1) = 2 distance of P from (2)

$$\therefore \quad \frac{2x - y + 3z + 5}{\sqrt{4 + 1 + 9}} = 2 \cdot \frac{x + 2y - 3z - 4}{\sqrt{1 + 4 + 9}}$$

i.e. $2x - y + 3z + 5 = 2(x + 2y - 3z - 4)$ or $5y - 9z - 13 = 0$ is the required equation of the locus.

4. Solution:

The condition is necessary. For, let P_1, P_2, P_3 be collinear, then one of the three points say P_3, divides the segment joining the other two in some ratio $m_1:m_2$. Hence

$$\underline{v}_3 = \frac{m_2\underline{v}_1 + m_1\underline{v}_2}{m_1 + m_2} \text{ or } m_2\underline{v}_1 + m_1\underline{v}_2 - (m_1 + m_2)\underline{v}_3 = 0$$

i.e. $k_1\underline{v}_1 + k_2\underline{v}_2 + k_3\underline{v}_3 = 0$ with $k_1 + k_2 + k_3 = 0$

The condition is also sufficient, for suppose $k_1\underline{v}_1 + k_2\underline{v}_2 + k_3\underline{v}_3 = 0$ where $k_1 + k_2 + k_3 = 0$ i.e. $k_3 = -(k_1 + k_2)$

This gives $\underline{v}_3 = \dfrac{k_1\underline{v}_1 + k_2\underline{v}_2}{k_1 + k_2}$ showing that P_3 divides the line segment joining P_1, P_2 in the ratio $k_2:k_1$ and so P_3 lies in this line.

5. Solution:

Equation of the line through (3, −4, 5) parallel to

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \text{ is } \frac{x - 3}{2} = \frac{y + 4}{1} = \frac{z - 5}{-2} = t$$

$$\therefore \quad x = 3 + 2t,\ y = -4 + t,\ z = 5 - 2t$$

This lies on $2x + 5y - 6z = 16$ if $2(3 + 2t) + 5(-4 + t) - 6(5 - 2t) = 16$

$$21t = 16 + 50 - 6 = 66 - 6 = 60,\ t = \frac{60}{21} = \frac{20}{7}$$

The point of intersection of the line with the plane is

$$x = 3 + \frac{40}{7} = \frac{61}{7},\ y = -4 + \frac{20}{7} = \frac{-28 + 20}{7} = -\frac{8}{7}$$

$$z = 5 - \frac{40}{7} = \frac{35 - 40}{7} = -\frac{5}{7}$$

Distance between (3, −4, 5) and $\left(\frac{61}{7}, -\frac{8}{7}, -\frac{5}{7}\right)$ is

$$\sqrt{\left(\frac{61}{7}-3\right)^2+\left(-\frac{8}{7}+4\right)^2+\left(-\frac{5}{7}-5\right)^2}$$

$$=\sqrt{\left(\frac{40}{7}\right)^2+\left(\frac{20}{7}\right)^2+\left(\frac{-40}{7}\right)^2}=\frac{60}{7}$$

6. Solution:

Let the equation of the plane be $lx + my + nz = p$, $l^2 + m^2 + n^2 = 1$

Then $\frac{x}{p/l} + \frac{y}{p/m} + \frac{z}{p/n} = 1$. Thus coordinates of A,B,C are respectively

$$\left(\frac{p}{l}, 0, 0\right), \left(0, \frac{p}{m}, 0\right), \left(0, 0, \frac{p}{n}\right)$$

(i) Centroid of ΔABC is

$$x = \frac{\frac{p}{l}+0+0}{3}, \; y = \frac{0+\frac{p}{m}+0}{3}, \; z = \frac{0+0+\frac{p}{n}}{3}$$

Thus $\frac{p}{3l} = x$ i.e. $l = \frac{p}{3x}$. Similarly $m = \frac{p}{3y}$, $n = \frac{p}{3z}$

Since $l^2 + m^2 + n^2 = 1$, $\frac{p^2}{9x^2} + \frac{p^2}{9y^2} + \frac{p^2}{9z^2} = 1$ i.e. $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

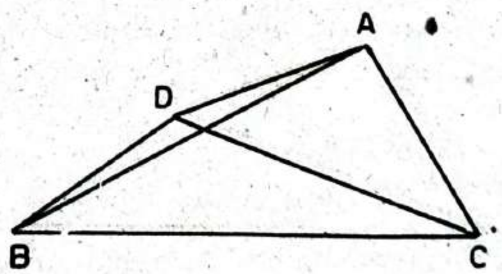
(ii) Here $x = \frac{p/l + 0 + 0 + 0}{4} \Rightarrow l = \frac{p}{4x}$

Similarly $m = \frac{p}{4y}$, $n = \frac{p}{4z}$. Since $l^2 + m^2 + n^2 = 1$

$$\frac{p^2}{16x^2} + \frac{p^2}{16y^2} + \frac{p^2}{16z^2} = 1 \text{ i.e. } x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$$

7. Solution:

Let the planes $y + z = 0$, $z + x = 0$, $x+y=0$ and $x + y + z = 0$ be ABC, ACD, ADB and BCD respectively.



$\therefore$ Equations of the line AC (being the intersects of ABC and ACD) are $y + z = 0 = z + x$.

or $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ ...... (1)

Edge opposite to AC is BD which is the intersection of the planes ABD and BCD. Its equations are $x + y = 0$ and $x + y + z = a$.

Any point on this line is (0, 0, a). If l, m, n are the direction ratios of this line then

$$1.l + 1m + 0.n = 0 \text{ and } 1.l + 1.m + 1.n = 0, \therefore \frac{l}{1} = \frac{m}{-1} = \frac{n}{0}$$

∴ Equations of BD, in the symmetric form, are

$$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-a}{0}$$

Now, if L, M, N are the direction cosines of the shortest distance between (1) and (2) then $L.1 + M.1 + N(-1) = 0$ and $L.1 + M(-1) + N(0) = 0$

$$\therefore \quad \frac{L}{-1} = \frac{M}{-1} = \frac{N}{2} \text{ which give}$$

$$L = \frac{-1}{\sqrt{6}}, \; M = \frac{-1}{\sqrt{6}}; \; N = \frac{2}{\sqrt{6}}$$

∴ Shortst distance between the opposite edge AC and BD

$$= L(x_2 - x_1) + M(y_2 - y_1) + N(z_2 - z_1) \mid$$

$$= \left| \frac{-1}{\sqrt{6}}(0) + \frac{-1}{\sqrt{6}}(0) + \frac{2}{\sqrt{6}}(a) \right| = \frac{2a}{\sqrt{6}}$$

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0 \; i.e., \; x - y = 0 \quad .....(i)$$

$$\begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{vmatrix} = 0 \; i.e., \; -x + y + z + a = 0 \quad .....(ii)$$

∵ (−a, −a, −a) satisfies *(i)* and *(ii)* ∴ This point lies on the shortest distance between AC and BD. Similarly (−a, −a, −a) lies on the other two shortest distances. Hence it lies on intersection of all these shortest distances.

8. Solution:

Let $f(x, y, z) = xyz - c^3 = 0$, $f_x = yz$, $f_y = zx$, $f_z = xy$

Let $P(x_1, y_1, z_1)$ be a point on the surface, then $x_1y_1z_1 = c^3$.

The equation of the tangent plane at P is

$$y_1z_1(x - x_1) + z_1x_1(y - y_1) + x_1y_1(z - z_1) = 0$$

i.e. $$y_1z_1x + z_1x_1y + x_1y_1z - 3x_1y_1z_1 = 0$$

This plane intersects the axes in the points A $(3x_1, 0, 0)$, B $(0, 3y_1, 0)$, C $(0, 0, 3z_1)$.

∴ The vertices of the tetrahedron are A, B, C and O (0, 0, 0).

∴ Volume of tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ 3x_1 & 0 & 0 & 1 \\ 0 & 3y_1 & 0 & 1 \\ 0 & 0 & 3z_1 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} 3x_1 & 0 & 0 \\ 0 & 3y_1 & 0 \\ 0 & 0 & 3z_1 \end{vmatrix} \text{ (numerically)}$$

$$= \frac{27\, x_1y_1z_1}{6} = \frac{9c^3}{2} = \text{constant.}$$

———— O ————