Chapter 4
Lecture 4

Two body central Force Problem

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4.11 Keppler's Laws

Keppler First Law: "Every planet describes an ellipse with the sun at one of the foci"

Let us consider a particle of mass "μ" in under inverse square law force. Since the inverse square attractive force

$$f_{(r)} = -\frac{k}{r^2} = -ku^2 \quad (\text{if } r = \frac{1}{u}) \qquad (4.11.1)$$

For gravitational force $k = GmM_s$

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2} \qquad (4.10.2)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k u^2}{l^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2} \qquad (4.11.2)$$

We will now solve these equations Eq. (4.11.2) to understand the nature of the orbit.

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2} \Rightarrow \frac{d^2u}{d\theta^2} + u - \frac{\mu k}{l^2} = 0 \qquad (4.11.3)$$

4.11 Keppler's Laws

Consider a function $y = u - \frac{\mu k}{l^2}$ (4.11.4)

Differentiating above equation

$$\frac{dy}{d\theta} = \frac{du}{d\theta}$$

Differentiating above equation again

$$\frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2} \qquad (4.11.5)$$

Now

$$\frac{d^2y}{d\theta^2} + y = \frac{d^2u}{d\theta^2} + u - \frac{\mu k}{l^2} = 0$$

$$\frac{d^2y}{d\theta^2} + y = 0 \qquad (4.11.6)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}$$

The general solution is;

$$u = Bcos\theta + Csin\theta + \frac{\mu k}{l^2}$$

Where $B = Acos\theta_o$ and $C = Asin\theta_o$

$$u = Acos(\theta - \theta_o) + \frac{\mu k}{l^2}$$

$$r = \frac{1}{\frac{\mu k}{l^2} + Acos(\theta - \theta_o)} = \frac{l^2/_{\mu k}}{1 + \frac{Al^2}{\mu k}cos(\theta - \theta_o)}$$

$$r = \frac{\alpha}{1 + ecos(\theta - \theta_o)}$$

$$\frac{\alpha}{r} = 1 + ecos\theta$$

4.11 Keppler's Laws

It is a second order differential equation where "y" is a function of "θ" And

$$y = A\cos(\theta - \theta_o) \qquad (4.11.7)$$

where A and θ_o are constants.

$$y = u - \frac{\mu k}{l^2} = A\cos(\theta - \theta_o)$$

$$\boxed{u = \frac{\mu k}{l^2} + A\cos(\theta - \theta_o)} \qquad (4.11.8)$$

$$u\left(\frac{l^2}{\mu k}\right) = 1 + \frac{Al^2}{\mu k}\cos(\theta - \theta_o)$$

Using Equation Eq. (4.9.2) $$\boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu(E-V)}{l^2}}$$

and using $V = -\frac{k}{r} = -ku$

4.11 Keppler's Laws

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E+ku)}{\mu h^2}$$

Using Eq. $\boxed{u = A\cos(\theta - \theta_o) + \frac{\mu k}{l^2}}$ and $\boxed{\frac{du}{d\theta} = -A\sin(\theta - \theta_o)}$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = [-A\sin(\theta - \theta_o)]^2 + \left[A\cos(\theta - \theta_o) + \frac{\mu k}{l^2}\right]^2 = \frac{2\mu}{l^2}(E + ku)$$

$$\Rightarrow A^2[\sin^2(\theta - \theta_o) + \cos^2(\theta - \theta_o)] + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu}{l^2}(E + ku)$$

$$\boxed{\Rightarrow A^2 + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu}{l^2}\left(E + kA\cos(\theta - \theta_o) + \frac{\mu k^2}{l^2}\right)}$$

$$\Rightarrow A^2 + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu E}{l^2} + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) + 2\left(\frac{\mu k}{l^2}\right)^2$$

$$\boxed{\Rightarrow A^2 = \frac{2\mu E}{l^2} + \left(\frac{\mu k}{l^2}\right)^2}$$

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$$\boxed{\Rightarrow A = \frac{\mu k}{l^2}\sqrt{\frac{2El^2}{\mu k^2}+1}}$$

And $$u\left(\frac{l^2}{\mu k}\right) = 1 + \frac{Al^2}{\mu k}cos(\theta - \theta_o) \qquad (4.11.8)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \frac{\mu k}{l^2}\left(\sqrt{\frac{2El^2}{\mu k^2}+1}\right)\frac{l^2}{\mu k}cos(\theta - \theta_o)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \left(\sqrt{\frac{2El^2}{\mu k^2}+1}\right)cos(\theta - \theta_o)$$

$$\boxed{\Rightarrow \frac{\alpha}{r} = 1 + ecos(\theta - \theta_o)} \qquad (4.11.9)$$

Eq. (4.11.9) is equation of conic which describe the motion of planet around the sun.

4.10 Equation of motion for a body under central force (inverse square law force)

For Eq.(4.10.11) & Eq.(4.10.12) if we assume $\theta_o = 0$ & $\theta = 0^o$ & 180^o

$$\boxed{r_1 = \frac{\alpha}{1+e}} \quad \& \quad \boxed{r_2 = \frac{\alpha}{1-e}} \qquad (4.10.13) \ \& \quad (4.10.14)$$

For $e > 1$ of $E > 0$, r_2 is negative

And $e = 1, E = 0$, r_2 is infinity

Both cases ⇨ motion is unbound

Therefore $e < 1$ and $E < 0$ is necessary to keep a bounded motion.

The finite and positive values of r_1 and r_2 represents the turning points.

4.10 Equation of motion for a body under central force (inverse square law force)

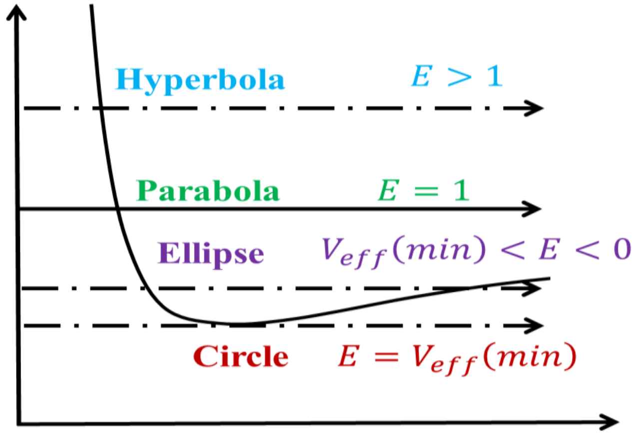
Nature of the Orbit

The nature of orbit is determined by eccentricity e which depend on energy

Value of E	Value of eccentricity	Nature of orbit
$E > 1$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$V_{eff}(min) < E < 0$	$0 < e < 1$	Ellipse
$E = V_{eff}(min)$	$e = 0$	Circle

we can always set $\theta_o = 0$ And $\frac{1}{c} = \alpha = \frac{L^2}{\mu k}$ $\Rightarrow \frac{1}{r} = C[1 + e\cos(\theta - \theta_o)]$

- Bound motion is possible only for Ellipse or circle.
- The motion of planets is either circular of elliptical.
- The variation of length of the day and seasonal changes suggest that the path of the planet is elliptical.



4.11 Keppler's Laws

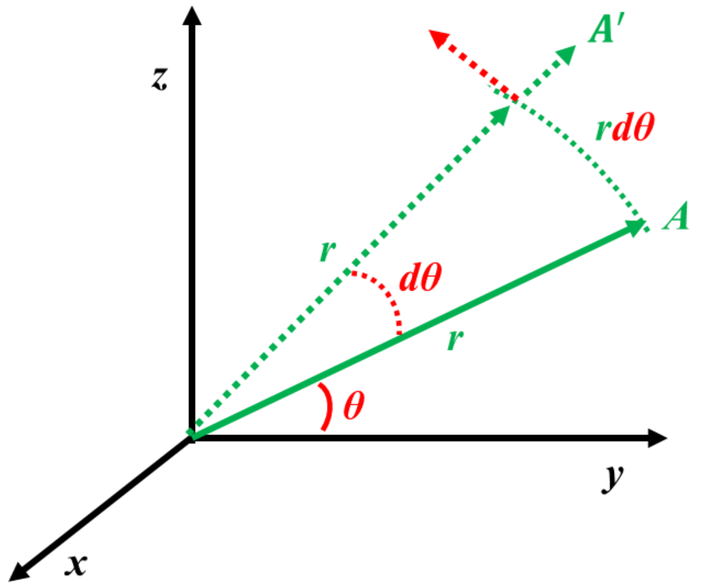
- Since the planet repeat its motion after a fixed period.
- During this period the variation in the length of day and night can only be explained if the orbit of the planet is elliptical.
- We conclude that the planet around the sun describe elliptical orbit with sun at one of its foci.
- Furthermore, the finite and positive values of r_1 and r_2 represents the turning points for the planet or the minimum and maximum radii of the planet during the motion which are called apogee and perigee for the earth orbit.

Keppler Second Law: "The position vector of particle drawn from the origin sweeps equal area in equal interval of times."   OR

"The Areal velocity of the body under the central force is constant."   OR

"The rate of change of area covered by the radial vector drawn from the centre to the planet under the central force is constant."

The area of Triangle $\Delta AOA'$ in given figure is



$$d\boldsymbol{A} = \frac{1}{2} r^2 d\theta \hat{n} \qquad \textbf{(4.11.12)}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \hat{n}$$

Multiplying both sides with mass "μ" of the body

$$m \frac{d\boldsymbol{A}}{dt} = \frac{1}{2} \mu r^2 \frac{d\theta}{dt} \hat{n} = \frac{1}{2} \mu r^2 \dot{\theta} \hat{n}$$

$$\mu \frac{d\boldsymbol{A}}{dt} = \frac{1}{2} \boldsymbol{l}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2\mu} \boldsymbol{l} = \boldsymbol{constant} \qquad \textbf{(4.11.13)}$$

4.11 Keppler's Laws

Kepler's Third Law: "The square of the time period of revolution of the planet is directly proportional to the cube of the semi-major axis of the orbit"

From the Kepler's second law, we know that Areal velocity of the body under the action of central force is constant

$$\dot{A} = \frac{l}{2\mu} = constant \qquad (4.11.13)$$

$$\frac{dA}{dt} = \frac{l}{2\mu} \Rightarrow \int \frac{dA}{dt} dt = \frac{l}{2\mu} \int dt$$

$$\Rightarrow \int_0^A dA = \frac{l}{2\mu} \int_0^\tau dt$$

Where τ is the time period of revaluation.

$$\boxed{\Rightarrow A = \frac{l}{2\mu} \tau} \qquad (4.11.14)$$

Since the area of the ellipse is $\boxed{\boldsymbol{A = \pi ab}}$

4.11 Keppler's Laws

And $b = a\sqrt{1-e^2}$

$\Rightarrow A = \pi a^2\sqrt{1-e^2}$ (4.11.15)

And we also know that by using $r_o = a = -\frac{k}{2E}$

$\Rightarrow E = -\frac{k}{2a}$ putting this in $e = \sqrt{1+\frac{2El^2}{\mu k^2}}$

$\Rightarrow e = \sqrt{1-\frac{l^2}{\mu k a}}$

$\Rightarrow e^2 = 1-\frac{l^2}{\mu k a} \Rightarrow \frac{l^2}{\mu k a} = 1-e^2$

$\Rightarrow \frac{l}{\sqrt{\mu k a}} = \sqrt{1-e^2}$ (4.11.16)

Therefore, $A = \pi a^2\sqrt{1-e^2} = \pi a^2\frac{l}{\sqrt{\mu k a}}$

4.11 Keppler's Laws

$$\Rightarrow \; A = \frac{\pi l}{\sqrt{\mu k}} a^{3/2} \qquad (4.11.17)$$

Comparing Equation for A

$$A = \frac{l}{2\mu}\tau = \frac{\pi l}{\sqrt{\mu k}} a^{3/2}$$

$$\Rightarrow \tau = 2\pi\sqrt{\frac{\mu}{k}}\, a^{3/2}$$

$$\Rightarrow \tau^2 = \frac{4\mu\pi^2}{k} a^3$$

$$\Rightarrow \tau^2 = (Constant) a^3$$

$\Rightarrow \tau^2 \propto a^3$ as desired. (4.11.18)

4.12 Virial Theorem

The **virial theorem** provides a general equation that relates the **average over time of the total Kinetic Energy (T)** of a system, bound by potential forces,

$$< T >= -\frac{1}{2} < \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} >$$

The word **virial** for the right-hand side of the equation derives from *vis*, the **Latin** word for "force" or "energy" and was given its technical definition by **Rudolf Clausius** in 1870.

significance : virial theorem is that it allows the average total kinetic energy to be calculated even for very complicated systems that defy an exact solution,

such as those considered in **Statistical mechanics**; this average total kinetic energy is related to the **Temperature** of the system by the **equipartition theorem**. H

4.12 Virial Theorem

Let us consider a system of points masses. Let the particle with mass "m_i", position vector "$\boldsymbol{r_i}$" and momentum "$\boldsymbol{P_i}$". We define a term "G" such that;

$$G = \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \boldsymbol{r_i} \qquad (4.12.1)$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \dot{\boldsymbol{P}}_i \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \dot{\boldsymbol{r}}_i$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} m_i \dot{r}_i^{\,2}$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + 2T \qquad (4.12.2)$$

$$I = \sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\left[\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \dot{\boldsymbol{r}}_i\right]$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{2}{2}\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{2}{2}\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \boldsymbol{r_i} = \boldsymbol{G}$$

The time average over the time interval is obtained by integrating both sides of the equation.

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\,dt = \frac{1}{\tau}[G(\tau) - G(0)] \qquad (4.12.3)$$

4.12 Virial Theorem

If the motion is periodic, all coordinates repeat itself after a certain time "τ"

$$\Rightarrow \frac{1}{\tau}\int_0^{\tau}\frac{dG}{dt}dt = 0 \qquad\qquad \text{because } G(\tau) = G(0)$$

If the motion is not periodic even, then for $\tau \gg$ the $\frac{1}{\tau}\,[G(\tau) - G(0)] \rightarrow 0$

In both cases right hand side is zero. Comparing Eq. (4.12.2) and (4.12.3)

$$\frac{1}{\tau}\int_0^{\tau}\frac{dG}{dt}dt =< \sum_{i=1}^{N}\boldsymbol{F_i}\cdot\boldsymbol{r_i} > +2 < T >= 0$$

$$\Rightarrow 2 < T >=-< \sum_{i=1}^{N}\boldsymbol{F_i}\cdot\boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < \sum_{i=1}^{N}\boldsymbol{F_i}\cdot\boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < \sum_{i=1}^{N}(-\boldsymbol{\nabla_r}V)\cdot\boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < -\frac{\boldsymbol{dV}}{\boldsymbol{dr}}\cdot\boldsymbol{r} > \qquad (4.12.4)$$

4.12 Virial Theorem

Since $V = \frac{k}{r}$ and $\frac{\boldsymbol{dV}}{\boldsymbol{dr}} = -\frac{k}{r^2}$ for central force

$\frac{\boldsymbol{dV}}{\boldsymbol{dr}} \cdot \boldsymbol{r} = -\frac{k}{r^2} \cdot r = -\frac{k}{r}$ putting this value in Eq. (4.12.4)

$$< T >= -\frac{1}{2} < -\frac{\boldsymbol{dV}}{\boldsymbol{dr}}.r >= -\frac{1}{2} < \frac{k}{\boldsymbol{r}} >$$

$$\Rightarrow < T >= -\frac{1}{2} < V > \qquad (4.12.5)$$

It is true for every system having potential

$V = kr^{n+1}$

$$\Rightarrow < T >= \frac{n+1}{2} < V > \qquad (4.12.6)$$

$V = -\frac{k}{r}$ & $\frac{\boldsymbol{dV}}{\boldsymbol{dr}} = \frac{k}{r^2}$ for central attractive force

$\frac{\boldsymbol{dV}}{\boldsymbol{dr}} \cdot \boldsymbol{r} = \frac{k}{r^2} \cdot r = \frac{k}{r}$ putting this value in Eq. (4.12.4)

$$< T >= -\frac{1}{2} < -\frac{dV}{dr}.r >= -\frac{1}{2} < -\frac{k}{\boldsymbol{r}} >$$

$$\Rightarrow < T >= -\frac{1}{2} < V >$$