Chapter 6

Lecture 2

Canonical Transformations

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6.1 Canonical Transformations

Properties of the Four basic canonical transformations

Generating function	Derivatives of generating function	Trivial special cases	Transformation
$F_1(q_i, Q_i, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$p_i = Q_i,$ $P_i = -q_i$
$F_2(q_i, P_i, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$p_i = P_i$ $Q_i = q_i$
$F_3(p_i, Q_i, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$q_i = -Q_i$ $P_i = -p_i$
$F_4(p_i, P_i, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$q_i = -P_i$ $Q_i = p_i$

6.2 Conditions for the transformation to be canonical

Conditions for the transformation to be canonical

For $F_1(q_i, Q_i, t) \Rightarrow \quad dF_1 = \sum p_i dq_i - \sum P_i dQ_i$

For $F_2(q_i, P_i, t) \Rightarrow \quad dF_2 = \sum p_i dq_i + \sum Q_i dP_i$

For $F_3(p_i, Q_i, t) \Rightarrow \quad dF_3 = -\sum q_i dp_i - \sum P_i dQ_i$

For $F_4(p_i, P_i, t) \Rightarrow \quad dF_4 = -\sum q_i dp_i + \sum Q_i dP_i$

6.2 Conditions for the transformation to be canonical

The transformation from (q_i, p_i) to (Q_i, P_i) will be canonical if

$$\sum \boldsymbol{p_i dq_i} - \sum \boldsymbol{P_i dQ_i}$$

is an exact differential

Solution: Consider the generating function $F_1(q_i, Q_i)$

$$dF_1 = \sum \frac{\partial F_1}{\partial q_i} dq_i + \sum \frac{\partial F_1}{\partial Q_i} dQ_i$$

Since $\qquad p_i = \frac{\partial F_1}{\partial q_i}$ and $P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore,

$$dF_1 = \sum p_i dq_i - \sum P_i dQ_i$$

which is an exact differential equation.

6.2 Conditions for the transformation to be canonical

Similarly, considering generating function $F_4(p_i, P_i, t)$

$$dF_4(p_i, P_i, t) = \sum \frac{\partial F_4}{\partial p_i} dp_i + \sum \frac{\partial F_4}{\partial P_i} dP_i$$

Since $\quad q_i = -\frac{\partial F_4}{\partial p_i} \quad$ and $\quad Q_i = \frac{\partial F_4}{\partial P_i}$

Therefore, $dF_4(p_i, P_i, t) = -\sum q_i dp_i + \sum Q_i dP_i$ which is an exact differential

Now subtracting dF_4 from dF_1

$$dF_1 - dF_4 = \sum p_i dq_i - \sum P_i dQ_i + \sum q_i dp_i - \sum Q_i dP_i$$

$$\Rightarrow dF_1 - dF_4 = \left(\sum q_i dp_i + \sum p_i dq_i\right) - \left(\sum Q_i dP_i + \sum P_i dQ_i\right)$$

6.2 Conditions for the transformation to be canonical

$$\Rightarrow \; dF_1 - dF_4 = d(q_i p_i) - d(Q_i P_i)$$

$$\Rightarrow d(F_1 - F_4) = d(q_i p_i - Q_i P_i)$$

Which is exact differential. Therefore the transformation is canonical.

And $\Rightarrow F_1 = F_4 + q_i p_i - Q_i P_i$

Examples [Conditions for the transformation to be canonical]

Show that transformation

$$P = \frac{1}{2}(p^2 + q^2) \text{ and } Q = tan^{-1}\frac{q}{p}$$

Solution: The transformation is canonical if $[pdq - PdQ]$ is an exact differential

$$pdq - PdQ = pdq - \frac{1}{2}(p^2 + q^2)\left[\frac{\frac{pdq-qdp}{p^2}}{1+\frac{q^2}{p^2}}\right]$$

$$\Rightarrow pdq - PdQ = pdq - \frac{1}{2}(p^2 + q^2)\left[\frac{pdq-qdp}{p^2} \times \frac{p^2}{p^2+q^2}\right]$$

$$\Rightarrow pdq - PdQ = pdq - \frac{1}{2}[pdq - qdp]$$

Examples [Conditions for the transformation to be canonical]

$$\Rightarrow pdq - PdQ = \frac{1}{2}pdq + \frac{1}{2}qdp$$

$$\Rightarrow pdq - PdQ = \frac{1}{2}(pdq + qdp)$$

$$\Rightarrow pdq - PdQ = \frac{1}{2}d(pq)$$

Hence the transformation is canonical.

Second Approach

Sympletic approach to canonical transformation

Let

$$\boldsymbol{Q_i = Q_i(q_j, p_j)}$$

$$\boldsymbol{P_i = P_i(q_j, p_j)}$$

The inverse transformation are

$$\boldsymbol{q_j = q_j(Q_i, P_i)}$$

$$\boldsymbol{p_j = p_j(Q_i, P_i)}$$

As the transformation does not involve time, therefore the Hamiltonian does not change in this case.

$$K(Q_i, P_i) = H(q_j, p_j)$$

To verify $\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial P_i}$

We Know that $H = H(q_j, p_j)$

$$\frac{\partial H}{\partial P_i} = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} \qquad (1)$$

Second Approach

Now $$\dot{Q}_i = \sum_j \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial Q_i}{\partial p_j}\dot{p}_j$$

$$\Rightarrow \dot{Q}_i = \sum_j \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_j \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \qquad (2)$$

Comparing eq (1) and eq (2) we concludes that

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}$$

The transformation is Canonical only if

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q_j,\,p_j} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q_i,\,P_i} \quad \text{and} \quad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q_j,\,p_j} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q_i,\,P_i}$$

Similarly we verify $$\dot{P}_i = -\frac{\partial H}{\partial Q_i}$$

Second Approach

$$\frac{\partial H}{\partial Q_i} = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_i} \qquad (3)$$

and $$\dot{P}_i = \sum_j \frac{\partial P_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial P_i}{\partial p_j}\dot{p}_j$$

$$\Rightarrow \dot{P}_i = \sum_j \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_j \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j} \qquad (4)$$

Comparing eq (3) and eq (4) we concludes that

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i}$$

The transformation is Canonical only if

$$\left(\frac{\partial P_i}{\partial p_j}\right)_{q_j, p_j} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q_i, P_i} \quad \text{and} \quad \left(\frac{\partial P_i}{\partial q_j}\right)_{q_j, p_j} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q_i, P_i}$$

Examples

Show that transformation

$$Q = \log\left(\frac{1}{q}\sin p\right) \text{ and } P = q\cot p$$

Solution: since

$$Q = \log\left(\frac{1}{q}\sin p\right)$$

$$\Rightarrow \dot{Q} = \frac{q}{\sin p}\frac{d}{dt}\left(\frac{1}{q}\sin p\right)$$

$$\Rightarrow \dot{Q} = \frac{q}{\sin p}\left[\frac{q\dot{p}\cos p - \dot{q}\sin p}{q^2}\right]$$

$$\Rightarrow \dot{Q} = \frac{\dot{p}\cos p}{\sin p} - \frac{\dot{q}}{q}$$

$$\Rightarrow \dot{Q} = \dot{p}\cot p - \dot{q}\frac{1}{q}$$

$$\Rightarrow \dot{Q} = -\frac{\partial H}{\partial q}\cot p - \frac{\partial H}{\partial p}\frac{1}{q}$$

Examples

$$\Rightarrow \dot{Q} = \left(-\cot p \frac{\partial P}{\partial q}\right) \frac{\partial H}{\partial P} - \frac{1}{q} \frac{\partial H}{\partial P} \frac{\partial P}{\partial p}$$

$$\Rightarrow \dot{Q} = (-\cot^2 p) \frac{\partial H}{\partial P} - \frac{1}{q} \frac{\partial H}{\partial P} (-q\, cosec^2\, p)$$

$$\Rightarrow \dot{Q} = (cosec^2\, p - \cot^2 p) \frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \frac{\partial H}{\partial P} \qquad (1)$$

Now

$$P = q \cot p$$

$$\Rightarrow \dot{P} = \dot{q} \cot p - q\dot{p}\, cosec^2 p$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial p} \cot p + q\, cosec^2 p \frac{\partial H}{\partial q}$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial p} \cot p + q\, cosec^2 p \frac{\partial H}{\partial Q} \frac{\partial Q}{\partial q}$$

Examples

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}\left(\frac{q}{\sin p}\frac{\cos p}{q}\right)\cot p + q\ cosec^2 p\,\frac{\partial H}{\partial Q}\left(-\frac{q}{\sin p}\frac{\sin p}{q^2}\right)$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}(\cot p)\cot p - cosec^2 p\,\frac{\partial H}{\partial Q}(-1)$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q} cot^2\, p - cosec^2 p\,\frac{\partial H}{\partial Q}$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}(cot^2\, p - cosec^2 p)$$

$$\Rightarrow \dot{P} = -\frac{\partial H}{\partial Q}(cosec^2 p - cot^2\, p)$$

$$\Rightarrow \dot{P} = -\frac{\partial H}{\partial Q} \qquad (2)$$

From equation (1) and (2) we conclude that the transformation is canonical.

Examples (From: Goldstein Page 378)

Solve simple hormonic oscillator in one dimension whose Hamiltonian

$$H = \frac{p^2}{2m} + \frac{mw^2}{2} q^2$$

And generating function $F_1 = \frac{m}{2} w q^2 \cot \mathrm{Q}$

Where m and w are constants.

Solution: Since the generating function $F_1 = F_1(q, Q)$

Therefore

$$p = \frac{\partial F_1}{\partial q} = mwq \, cot \, Q \qquad (1)$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{mwq^2}{2} \left(-cosec^2 Q\right)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{mwq^2}{2} \, cosec^2 Q \qquad (2)$$

$$\Rightarrow q = \sqrt{\frac{2P}{mw}} \sin Q$$

Putting in equation 1

Examples (From: Goldstein Page 378)

$$p = \frac{\partial F_1}{\partial q} = \sqrt{2mwP}\cos Q \qquad (3)$$

Since $$H = \frac{p^2}{2m} + \frac{mw^2}{2}q^2$$

Putting eq (2) and eq (3) in above equation

$$\Rightarrow H = \frac{2mwP cos^2 Q}{2m} + \frac{mw^2}{2}\frac{2P}{mw} sin^2 Q$$

$$\Rightarrow H = Pw(cos^2 Q + sin^2 Q)$$

$$\Rightarrow H = Pw$$

Since $$\dot{Q} = \frac{\partial H}{\partial P} = \mathrm{w}$$

Integrating above equation $Q = \mathrm{wt} + \alpha$

Examples (From: Goldstein Page 378)

Now Putting in equation (2) $$q = \sqrt{\frac{2P}{mw}} \sin(wt + \alpha)$$

since $$P = \frac{H}{w}$$

Therefore above equation is $$q = \sqrt{\frac{2H}{mw^2}} \sin(wt + \alpha)$$

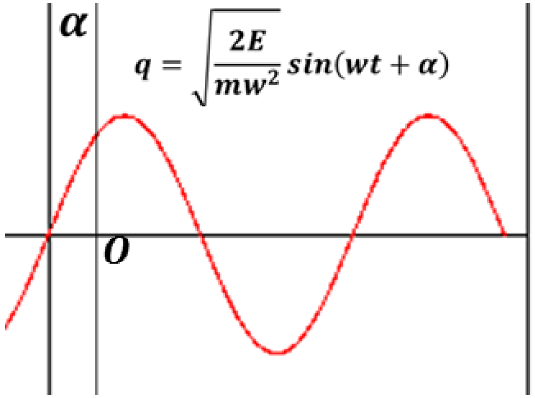
Since H is not exploit function of time therefore $H = E = constant$

Therefore above equation is $$q = \sqrt{\frac{2E}{mw^2}} \sin(wt + \alpha)$$

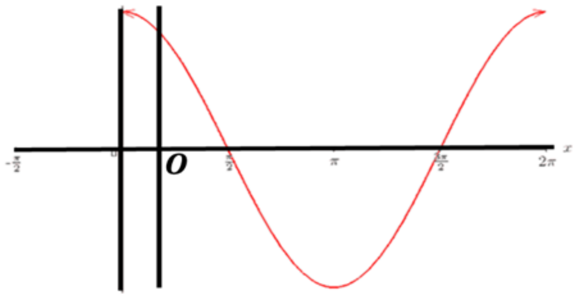
This is the one-dimensional solution of Simple Hormonic Oscillator

Examples (From: Goldstein Page 378)

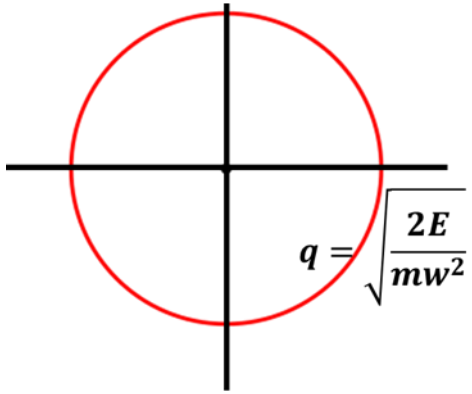
Since $q = \sqrt{\frac{2E}{mw^2}}\sin(wt + \alpha)$ & $p = \sqrt{2mE}\cos(wt + \alpha)$



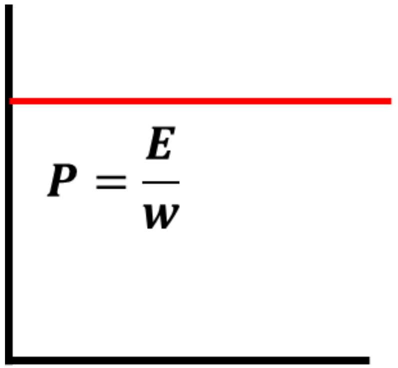
$p = \sqrt{2mE}\cos(wt + \alpha)$

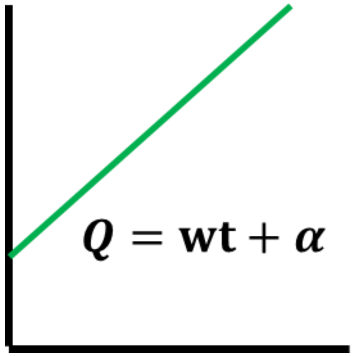


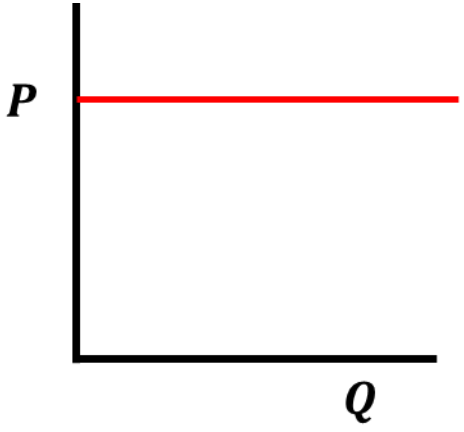
$p = \sqrt{2mE}$



$P = \frac{H}{w} = \frac{E}{w}$ & $Q = wt + \alpha$







Examples

For what value of α and β, equations

$$Q = q^{\alpha}\cos\beta p \qquad \text{and} \qquad P = q^{\alpha}\sin\beta p$$

Represents a canonical transformation. Find the generating Function F_3

Solution: The transformations will be canonical if it satisfies the following conditions.

$$\dot{Q} = \frac{\partial H}{\partial P} \qquad \& \qquad \dot{P} = -\frac{\partial H}{\partial Q}$$

Now if we take derivative of $Q = q^{\alpha}\cos\beta p$

$$\dot{Q} = \alpha q^{\alpha-1}\dot{q}\cos\beta p - \beta q^{\alpha}\dot{p}\sin\beta p$$

And

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \& \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\Rightarrow \dot{Q} = \alpha q^{\alpha-1}\frac{\partial H}{\partial p}\cos\beta p + \beta q^{\alpha}\frac{\partial H}{\partial q}\sin\beta p$$

Examples

$$\Rightarrow \dot{Q} = \alpha q^{\alpha-1}\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\cos\beta p + \beta q^{\alpha}\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} sin\,\beta p$$

$$\Rightarrow \dot{Q} = \left(\alpha q^{\alpha-1}\frac{\partial P}{\partial p}\cos\beta p + \beta q^{\alpha}\frac{\partial P}{\partial q}\sin\beta p\right)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = [\alpha q^{\alpha-1}(\beta q^{\alpha}\cos\beta p)\cos\beta p + \beta q^{\alpha}(\alpha q^{\alpha-1} sin\,\beta p)\, sin\,\beta p]\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = [\alpha\beta q^{2\alpha-1}\cos^2\beta p + \alpha\beta q^{2\alpha-1} sin^2\beta p]\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \alpha\beta q^{2\alpha-1}[\cos^2\beta p + sin^2\beta p]\frac{\partial H}{\partial P} = \alpha\beta q^{2\alpha-1}\frac{\partial H}{\partial P}$$

The transformation is canonical if

$$\alpha\beta q^{2\alpha-1} = 1$$

$$\Rightarrow q^{2\alpha-1} = 1$$

Examples

$$\Rightarrow 2\alpha - 1 = 0$$

$$\Rightarrow \alpha = {}^{1}/_{2} \qquad \& \qquad \Rightarrow \beta = 2$$

Therefore, $Q = q^{1/2}\cos 2p$ and $P = q^{1/2}\sin 2p$

Now the generating function $\frac{\partial F_3}{\partial p} = -q$

$$\Rightarrow F_3 = -qp$$

Since

$$Q^2 + P^2 = q(\cos^2 2p + sin^2 2p) = q \qquad \& \qquad \frac{P}{Q} = \tan 2p \Rightarrow p = \frac{1}{2}tan^{-1}\frac{P}{Q}$$

Therefore the generating function

$$F_3 = -qp = -\frac{1}{2}(Q^2 + P^2)tan^{-1}\frac{P}{Q}$$

Examples (Book: Classical mechanics by Takwal)

Show that the transformation is canonical

$$Q = \log\left(1 + q^{1/2}\cos p\right) \quad \text{and} \quad P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p$$

Also Show that the generating function $F_3 = -(e^Q - 1)^2 \tan p$

Solution: The transformations will be canonical if it satisfies the following conditions.

$$\dot{Q} = \frac{\partial H}{\partial P} \quad \& \quad \dot{P} = -\frac{\partial H}{\partial Q}$$

Now if we take derivative of $Q = q^{\alpha}\cos \beta p$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\frac{d}{dt}\left(1 + q^{1/2}\cos p\right)$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\dot{q}\cos p - q^{1/2}\dot{p}\sin p\right)$$

Examples (Book: Classical mechanics by Takwal)

And $\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$ & $\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$

Therefore, $\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\cos p + q^{1/2}\frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\sin p\right)$

$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial P}{\partial p}\cos p + q^{1/2}\frac{\partial P}{\partial q}\sin p\right)\frac{\partial H}{\partial P}$

$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial}{\partial p}\left(2q^{1/2}\sin p + q\sin 2p\right)\cos p + q^{1/2}\frac{\partial}{\partial q}\left(2q^{1/2}\sin p + q\sin 2p\right)\sin p\right)\frac{\partial H}{\partial P}$

$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\left(2q^{1/2}\cos p + 2q\cos 2p\right)\cos p + q^{1/2}\left(2\frac{1}{2}q^{-1/2}\sin p + \sin 2p\right)\sin p\right)\frac{\partial H}{\partial P}$

$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\cos^2 p + q^{1/2}\cos 2p\cos p + \sin^2 p + q^{1/2}\sin 2p\sin p\right)\frac{\partial H}{\partial P}$

Examples (Book: Classical mechanics by Takwal)

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\left(\cos^2 p + \sin^2 p\right) + q^{1/2}\left(\cos 2p \cos p + \sin 2p \sin p\right)\right)\frac{\partial H}{\partial P}$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(1 + q^{1/2}\cos(2p - p)\right)\frac{\partial H}{\partial P}$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(1 + q^{1/2}\cos p\right)\frac{\partial H}{\partial P}$$

$$\dot{Q} = \frac{\partial H}{\partial P}$$

Similarly

$$\dot{P} = 2\left(1 + q^{1/2}\cos p\right)\left(\frac{1}{2}q^{-1/2}\dot{q}\sin p + q^{1/2}\dot{p}\cos p\right) + 2q^{1/2}\sin p\left(\frac{1}{2}q^{-1/2}\dot{q}\cos p - q^{1/2}\dot{p}\sin p\right)$$

$$\Rightarrow \dot{P} = \left(q^{-1/2}\sin p + \sin 2p\right)\dot{q} + 2\left(q^{1/2}\cos^2 p + q\cos 2p\right)\dot{p}$$

And $\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{dp}$ & $\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial Q}\frac{\partial Q}{dq}$

Examples (Book: Classical mechanics by Takwal)

Putting and solving we get $\dot{P} = -\frac{\partial H}{\partial Q}$ (Home work)

Hence the transformation is canonical.

Now $\frac{\partial F_3}{\partial Q} = \frac{\partial}{\partial Q}\left[-(e^Q - 1)^2 \tan p\right]$

$$\frac{\partial F_3}{\partial Q} = \left[-2(e^Q - 1)e^Q \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(e^{\log\left(1+q^{1/2}\cos p\right)} - 1\right)e^{\log\left(1+q^{1/2}\cos p\right)} \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(1 + q^{1/2}\cos p - 1\right)\left(1 + q^{1/2}\cos p\right) \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(q^{1/2}\cos p\right)\left(1 + q^{1/2}\cos p\right) \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = -\left[2\left(1 + q^{1/2}\cos p\right) q^{1/2}\sin p\right] = -P$$