Chapter 4

Lecture 3

Two body central Force Problem

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4.8 Prove that for central force field the equation of motion can be written as;

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu f_{(u)}}{l^2u^2} \qquad \textbf{And} \qquad \frac{d^2u}{d\theta^2} + u = -\frac{f_{(u)}}{\mu h^2u^2}$$

where $h = r^2\dot{\theta}$

Solution: Consider a particle of mass "μ" is at a distance "$\boldsymbol{r}$" from the origin. The acceleration of the particle can have two components in the polar coordinates

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad (4.8.1)$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad (4.8.2)$$

Since the central force is always directed along the radial vector "$\boldsymbol{r}$". The radial force is responsible for the motion. Therefore;

$$f_{(r)} = \mu\left(\ddot{r} - r\dot{\theta}^2\right) \qquad (4.8.3)$$

$$f_{(\theta)} = 0 \qquad (4.8.4)$$

Let us consider a function "u" such that $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \dot{r} = -r^2\dot{\theta}\frac{du}{d\theta}$$

$$\boxed{\Rightarrow \dot{r} = -h\frac{du}{d\theta}} \qquad (4.8.5)$$

Differentiating above equation with respect to t

$$\frac{d\dot{r}}{dt} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{dt}\right)$$

$$\Rightarrow \ddot{r} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\frac{d\theta}{dt}\right)$$

$$\Rightarrow \ddot{r} = -h\dot{\theta}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = -h\dot{\theta}\frac{d^2u}{d\theta^2} \qquad (4.8.6)$$

Since $h^2 = r^2\dot{\theta} \Rightarrow \dot{\theta} = {}^{h}/_{r^2}$ $or\ \dot{\theta} = hu^2$, Putting in Eq. (4.8.6)

$$\boxed{\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}} \qquad (4.8.7)$$

Putting $r = \frac{1}{u}$, $\dot{\theta} = hu^2$ and Eq. (4.8.7) in Eq. (4.8.3)

$$f_{(r)} = \mu(\ddot{r} - r\dot{\theta}^2) \Rightarrow f_{(u)} = \mu\left(-h^2u^2\frac{d^2u}{d\theta^2}\right) - \mu\left(\frac{1}{u}\right)(hu^2)^2$$

$$\Rightarrow f_{(u)} = -\mu h^2u^2\frac{d^2u}{d\theta^2} - \mu h^2u^3$$

$$\Rightarrow f_{(u)} = -\mu h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\Rightarrow -\frac{f_{(u)}}{\mu h^2u^2} = \left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\boxed{\Rightarrow \left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{f_{(u)}}{\mu h^2u^2}} \qquad (4.8.8)$$

As required.

Since $l = \mu r^2\dot{\theta} = \mu h$ putting in Eq. (4.8.8)

$$\boxed{\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2}} \qquad (4.8.9)$$

As desired.

4.9 Show That: a) $\boldsymbol{v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)}$

b) Using results from part "a" also prove that the conservation of energy equation will be

$\boldsymbol{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{\mu h^2}}$ **if** $\boldsymbol{u = \frac{1}{r}}$

Solution: Let us consider a particle of mass "μ" and position vector "$\boldsymbol{r}$".

Since $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} \quad -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \dot{r} = -r^2\dot{\theta}\frac{du}{d\theta} \Rightarrow \dot{r} = -h\frac{du}{d\theta}$$

Therefore, $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$$\Rightarrow v^2 = \left(-h\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(hu^2)^2 \quad = h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2$$

$$\Rightarrow v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \qquad (4.9.1)$$

Since $E = T + V \Rightarrow T = E - V$

$$\Rightarrow \frac{1}{2}\mu v^2 = E - V$$

$$\Rightarrow \frac{1}{2}\mu h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) = E - V$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{\mu h^2} \qquad (4.9.2)$$

Eq. (4.9.1) and Eq. (4.9.2) are as desired.

Problem (Page 293, Classical Mechanics by Marion)

Find the force law for a central force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where "k" and "α" are constants. Also find value of $\theta_{(t)}$ and $r_{(t)}$. Also find Energy of the orbit.

Solution. Since we have verified that

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2} = -\frac{\mu r^2 f_{(r)}}{l^2} \qquad (1)$$

Now using

$$r = ke^{\alpha\theta} \;\Rightarrow \mathrm{u} = \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiating Twice with respect to θ

$$\frac{d^2u}{d\theta^2} = \frac{\alpha^2}{k}e^{-\alpha\theta} = \alpha^2 u \qquad (2)$$

Putting value of u and $\frac{d^2u}{d\theta^2}$ in equation 1

$$\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow \alpha^2 u+u=-\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow f_{(r)}=-\frac{l^2}{\mu r^3}(\alpha^2+1) \qquad (3)$$

Eq. 3 represents the force responsible for motion.

Now the central potential responsible for the motion of the particle will be

$$V=-\int f_{(r)}\,dr=-\frac{l^2}{2\mu r^2}(\alpha^2+1) \qquad (4)$$

Total energy of the system is $\quad E=T+V=\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+V \quad (5)$

Now $\quad \dot{r}=\frac{dr}{d\theta}\frac{d\theta}{dt}$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \quad = \frac{dr}{d\theta}\frac{l}{\mu r^2}$$

$$\dot{r} = k\alpha e^{\alpha\theta}\frac{l}{\mu r^2} = r\alpha\frac{l}{\mu r^2}$$

$$\dot{r} = \alpha\frac{l}{\mu r} \qquad (6)$$

Now $$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V$$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{l\alpha}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2}{2\mu r^2}(\alpha^2 + 1)$$

$$\Rightarrow E = \frac{l^2}{2\mu r^2}(\alpha^2 + 1) - \frac{l^2}{2\mu r^2}(\alpha^2 + 1) = 0 \qquad (7)$$

Eq. 7 gives the total energy of the system. Zero value represent a bound system. Now we will determine of $\theta_{(t)}$ and $r_{(t)}$

Since $$\dot{\theta} = \frac{l}{\mu r^2}$$

$$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \Rightarrow e^{2\alpha\theta} d\theta = \frac{l}{\mu k^2}\ dt$$

Integrating both sides we get $\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C$

$$e^{2\alpha\theta} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right)$$

$$\Rightarrow \theta_{(t)} = \frac{1}{2\alpha}\ln\left[2\alpha\left(\frac{lt}{\mu k^2} + C\right)\right] \qquad (9)$$

Now $r = ke^{\alpha\theta} \Rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$

$$\Rightarrow \frac{r^2}{k^2} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right)$$

$$\Rightarrow r_{(t)} = \sqrt{2\alpha k^2\left(\frac{lt}{\mu k^2} + C\right)} \qquad (10)$$

4.10 Equation of motion for a body under central force (inverse square law force)

Solve $\left(\boldsymbol{\frac{d^2u}{d\theta^2}+u}\right)=-\boldsymbol{\frac{\mu f_{(u)}}{L^2u^2}}$ **and** $\boldsymbol{\theta=\theta_o+\int\frac{l/_{r^2}}{\sqrt{2\mu\left(E-V_{(r)}-\frac{l^2}{2\mu r^2}\right)}}dr}$ **and prove that the solution is the equation of conic. i.e. the motion under the inverse square law force represent motion on conic path. Also discuss the possibilities of bound and unbound system.**

Let us consider a particle of mass "μ" in under inverse square law force. The equation of motion can be written as

$$\left(\frac{d^2\boldsymbol{u}}{d\theta^2}+\boldsymbol{u}\right)=-\frac{\mu f_{(u)}}{l^2u^2} \qquad \textbf{(4.10.1)}$$

Since the inverse square attractive force

$$f_{(r)}=-\frac{k}{r^2}=-ku^2$$

$$\frac{d^2\boldsymbol{u}}{d\theta^2}+\boldsymbol{u}=\frac{\mu ku^2}{l^2u^2}$$

$$\boxed{\boldsymbol{\frac{d^2u}{d\theta^2}+u=\frac{\mu k}{l^2}}} \qquad \textbf{(4.10.2)}$$

4.10 Equation of motion for a body under central force (inverse square law force)

Starting with equation Eq. (4.10.2)

$$\frac{d^2\boldsymbol{u}}{d\theta^2} + \boldsymbol{u} = \frac{\mu k}{L^2} \boxed{\Rightarrow \frac{d^2\boldsymbol{u}}{d\theta^2} + \boldsymbol{u} - \frac{\mu k}{l^2} = 0} \qquad (4.10.2)$$

Consider a function

$$\boxed{y = \boldsymbol{u} - \frac{\mu k}{l^2}} \qquad (4.10.3)$$

Differentiating above equation Twice

$$\boxed{\frac{d^2y}{d\theta^2} = \frac{d^2\boldsymbol{u}}{d\theta^2}} \qquad (4.10.4)$$

Now

$$\frac{d^2y}{d\theta^2} + y = \frac{d^2u}{d\theta^2} + \boldsymbol{u} - \frac{\mu k}{l^2} = 0$$

$$\boxed{\frac{d^2y}{d\theta^2} + y = 0} \qquad (4.10.5)$$

4.10 Equation of motion for a body under central force (inverse square law force)

It is a second order differential equation where “y” is a function of “θ”

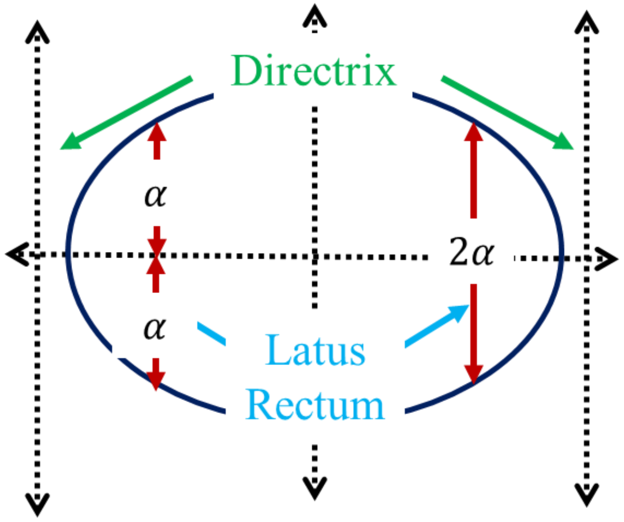
And $$y = A\cos(\theta - \theta_o) \tag{4.10.6}$$

$$y = \boldsymbol{u} - \frac{\mu k}{l^2} = A\cos(\theta - \theta_o)$$

$$\frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_o)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \frac{Al^2}{\mu k}\cos(\theta - \theta_o) \tag{4.10.7}$$

Equation of conic. $$\boxed{\frac{\alpha}{r} = 1 + e\cos(\theta - \theta_o)} \qquad (4.10.7)a$$



where $\alpha = \frac{l^2}{\mu k}$ Semi latus rectum.

and $e = \frac{Al^2}{\mu k}$ is eccentricity which is defined as the measure of deviation from circular shape.

4.10 Equation of motion for a body under central force (inverse square law force)

Now consider the first integral for the motion under central force

$$\theta = \theta_o + \int \frac{l/r^2}{\sqrt{2\mu\left(E - V_{(r)} - \frac{l^2}{2\mu r^2}\right)}} dr \qquad (4.10.8)$$

Since $\boxed{du = -\frac{1}{r^2} dr}$ & $\boxed{V = -\frac{k}{r} = -ku}$ (4.10.9) & (4.10.10)

Putting Eq. (4.10.4) and Eq. (4.10.5) in Eq. (4.10.3)

$$\theta = \theta_o - \int \frac{du}{\sqrt{\left(\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2}u - u^2\right)}} \qquad (4.10.11)$$

Let $\frac{2\mu E}{l^2} = a, \frac{2\mu k}{l^2} = b$ and $-1 = c$

Then $$\theta - \theta_o = -\int \frac{du}{\sqrt{\left(\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2}u - u^2\right)}} = -\int \frac{du}{\sqrt{(a + bu + cu^2)}}$$

4.10 Equation of motion for a body under central force (inverse square law force)

$$\theta - \theta_o = -\left[\frac{1}{\sqrt{-c}} cos^{-1}\left\{-\left(\frac{b+2cu}{\sqrt{b^2-4ac}}\right)\right\}\right]$$

$$\theta_o - \theta = \left[cos^{-1}\left\{\frac{-\frac{\mu k}{l^2}+u}{\sqrt{\left(\frac{\mu k}{l^2}\right)^2+\left(\frac{2\mu E}{l^2}\right)}}\right\}\right]$$

$$u = \frac{\mu k}{l^2} + \frac{\mu k}{l^2}\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)} cos(\theta_o - \theta)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = \left[1+\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)} cos(\theta_o - \theta)\right]$$

$$\boxed{\Rightarrow \frac{\alpha}{r} = [1 + ecos(\theta_o - \theta)] = [1 + ecos(\theta - \theta_o)]} \qquad (4.10.12)$$

We have shown that the solution of the first integral is an equation of conic

$\alpha = \frac{l^2}{\mu k}$ ⇨ semi latus rectum and $e = \sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)}$ is the eccentricity

4.10 Equation of motion for a body under central force (inverse square law force)

For Eq.(4.10.11) & Eq.(4.10.12) if we assume $\theta_o = 0$ &$\theta = 0^o$ & 180^o

$$\boxed{r_1 = \frac{\alpha}{1+e} = \frac{\alpha}{1+\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)}}} \quad \& \quad \boxed{r_2 = \frac{\alpha}{1-e} = \frac{\alpha}{1-\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)}}} \qquad (4.10.13)\ \&\ (4.10.14)$$

For $e > 1$ of $E > 0$, r_2 is negative

And $e = 1, E = 0,$ r_2 is infinity

Both cases ⇨ motion is unbound

Therefore $e < 1$ and $E < 0$ is necessary to keep a bounded motion.

The finite and positive values of r_1 and r_2 represents the turning points.

Comparing the equation of eccentricity

$$\boldsymbol{A = \frac{\mu k}{l^2}\sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)}} \qquad (4.10.15)$$

4.10 Equation of motion for a body under central force (inverse square law force)

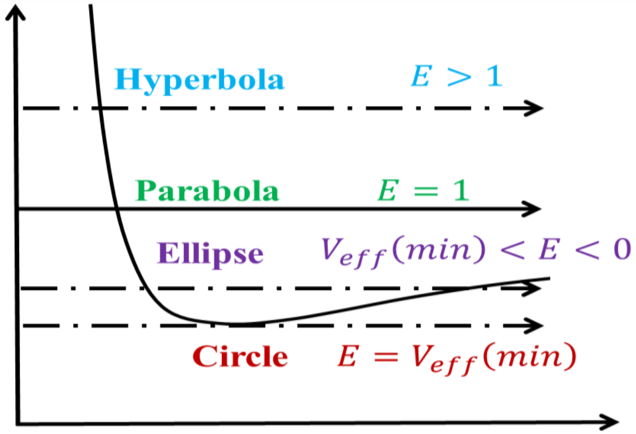
Nature of the Orbit

The nature of orbit is determined by eccentricity e which depend on energy

Value of E	Value of eccentricity	Nature of orbit
$E > 1$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$V_{eff}(min) < E < 0$	$0 < e < 1$	Ellipse
$E = V_{eff}(min)$	$e = 0$	Circle

we can always set $\theta_o = 0$ And $\frac{1}{c} = \alpha = \frac{L^2}{\mu k} \quad \Rightarrow \frac{1}{r} = C[1 + e\cos(\theta - \theta_o)]$

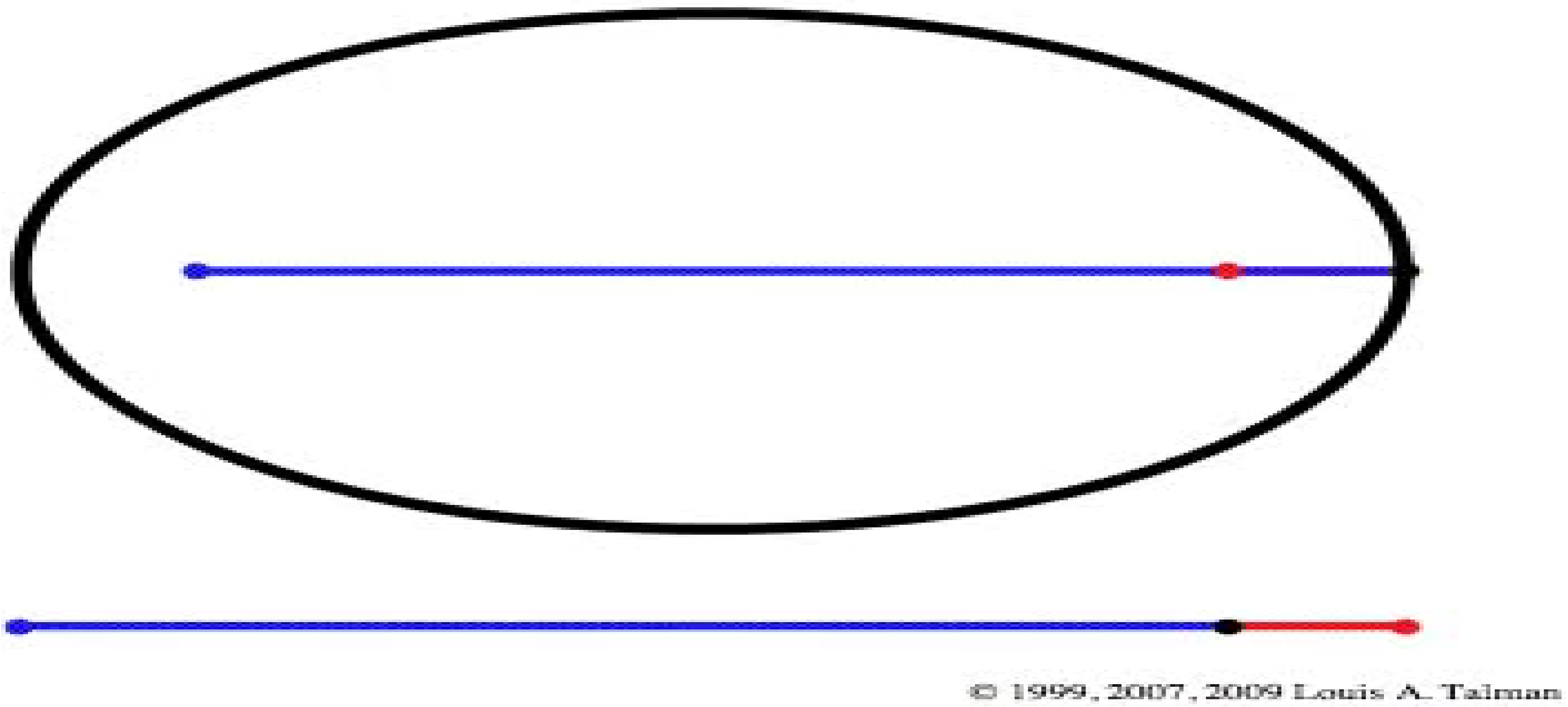
- Bound motion is possible only for Ellipse or circle.
- The motion of planets is either circular of elliptical.
- The variation of length of the day and seasonal changes suggest that the path of the planet is elliptical.



4.10 Equation of motion for a body under central force (inverse square law force)

Elliptic Orbit

The ellipse is a curve traced out by a particle moving in such a way that the sum of its distance from two fixed foci O and O' is always constant.



4.10 Equation of motion for a body under central force (inverse square law force)

Elliptic Orbit

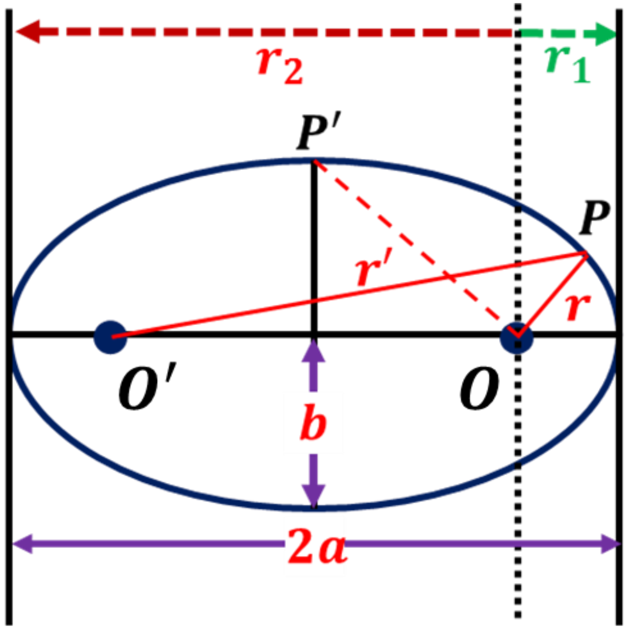
Not the distance between two foci $\qquad \overline{OO'} = r_2 - r_1$

$$\Rightarrow \overline{OO'} = \frac{2e\alpha}{(1-e^2)} = 2ae$$

$$\Rightarrow \frac{\overline{OO'}}{2} = ae \qquad (4.10.16)$$

From the figure it is clear that $\overline{OP'} = \overline{O'P'}$ and

$\overline{OP'} + \overline{O'P'} = 2a \qquad \& \qquad \overline{OP'} = a$



Now from figure $\qquad b^2 = (\overline{OP'})^2 - \left(\frac{\overline{OO'}}{2}\right)^2$

$$\Rightarrow b^2 = a^2 - a^2e^2 = a^2(1-e^2)$$

$$\Rightarrow b = a\sqrt{(1-e^2)}$$

4.10 Equation of motion for a body under central force (inverse square law force)

If $\boldsymbol{e \neq 0}$ then,

Since $$e = \sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)}$$

Therefore, $$b = a\sqrt{\left(1 - 1 - \left(\frac{2l^2E}{\mu k^2}\right)\right)}$$

$$b = a\sqrt{\left(-\left(\frac{2l^2E}{\mu k^2}\right)\right)}$$

The energy of the bounded system is less than zero therefore it will give a real value solution.

4.10 Equation of motion for a body under central force (inverse square law force)

If $\boldsymbol{e = 0}$ ellipse will become circle $\boldsymbol{b = a}$

(Note in the region when body passes through closest distance the curve is arc of a circle)

And
$$1 + \left(\frac{2l^2E}{\mu k^2}\right) = 0$$

$$E = -\left(\frac{\mu k^2}{2l^2}\right)$$

Eq. (4.10.15) will be reduced to $a = \alpha$, therefore the radius of the circle is;

$$r_o = a = \frac{l^2}{\mu k} = \frac{-\left(\frac{\mu k^2}{2E}\right)}{\mu k}$$

And
$$r_o = -\frac{k}{2E}$$

And
$$E = -\frac{k}{2a}$$

4.10 Equation of motion for a body under central force (inverse square law force)

Putting this value in equation of eccentricity we get

$$e = \sqrt{1 - \left(\frac{l^2}{\mu k a}\right)}$$

Using this value the semi-minor axis b can be written as.

$$b = a\sqrt{\left(1 - 1 + \left(\frac{l^2}{\mu k^2 a}\right)\right)}$$

$$b = a\sqrt{\frac{l^2}{\mu k^2 a}}$$

$$b = a^{1/2}\,\frac{l}{k\sqrt{\mu}}$$