Chapter 4

Lecture 2

Two body central Force Problem

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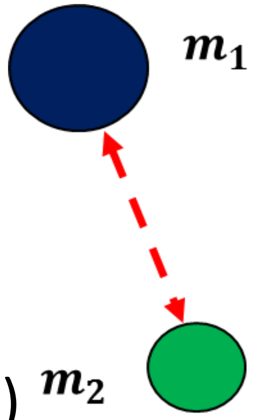
4.3 Equation of Motion for a body under the action of central force and First Integrals

Consider a conservative, where force can be drivable from potential "$V_{(r)}$".

The problem has spherical symmetry & angular momentum $(\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P})$ conserved.

Lagrangian of the system $$L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V_{(r)} \qquad \textbf{(4.3.1)}$$

Using Lagrange's equation $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$



And $$\frac{\partial L}{\partial \dot{\theta}} = P_\theta = \mu r^2 \dot{\theta},$$ and $$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\mu r^2 \dot{\theta}\right) = 0 \qquad \textbf{(4.3.2)}$$

$$\Rightarrow \left(\mu r^2 \dot{\theta}\right) = P_\theta = \boldsymbol{l} = constant \qquad \textbf{(4.3.3)}$$

Eq. (4.3.3) is **first integral of motion**

4.3 Equation of Motion for a body under the action of central force and First Integrals

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0 \Rightarrow A = \left(\frac{1}{2}r^2\dot{\theta}\right) = \left(\frac{l}{2\mu}\right) = constant \qquad \textbf{(4.3.4)}$$

Thus, the areal velocity is constant in time. **(Kepler's Second Law)**

- Areal velocity conservation is a general property of central force motion
- It is not restricted to the inverse-square law force involved in planetary motion.

Lagrange's equation for radial part

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}(\mu\dot{r}) - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad \textbf{(4.3.5)}$$

Since $f_{(r)} = -\frac{\partial V}{\partial r}$ & $\dot{\theta} = \frac{\boldsymbol{l}}{\mu r^2}$ {from(**4.3.3**)}, Therefore, Eq. (**4.3.5**)

Since $\Rightarrow \mu\ddot{r} - \frac{L^2}{\mu r^3} = f_{(r)}$ **(4.3.7)**

4.3 Equation of Motion for a body under the action of central force and First Integrals

$$\Rightarrow \mu\ddot{r} = \frac{l^2}{\mu r^3} - \frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{l^2}{2\mu r^2} + V\right)$$

Multiplying Both sides with "$\dot{r}$" $\quad \Rightarrow \mu\dot{r}\ddot{r} = -\dot{r}\frac{\partial}{\partial r}\left(\frac{l^2}{2\mu r^2} + V\right)$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{dr}{dt}\frac{\partial}{\partial r}\left(\frac{l^2}{2\mu r^2} + V\right) = -\frac{d}{dt}\left(\frac{l^2}{2\mu r^2} + V\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + \frac{l}{2\mu r^2} + V\right) = 0$$

$$\Rightarrow \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V = Constant \qquad \textbf{(4.3.8)}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + V_{(r)} \qquad \textbf{(4.3.10)}$$

From eq. (**4.3.8**) and Eq. (**4.3.10**), total energy of a body under the action of central force is constant.

4.4 First Integral

The Angular momentum, of the system is

$$\boldsymbol{l} = \mu r^2 \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{\boldsymbol{l}}{\mu r^2} \Rightarrow \frac{d\theta}{dt} = \frac{\boldsymbol{l}}{\mu r^2}$$

$$\Rightarrow d\theta = \frac{\boldsymbol{l}}{\mu r^2} dt$$

Integrating above equation

$$\Rightarrow \int_{\theta_o}^{\theta} d\theta = \int_0^t \frac{\boldsymbol{l}}{\mu r^2} dt \qquad \textbf{(4.4.1)}$$

Now the total energy of a body moving under central force is given by

$$E = T + V = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{\boldsymbol{l}^2}{\mu r^2} + V_{(r)} \qquad \textbf{(4.4.3)}$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{2\mu r^2} - V_{(r)}\right)} \qquad \textbf{(4.4.4)}$$

4.4 First Integral

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{\mu r^2} - V_{(r)}\right)}$$

$$\Rightarrow t = \int_{r_o}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{2\mu r^2} - V_{(r)}\right)}} \qquad (\mathbf{4.4}.5)$$

Eq. (**4.4.1**) & Eq. (**4.4.5**) are known as first integral for the motion in central force field. where $\boldsymbol{l}$, E, θ_o and r_o must be known initially.

Eq. (**4.4.1**) & Eq. (**4.4.5**) gives "r" and "θ" in terms of t. We are often interested to find "θ" in terms of "r" which will determine the shape of the orbit of the body.

4.4 First Integral

Since $\frac{d\theta}{dt} = \frac{\boldsymbol{l}}{\mu r^2} \quad \Rightarrow \frac{d\theta}{dt}\frac{dr}{dr} = \frac{\boldsymbol{l}}{\mu r^2}$

$$\Rightarrow d\theta = \frac{\boldsymbol{l}}{\mu r^2 \dot{r}} dr \qquad \textbf{(4.4.6)}$$

From Eq. (**4.4.4**) we know that

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{\mu r^2} - V_{(r)}\right)} \qquad \textbf{(4.4.7)}$$

$$\Rightarrow d\theta = \frac{\boldsymbol{l}}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{\mu r^2} - V_{(r)}\right)}} dr$$

$$\Rightarrow \theta = \theta_o + \int_{r_o}^{r} \frac{\boldsymbol{l}/_{r^2}}{\sqrt{2\mu\left(E - \frac{\boldsymbol{l}^2}{2\mu r^2} - V_{(r)}\right)}} dr \qquad \textbf{(4.4.8)}$$

Eq. 47 gives "θ" in terms of "r" which determine the shape of the orbit of the body under the action of central force field.

4.5 General Features of Motion Under Central Force

$$\begin{cases} \mu(\ddot{r} - r\dot{\theta}^2) = F_r \\ \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta \end{cases} \qquad \textbf{(4.5.1)}$$

The tangential component "F_θ" is zero because the force is radial

$$\mu(\ddot{r} - r\dot{\theta}^2) = F_r$$

$$\Rightarrow \mu\ddot{r} = F_r + \mu r\dot{\theta}^2$$

$$\boxed{\Rightarrow \mu\ddot{r} = F_r + \frac{\boldsymbol{l}^2}{\mu r^3}} \qquad \textbf{(4.5.2)}$$

$\frac{L^2}{\mu r^3}$ is known as centrifugal force. It is a pseudo or false force since it does not arise from the interaction between the particles in the orbit. It appears due to accelerated motion of the body.

Since $\boldsymbol{l}^2 = \mu^2 r^4 \dot{\theta}^2$

$$\Rightarrow \frac{\boldsymbol{l}^2}{\mu r^3} = \mu r\dot{\theta}^2 = \frac{\mu(r^2\dot{\theta}^2)}{r} = \frac{\mu v^2}{r} \text{ or } \frac{mv^2}{r}$$

4.5 General Features of Motion Under Central Force

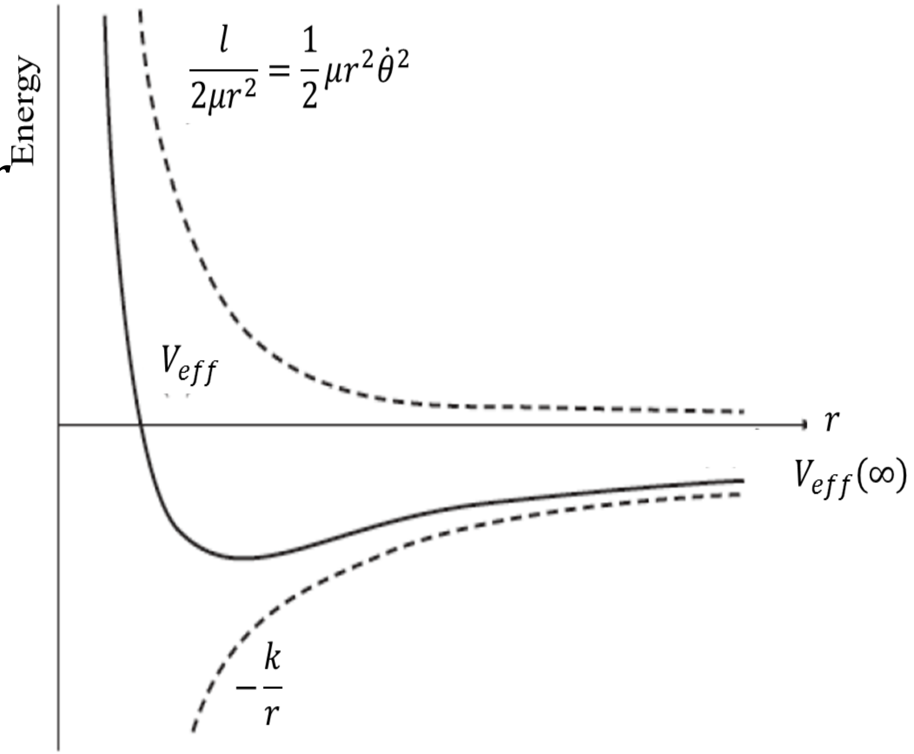
"$\mu\ddot{r}$" is the effective force responsible for the motion and can be derived from potential "V_{eff}"

$$\mu\ddot{r} = -\frac{dV_{eff}}{dr} \qquad (4.5.3)$$

Therefore Eq. (4.5.2) can be written as

$$-\frac{dV_{eff}}{dr} = F_r + \frac{l^2}{\mu r^3} \qquad \Rightarrow V_{eff} = -\int\left(-\frac{dV}{dr} + \frac{l^2}{\mu r^3}\right)dr$$

$$\Rightarrow V_{eff} = V + \frac{l^2}{2\mu r^2} \qquad (4.5.4)$$



For an inverse square law (gravitational or electrostatic force)

$$F_r = -\frac{k}{r^2} \quad \Rightarrow V = -\frac{k}{r}$$

Therefore, $\qquad V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad (4.5.5)$

Note that the centrifugal potential reduces the effect of the inverse square law

4.5 General Features of Motion Under Central Force

Not the total energy of the system is

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}$$

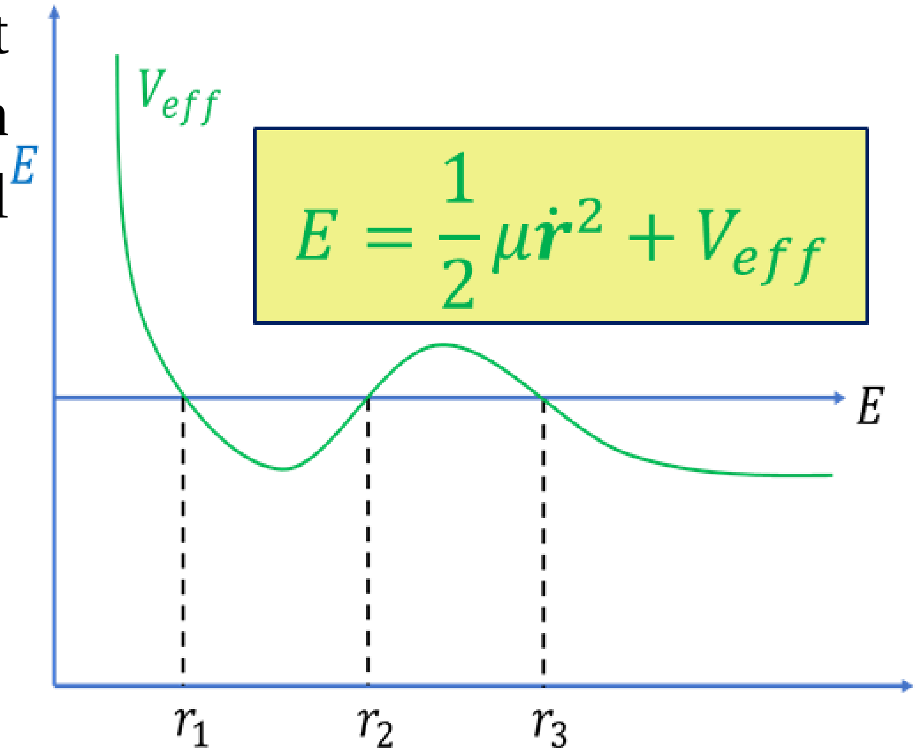
$$\Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}\right)} \qquad \textbf{(4.5.6)}$$

The centrifugal part gives a repulsive potential while the inverse square law part gives an attractive potential.

Centrifugal part decreases much faster with distance "r" as compared to the inverse square attractive part. The combine potential is given as the V_{eff} which decrease sharply from positive value to negative and then increase with r. The V_{eff} approaches to zero value at infinite value of r.

4.6 Motion in arbitrary potential Field

Let an arbitrary potential V_{eff} which may or may not be same as the real problem and it might appear in different problems. The Energy and potential curves intersect at "r_1", "r_2" and "r_3".



$$E = V_{eff} \qquad \textbf{(4.6.1)}$$

And $\frac{1}{2}\mu\dot{r}^2 = 0$ & $\dot{r} = 0$

The curve can be divided into three regions.

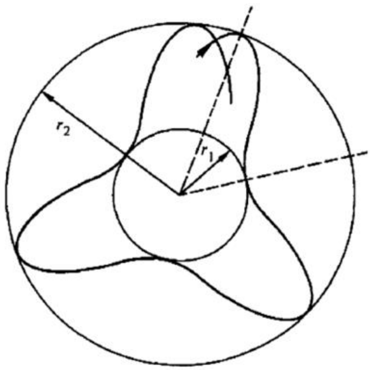
<u>Region for $r < r_1$</u>

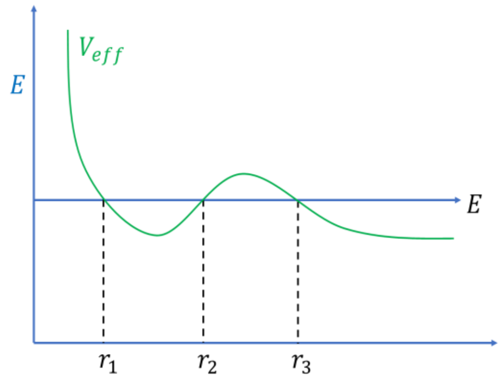
$$E < V_{eff} \qquad \textbf{(4.6.2)}$$

$\& \; T = \frac{1}{2}\mu\dot{r}^2 < 0$

& velocity has imaginary value. Hence motion in this region is not possible.

4.6 Motion in arbitrary potential Field





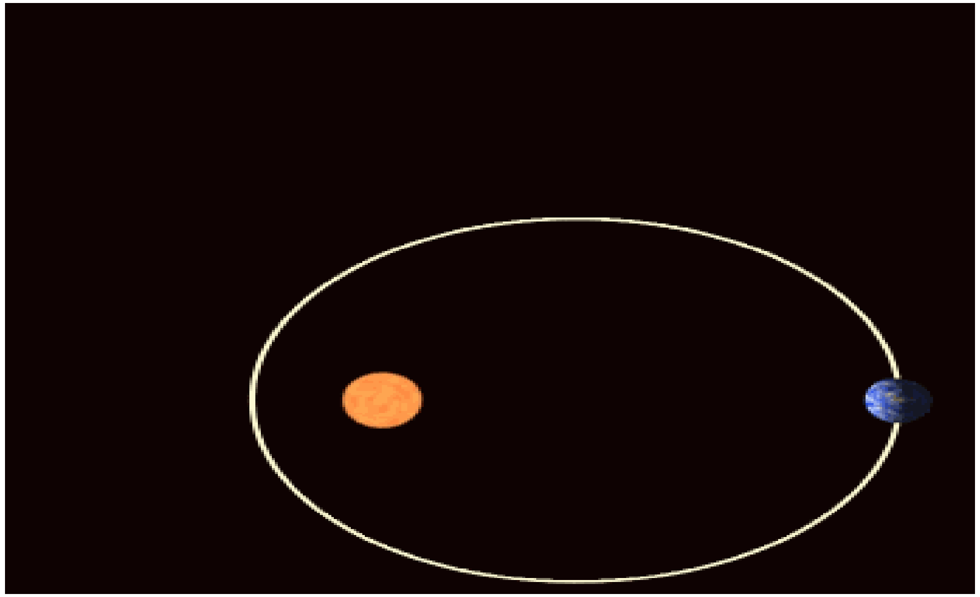
Region for $r_1 < r < r_2$

In this region $E > V_{eff}$

for $r < r_1$ and $r_2 < r$,

The kinetic energy $T = \frac{1}{2}\mu\dot{r}^2 < 0$

Which is not possible therefore the body will turn back on r_1 and r_2.



Region for $r_2 < r < r_3$

In this region $E < V_{eff}$

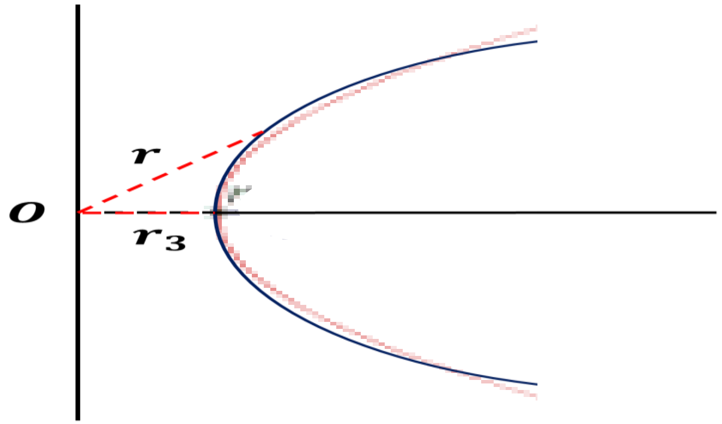
& $T = \frac{1}{2}\mu\dot{r}^2 < 0$ Therefore, the motion in this region is not possible.

4.6 Motion in arbitrary potential Field

Region for $r > r_3$

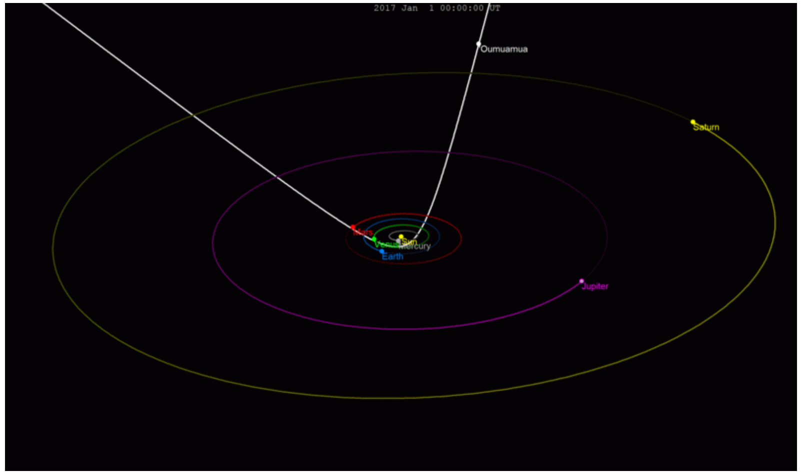
Turning point is $r = r_3$.

The particle approaches to r_3 and rebounded.



$$E = T + V_{eff} = 0 \Rightarrow T = -V_{eff}$$

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(-V_{eff}\right)} \qquad \textbf{(4.6.3)}$$



$\dot{r}$ = escape velocity; the initial velocity required to escape from the potential field V_{eff}.

The nature of motion of the particle discussed earlier with help of arbitrary potential will help to understand the nature of orbit.

4.7 Motion in Inverse Square Law Force Field

$$F_r = \frac{k}{r^2} \qquad \textbf{(4.7.1)}$$

$$\Rightarrow V_r = \frac{k}{r} \qquad \textbf{(4.7.2)}$$

Therefore, the effective potential V_{eff} is given by

$$V_{eff} = \frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad \textbf{(4.7.3)}$$

The value of "k" depends on the nature of physical problem. For example,

i) gravitational force between two spherical bodies of mass m_1 and m_2

$$k = -Gm_1m_2 \qquad \textbf{(4.7.4)}$$

ii) Electrostatic force

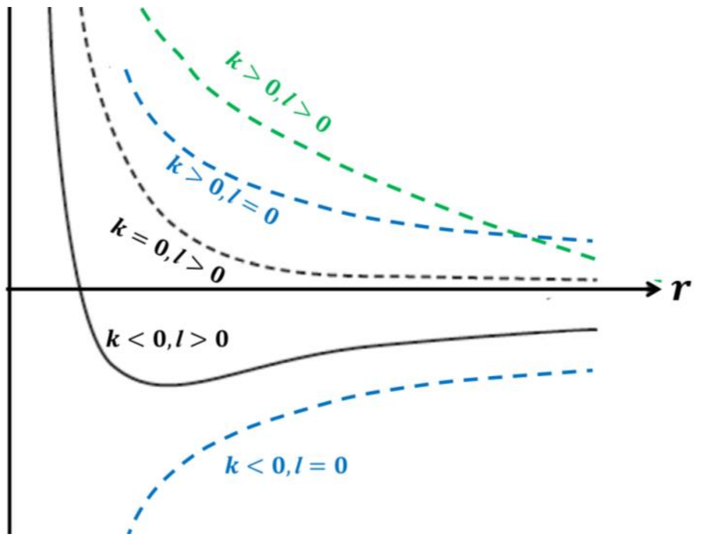
$$k = \frac{q_1 q_2}{4\pi\epsilon_o} \qquad \textbf{(4.7.5)}$$

The nature of the orbit depends on sign of "k". If k > 0 ⇒repulsive & for k < 0 ⇒ attractive.

4.7 Motion in Inverse Square Law Force Field

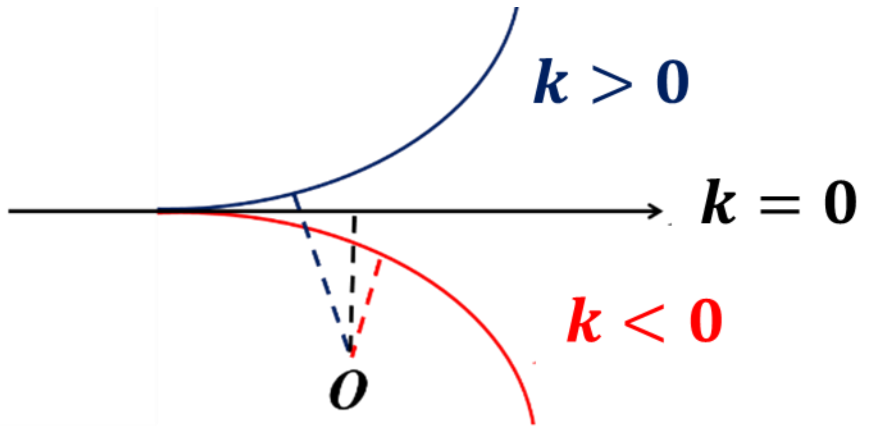
If effective potential V_{eff} is plotted against "r" for different values of "k" and "L" following curves are obtained.

Case I	$k > 0, l > 0$
Case II	$k > 0, l = 0$
Case III	$k = 0, l > 0$
Case IV	$k < 0, l > 0$
Case V	$k < 0, l = 0$



These curves can be very helpful in understanding the nature of the orbit.

A body with total energy $E > V_{eff}$ approaching to the centre of force from infinite distance. The particle will be deflected as given in figure.



4.7 Motion in Inverse Square Law Force Field

(1)<u>For E_1 at $r = r_1$</u>

$$E_1 = V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

Turning point at $r = r_1$.

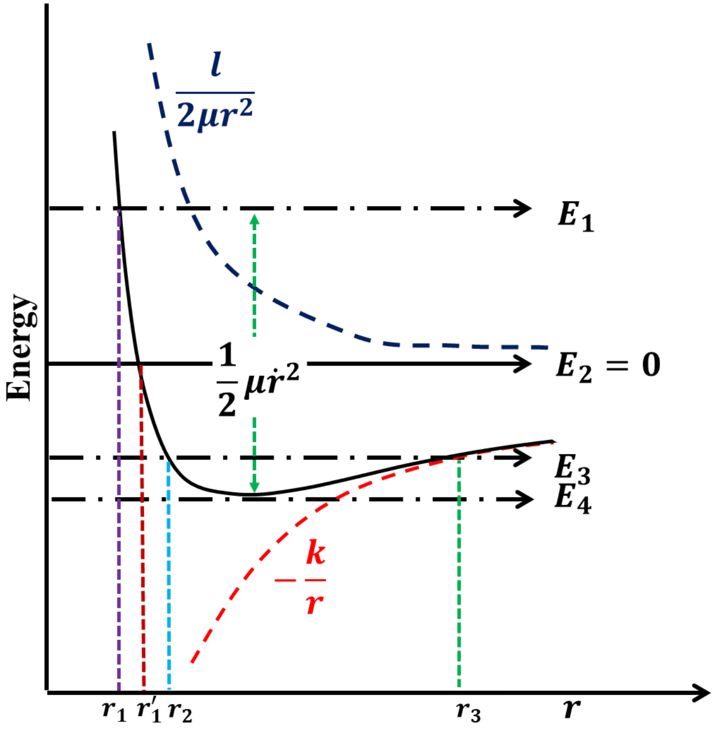
Motion represents scattering, where body is not bound to the centre and deflected away.

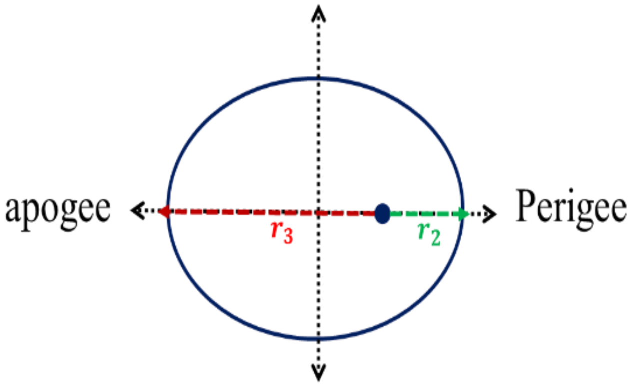
(ii) <u>For $E_2 = 0$</u>

Possible roots are $r = r_1'$ and $r = \infty$. The particle moves away & radial velocity fall continuously.

(iii) <u>For $E_3 < 0$</u>

Two roots $r = r_2$ and $r = r_3$ of equation are real and distinct.





4.7 Motion in Inverse Square Law Force Field

(iv) **For $E_4 = V_{eff}$,**

which is tangent of the potential energy curve.

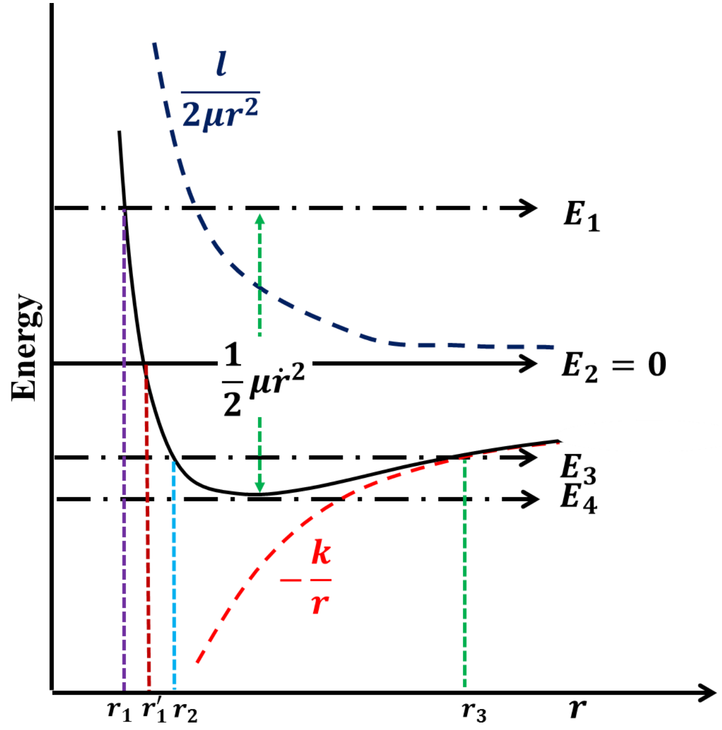
Therefore $$\frac{dV_{eff}}{dr} = 0$$

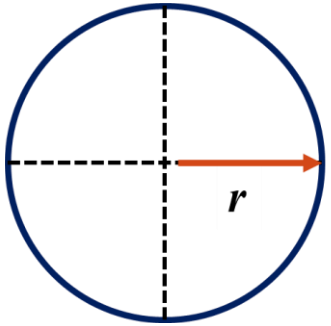
$$\Rightarrow \frac{dV}{dr} - \frac{l^2}{\mu r^3} = 0$$

$$\Rightarrow F_r = -\frac{dV}{dr} = -\frac{l^2}{\mu r^3} = -\mu r\dot{\theta}^2$$

$$\Rightarrow F_r = -\frac{\mu r^2\dot{\theta}^2}{r} = -\frac{\mu v^2}{r}$$

Thus F_r is equal to the centrifugal force required to maintain circular motion of the body around the centre of the force. Thus F_r is centripetal force that maintain the orbit.





4.9 Show That: a) $\boldsymbol{v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)}$

b) Using results from part "a" also prove that the conservation of energy equation will be

$\boldsymbol{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{\mu h^2}}$ **if** $\boldsymbol{u = \frac{1}{r}}$

Solution: Let us consider a particle of mass "μ" and position vector "$\boldsymbol{r}$".

Since $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} - \frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \dot{r} = -r^2\dot{\theta}\frac{du}{d\theta} \Rightarrow \dot{r} = -h\frac{du}{d\theta}$$

Therefore, $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$$\Rightarrow v^2 = \left(-h\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(hu^2)^2 = h\left(\frac{du}{d\theta}\right)^2 + h^2u^2$$

$$\Rightarrow v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \qquad (4.9.1)$$

Since $E = T + V \Rightarrow T = E - V$

$$\Rightarrow \frac{1}{2}\mu v^2 = E - V$$

$$\Rightarrow \frac{1}{2}\mu h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) = E - V$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{\mu h^2} \qquad (4.9.2)$$

Eq. (4.9.1) and Eq. (4.9.2) are as desired.

Problem (Page 293, Classical Mechanics by Marion)

Find the force law for a central force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where "k" and "α" are constants. Also find value of $\theta_{(t)}$ and $r_{(t)}$. Also find Energy of the orbit.

Solution. Since we have verified that

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{\left(\frac{1}{u}\right)}}{l^2u^2}$$

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2} \qquad (1)$$

Now using

$$r = ke^{\alpha\theta} \Rightarrow \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiating Twice with respect to θ

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k}e^{-\alpha\theta}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{\alpha^2}{k}e^{-\alpha\theta} = \alpha^2 u \qquad (2)$$

Putting value of u and $\frac{d^2u}{d\theta^2}$ in equation 1

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow \alpha^2 u + u = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow f_{(r)} = -\frac{l^2}{\mu r^3}(\alpha^2 + 1) \tag{3}$$

Eq. 3 represents the force responsible for motion.

Now the central potential responsible for the motion of the particle will be

$$V = -\int f_{(r)}\, dr = -\frac{l^2}{2\mu r^2}(\alpha^2 + 1) \tag{4}$$

Total energy of the system is

$$E = T + V = \frac{1}{2}\mu \dot{r}^2 + \frac{l^2}{2\mu r^2} + V \tag{5}$$

Now

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \ = \frac{dr}{d\theta}\frac{l}{\mu r^2}$$

$$\dot{r} = k\alpha e^{\alpha\theta}\frac{l}{\mu r^2} = r\alpha \frac{l}{\mu r^2}$$

$$\dot{r} = \alpha \frac{l}{\mu r} \tag{6}$$

Now $E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{l\alpha}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2}{2\mu r^2}(\alpha^2+1)$$

$$\Rightarrow E = \frac{l^2}{2\mu r^2}(\alpha^2+1) - \frac{l^2}{2\mu r^2}(\alpha^2+1) = 0 \qquad (7)$$

Eq. 7 gives the total energy of the system. Zero value of the system represent a bound system.

Now we will determine of $\theta_{(t)}$ and $r_{(t)}$

Since $\dot{\theta} = \frac{l}{\mu r} \Rightarrow \frac{d\theta}{dt} = \frac{l}{\mu r}$

$$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \Rightarrow e^{2\alpha\theta}d\theta = \frac{l}{\mu k^2}\,dt$$

Integrating both sides we get $\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C$

$$e^{2\alpha\theta} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow \theta_{(t)} = \frac{1}{2\alpha}\ln\left[2\alpha\left(\frac{lt}{\mu k^2} + C\right)\right] \qquad (9)$$

Now $r = ke^{\alpha\theta}$

$$\Rightarrow \frac{r}{k} = e^{\alpha\theta} \Rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$$

$$\Rightarrow \frac{r^2}{k^2} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow r_{(t)} = \sqrt{2\alpha k^2\left(\frac{lt}{\mu k^2} + C\right)} \qquad (10)$$