Chapter 4
Lecture 1

Two body central Force Problem

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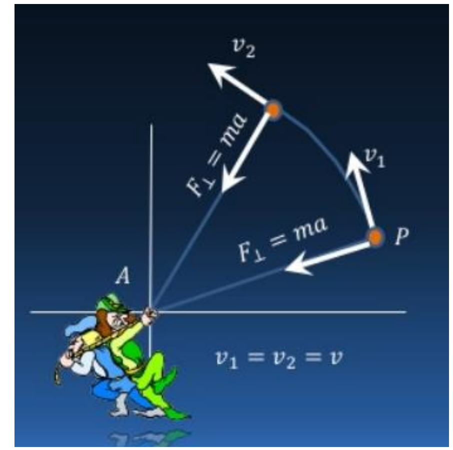
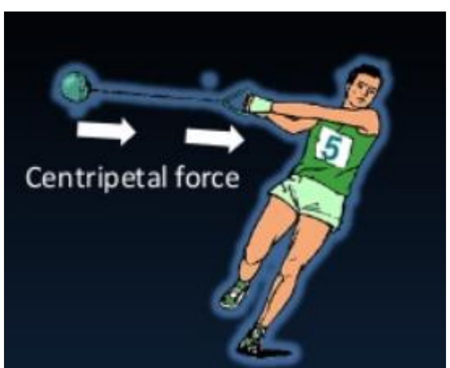
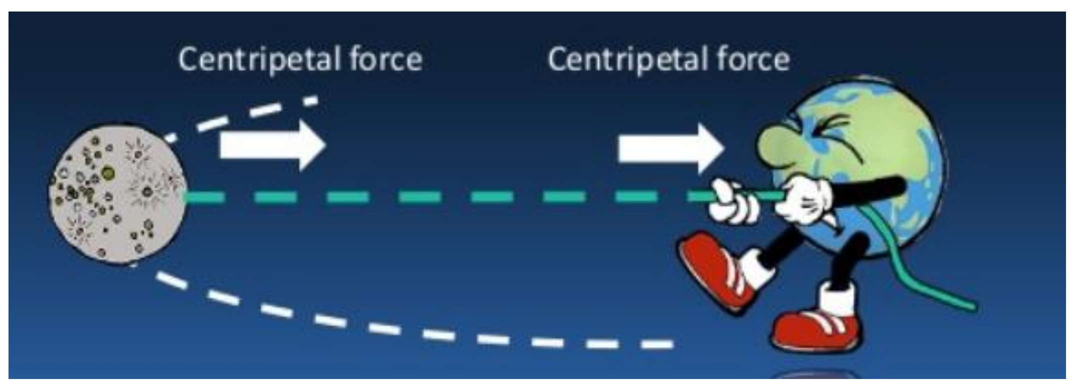
TWO BODY CENTRAL FORCE PROBLEM

4.1 Introduction

- One of the most important problems of classical mechanics is to understand the motion of a body moving under the influence of a central force field.
- Force which is always directed towards the centre or line joining two bodies
- The motion of the planets around the sun.





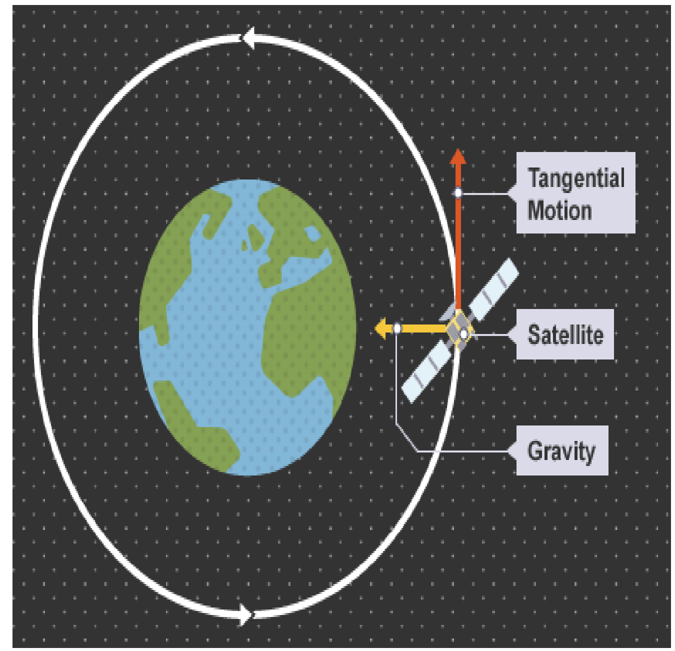


TWO BODY CENTRAL FORCE PROBLEM

- Motion of satellites around the earth

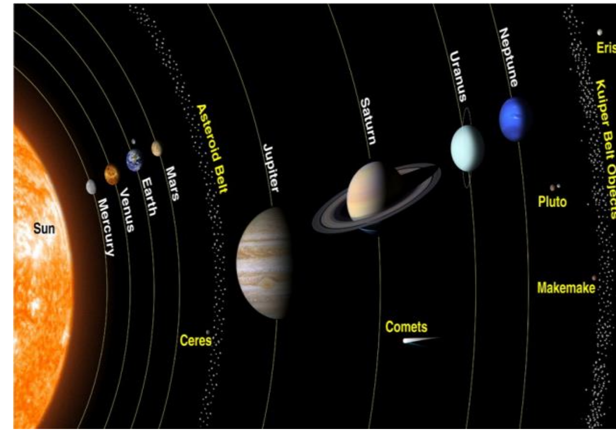
- Motion of two charge particles with respect to each other

- In this chapter, we study the two-body problem, which is reduced to an equivalent one-body problem.



TWO BODY CENTRAL FORCE PROBLEM

The motion of a particle in central force field can be classified as;

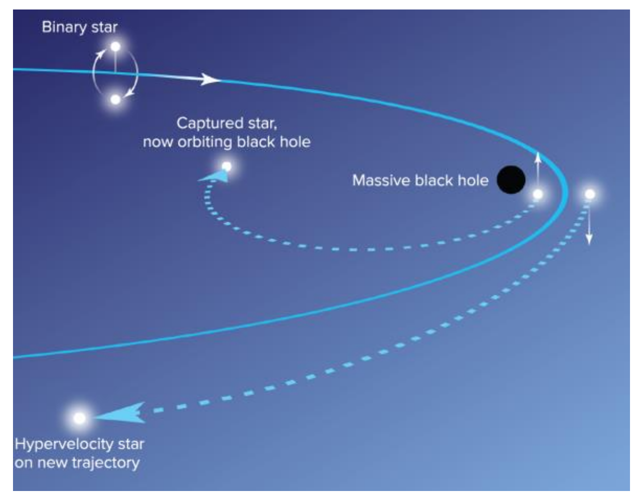


1) Bound motion

The distance between two bodies never exceeds a finite limit,

e.g. motion of planets around the sun.

2) Unbound motion



The distance between two particles or bodies is infinite at initial and final stage.

The bodies move from infinite distance and approach to interact in close proximity and finally move far from each other to an infinite distance.

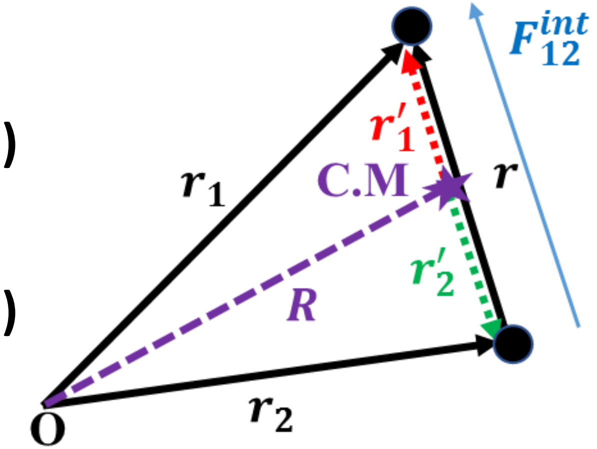
For example, scattering of alpha particles by gold nuclei as studied by Rutherford.

TWO BODY CENTRAL FORCE PROBLEM

- It is always possible to reduce the motion of two bodies to that of an equivalent single-body problem.
- The exact solution and understanding of two bodies motion problem is possible.
- However, the presence of the third body complicates the situation and an exact solution to the problem become an impossibility.
- Therefore, on must adopt the approximate methods to solve the many bodies problem.
- We can always reduce many body systems to a two-body problem either by neglecting the effects of the others or by some other screening methods, where the effects of the other bodies don't play prominent role.
- Such as the motion of a planets around the sun, where the effect due to the presence of other planets is neglected. However, we will restrict ourselves to the two bodies problem only.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Consider the motion of two particles. Let $F^{(ext)}$ be the total external force acting on the system. Let F^{int} be the total internal force due to the interaction between two particles.



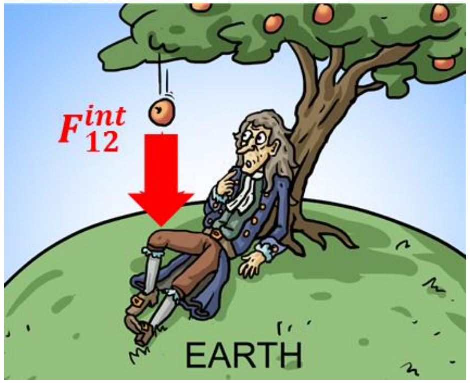
Total external force will be

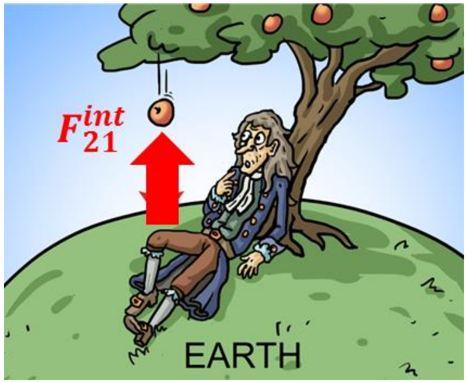
$$\boldsymbol{F}^{ext} = \boldsymbol{F}_1^{ext} + \boldsymbol{F}_2^{ext} \quad \textbf{(4.1.1)}$$

Further according to the Newton's 3rd law

$$\boldsymbol{F}_{12}^{int} = -\boldsymbol{F}_{21}^{int}\text{`} \quad \textbf{(4.1.2)}$$

Action and reaction forces.





If the action and reaction forces are same Why only apple falls for earth?

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

The equations of motion can be written as

Force on Particle 1

$$m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1^{ext} + \boldsymbol{F}_{12}^{int} \qquad \textbf{(4.1.3)}$$

Force on Particle 2

$$m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2^{ext} + \boldsymbol{F}_{21}^{int} \qquad \textbf{(4.1.4)}$$

Total force on system of Particles

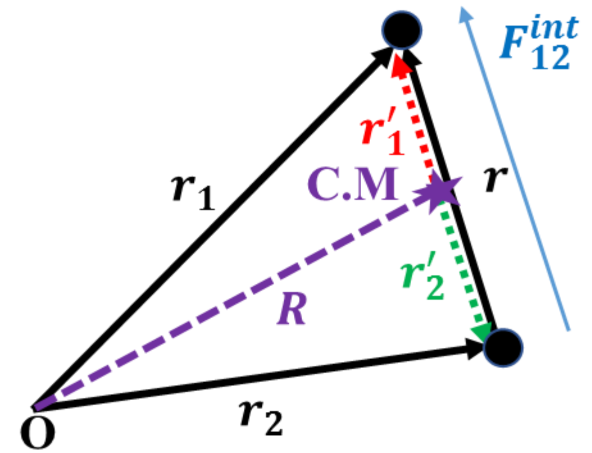
$$\boldsymbol{F}^{ext} = M\ddot{\boldsymbol{R}} \qquad \textbf{(4.1.5)}$$

Total mass of the system

$$M = m_1 + m_2 \qquad \textbf{(4.1.6)}$$

Position vector of the centre of mass of the system is

- $\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}$ **(4.1.7)**

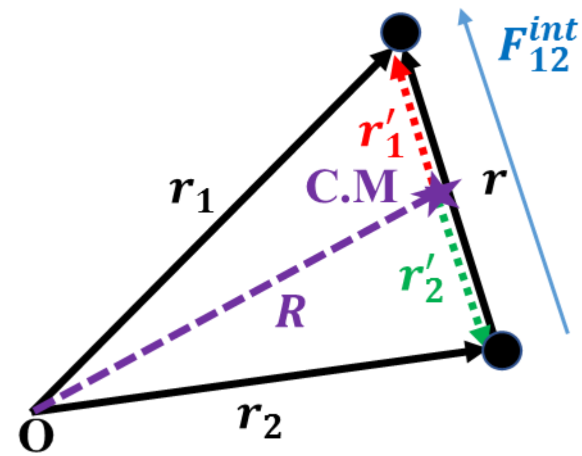


REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Position vector of particle 1 relative to particle 2 be

$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ **(4.1.8)**

$\boldsymbol{r}_1 = \boldsymbol{r} + \boldsymbol{r}_2$ **(4.1.9)**



Putting in Eq. (**4.1.9**) in Eq. (**4.1.7**)

$\boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1 \boldsymbol{r}}{m_1+m_2}$ **(4.1.10)**

Similarly, Eq. (**4.1.8**) can be written as

$\boldsymbol{r}_2 = \boldsymbol{r}_1 - \boldsymbol{r}$ **(4.1.11)**

Putting in equation **(4.1.11)** in equation **(4.1.7)**

$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2 \boldsymbol{r}}{m_1+m_2}$ **(4.1.12)**

Multiplying Eq. **(4.1.3)** by m_2 & Eq. (**4.1.**4) by m_1 and subtracting,

$$m_1 m_2(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = m_2 \boldsymbol{F}_{12}^{int} - m_1 \boldsymbol{F}_{21}^{int} + m_1 m_2 \left(\frac{\boldsymbol{F}_1^{ext}}{m_1} - \frac{\boldsymbol{F}_2^{ext}}{m_2}\right)$$

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Dividing the above equation by $(m_1 + m_2)$ and using $\boldsymbol{F}_{\boldsymbol{12}}^{\boldsymbol{int}} = -\boldsymbol{F}_{\boldsymbol{21}}^{\boldsymbol{int}}$

$$\frac{m_1 m_2}{(m_1+m_2)}(\ddot{\boldsymbol{r}}_{\boldsymbol{1}} - \ddot{\boldsymbol{r}}_{\boldsymbol{2}}) = \frac{(m_1+m_2)}{(m_1+m_2)}\boldsymbol{F}_{\boldsymbol{12}}^{\boldsymbol{int}} + \frac{m_1 m_2}{(m_1+m_2)}\left(\frac{\boldsymbol{F}_{\boldsymbol{1}}^{\boldsymbol{ext}}}{m_1} - \frac{\boldsymbol{F}_{\boldsymbol{2}}^{\boldsymbol{ext}}}{m_2}\right)$$

$$\Rightarrow \mu(\ddot{\boldsymbol{r}}_{\boldsymbol{1}} - \ddot{\boldsymbol{r}}_{\boldsymbol{2}}) = \boldsymbol{F}_{\boldsymbol{12}}^{\boldsymbol{int}} + \mu\left(\frac{\boldsymbol{F}_{\boldsymbol{1}}^{\boldsymbol{ext}}}{m_1} - \frac{\boldsymbol{F}_{\boldsymbol{2}}^{\boldsymbol{ext}}}{m_2}\right) \qquad \textbf{(4.1.13)}$$

Where μ is reduce mass of the system.

$$\mu = \frac{m_1 m_2}{(m_1+m_2)} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \textbf{(4.1.14)}$$

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Special case

If no external force is acting

$$\boldsymbol{F_1^{ext}} = \boldsymbol{F_2^{ext}} = \boldsymbol{0} \qquad \textbf{(A)}$$

equation **(4.1.13)** will be reduced to

$$\Rightarrow \mu(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \boldsymbol{F_{12}^{int}}$$

$$\Rightarrow \mu\ddot{\boldsymbol{r}} = \boldsymbol{F_{12}^{int}} \qquad \textbf{(4.1.15)a}$$

If the forces produce same acceleration

$$\frac{\boldsymbol{F_1^{ext}}}{\boldsymbol{m_1}} = \frac{\boldsymbol{F_2^{ext}}}{\boldsymbol{m_2}} \Rightarrow \ddot{\boldsymbol{r}}_1 = \ddot{\boldsymbol{r}}_2 \qquad \textbf{(B)}$$

The condition **B** is realized if centre producing the external forces is at a considerable distance from the system and the force due to it on any mass is proportional to that of the mass.

Such as gravitational force. In Earth-moon mutual motion, force due to the sun is assumed such that it satisfy the condition mentioned in Eq. **B**.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Equation will be reduced to

$$\Rightarrow \mu(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \boldsymbol{F}_{12}^{int}$$

$$\Rightarrow \mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}^{int} \qquad \textbf{(4.1.15)b}$$

Eq. **(4.1.15)b** represent motion of a particle of mass equal μ and moving under the action of force $\boldsymbol{F}_{12}^{int}$.

The reduction is equivalent to replace the system of two bodies by a mass μ and considering the acceleration produced is due to the internal force.

Eq. **(4.1.15)a** ($\mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}^{int}$) together with Eq. **(4.1.5)** ($\boldsymbol{F}^{ext} = M\ddot{\boldsymbol{R}}$) represents the motion of a two body system under the action of internal and external forces as long as the conditions mentioned in equations **A & B** are valid.

If the internal forces are attractive and these are the only forces acting on the system, the two bodies move around the centre of mass which acts as centre of force. i.e. directed towards the centre.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Condition on mass

If the mass of one of the particles is extremely large as compared to that of the other, say $m_1 >> m_2$, then the reduced mass is simply

$$\mu = \frac{m_1 m_2}{(m_1+m_2)} = \frac{m_1 m_2}{m_1(1+{}^{m_2}/_{m_1})}$$

$$\Rightarrow \mu = \frac{m_2}{(1+{}^{m_2}/_{m_1})} \qquad \text{as} \qquad {}^{m_2}/_{m_1} \approx 0$$

$$\Rightarrow \mu = m_2$$

In this case the centre of mass of the system coincides with the centre of mass of the heavier body.

This approximation is equivalent to neglecting the recoil of mass m_1. This is used in Bohr’s theory of hydrogen atom and motion of satellites around the earth. It can be assumed for the motion of earth around the Sun.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Since mass $m_1 >> m_2$,
acceleration in mass m_1

$$\boldsymbol{a_1 = \frac{F_{21}^{int}}{m_1} \approx 0}$$

acceleration in mass m_2

$$\boldsymbol{a_2 = \frac{F_{21}^{int}}{m_2} > 0}$$

That's is why

"An apple appears to fall towards the earth and not the earth towards the apple".

Lagrangian of the System

If $U(\boldsymbol{r}, \dot{\boldsymbol{r}})$ is the function of "$\boldsymbol{r}$"and higher derivative of "$\dot{\boldsymbol{r}}$". Then Lagrangian of the system can be written as

$L = T(\dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) - U(\boldsymbol{r}, \dot{\boldsymbol{r}})$ **(4.1.16)**

Where $T(\dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) = {}^1\!/_2\, M\dot{\boldsymbol{R}}^{\boldsymbol{2}} + T' = {}^1\!/_2\,(m_1+m_2)\dot{\boldsymbol{R}}^{\boldsymbol{2}} + T'$ **(4.1.17)**

And $T' = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1'^{\,\boldsymbol{2}} + \frac{1}{2}m_2\dot{\boldsymbol{r}}_{\boldsymbol{2}}'^{\,\boldsymbol{2}}$ **(4.1.18)**

Where $\boldsymbol{r}_{\boldsymbol{1}}' = \boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{R}$

$$\Rightarrow \boldsymbol{r}_1' = \boldsymbol{r}_{\boldsymbol{1}} - \frac{m_1\boldsymbol{r}_{\boldsymbol{1}} + m_2\boldsymbol{r}_{\boldsymbol{2}}}{(m_1+m_2)} = \frac{m_2(\boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{r}_{\boldsymbol{2}})}{(m_1+m_2)}$$

$$\Rightarrow \boldsymbol{r}_1' = \frac{m_2}{(m_1+m_2)}\boldsymbol{r} \qquad \textbf{(4.1.19)}$$

Similarly, $\boldsymbol{r}_2' = \boldsymbol{r}_{\boldsymbol{2}} - \boldsymbol{R}$

$$\Rightarrow \boldsymbol{r}_2' = \boldsymbol{r}_{\boldsymbol{2}} - \frac{m_1\boldsymbol{r}_{\boldsymbol{1}} + m_2\boldsymbol{r}_{\boldsymbol{2}}}{(m_1+m_2)} = -\frac{m_1(\boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{r}_{\boldsymbol{2}})}{(m_1+m_2)}$$

$$\Rightarrow \boldsymbol{r}_2' = -\frac{m_1}{(m_1+m_2)}\boldsymbol{r} \qquad \textbf{(4.1.20)}$$

Lagrangian of the System

Therefore, the kinetic energy from Eq **(4.1.18)** can be written as

$$T' = \frac{1}{2} m_1 \left(\frac{m_2}{m_1+m_2} \dot{\boldsymbol{r}} \right)^{\mathbf{2}} + \frac{1}{2} m_2 \left(- \frac{m_1}{m_1+m_2} \dot{\boldsymbol{r}} \right)^{\mathbf{2}}$$

$$\Rightarrow T' = \frac{1}{2} (m_2 + m_1) \frac{m_1 m_2}{(m_1+m_2)^2} \dot{\boldsymbol{r}}^{\mathbf{2}}$$

$$\Rightarrow T' = \frac{1}{2} \frac{m_1 m_2}{(m_2+m_1)} \dot{\boldsymbol{r}}^{\mathbf{2}} \qquad \textbf{(4.1.21)}$$

The Lagrangian of the system can be written as;

$$L = T\left(\dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}\right) - U(\boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$L = {}^{1}/_{2}\, (m_1 + m_2) \dot{\boldsymbol{R}}^{\mathbf{2}} + \frac{1}{2} \frac{m_1 m_2}{(m_1+m_2)} \dot{\boldsymbol{r}}^{\mathbf{2}} - U(\boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$L = {}^{1}/_{2}\, M \dot{\boldsymbol{R}}^{\mathbf{2}} + \frac{1}{2} \mu \dot{\boldsymbol{r}}^{\mathbf{2}} - U(\boldsymbol{r}, \dot{\boldsymbol{r}}) \qquad \textbf{(4.1.22)}$$

Where M is the total mass of the system and μ is the reduce mass of the system.

4.2 Properties of central Force

4.2.1a Under the central force, the angular momentum of the particle is conserved

a. In cartesian coordinates

The Torque on the system (if any) can be written as; $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$ **(4.2.1)**

and the angular momentum of the body is $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P}$ **(4.2.2)**

We know that; $$\frac{d\boldsymbol{l}}{dt} = \boldsymbol{N} \qquad \textbf{(4.2.3)}$$

Since the force acting on the body is central force and always directed towards the line joining the body with the centre therefore

$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F} = r\hat{r} \times F_r\hat{r} = rF_r(\hat{r} \times \hat{r}) = 0 \qquad \textbf{(4.2.4)}$$

$$\Rightarrow \frac{d\boldsymbol{l}}{dt} = \boldsymbol{N} = \boldsymbol{0} \Rightarrow \boldsymbol{l} = Constant \qquad \textbf{(4.2.5)}$$

Eq.**(4.2.4) & (4.2.5)** suggests that the total torque "***N***" acting on the system will be zero and angular momentum "***L***" of the body will be constant.

4.2 Properties of central Force

b. In Polar coordinates

$$\boldsymbol{F} = F_r \hat{r} + F_\theta \hat{\theta} \qquad \textbf{(4.2.6)}$$

And similarly, the torque acting on a particle in polar coordinates is

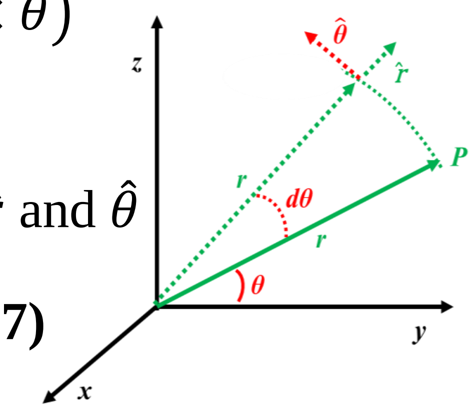
$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F} = r\hat{r} \times \left[\left(\mu\ddot{r} - mr\dot{\theta}^2\right)\hat{r} + \left(\mu r\ddot{\theta} + 2\mu\dot{r}\dot{\theta}\right)\hat{\theta}\right]$$

$$\Rightarrow \boldsymbol{N} = r\left(\mu\ddot{r} - \mu r\dot{\theta}^2\right)(\hat{r} \times \hat{r}) + r\left(\mu r\ddot{\theta} + 2\mu\dot{r}\dot{\theta}\right)\left(\hat{r} \times \hat{\theta}\right)$$

$$\Rightarrow \boldsymbol{N} = 0 + r\left(\mu r\ddot{\theta} + 2\mu\dot{r}\dot{\theta}\right)\left(\hat{r} \times \hat{\theta}\right)$$

$$\Rightarrow \boldsymbol{N} = \left(\mu r^2\ddot{\theta} + 2\mu r\dot{r}\dot{\theta}\right)\hat{n}$$ where $\hat{n}$ is $\perp$ to both $\hat{r}$ and $\hat{\theta}$

$$\Rightarrow \boldsymbol{N} = \frac{d}{dt}\left(\mu r^2\dot{\theta}\right)\hat{n} \qquad \textbf{(4.2.7)}$$



For Radial force, the angular part of the force is zero

$$\boldsymbol{N} = \frac{d\boldsymbol{l}}{dt} = 0 \Rightarrow \boldsymbol{l} = \mu r^2\dot{\theta} = Constant \qquad \textbf{(4.2.8)}$$

Note: Also $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P} = \boldsymbol{r} \times \mu\boldsymbol{v} = \boldsymbol{r} \times \mu r\boldsymbol{\omega} = \boldsymbol{r} \times \mu r\dot{\boldsymbol{\theta}} \Rightarrow |\boldsymbol{L}| = \mu r^2\dot{\theta}$

4.2 Properties of central Force

4.2.2 The path of a particle moving under the central force must be a Plane

Consider the central force $\boldsymbol{F} = F_r\hat{r}$ **(4.2.9)**

Taking cross product with radius vector of above equation

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r}F_r(\hat{r} \times \hat{r}) = 0$$

$$\Rightarrow \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times \mu\frac{d\boldsymbol{v}}{dt} = 0$$

$$\Rightarrow \boldsymbol{r} \times \mu\frac{d\boldsymbol{v}}{dt} = \mu\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{v}) = 0$$

$$\Rightarrow \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{v}) = 0 \qquad \textbf{(4.2.10)}$$

Integrating above equation $\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{q} = constant$ **(4.2.11)**

Since the vector "$\boldsymbol{q}$" is perpendicular to both "$\boldsymbol{r}$" and "$\boldsymbol{v}$"is zero

$$\boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{v}) = \boldsymbol{r} \cdot \boldsymbol{q} = 0$$

Therefore, the particle is in Plane.

4.2 Properties of central Force

4.2.3 The Areal velocity of the body under the central force is constant OR

The position vector of particle drawn from the origin sweeps equal area in equal interval of times. OR The rate of change of area is constant.

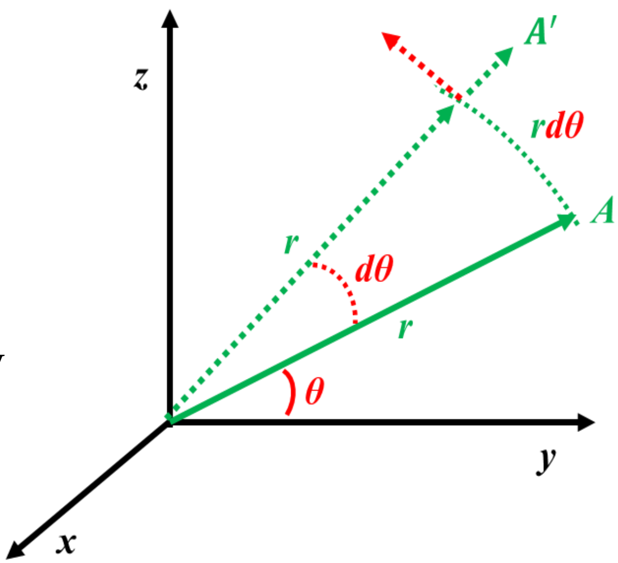
If the body move from position "A" to position "A'" and cover and angular displacement of "$d\theta$" and arc length "$rd\theta$".

The area of Triangle $\Delta AOA'$ in given figure is

$$d\boldsymbol{A} = \frac{1}{2}(\boldsymbol{r} \times rd\boldsymbol{\theta}) = \frac{1}{2}\left(r\hat{r} \times rd\theta\hat{\theta}\right)$$

$$d\boldsymbol{A} = \frac{1}{2}r^2 d\theta\hat{n} \qquad \textbf{(4.2.12)}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}\hat{n}$$



Multiplying both sides with mass "μ" of the body

$$\mu\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}\mu r^2\frac{d\theta}{dt}\hat{n} = \frac{1}{2}\mu r^2\dot{\theta}\hat{n}$$

$$\mu\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}\boldsymbol{l}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2\mu}\boldsymbol{l} = \boldsymbol{constant} \quad \textbf{(As required)}$$