CLASSICAL MECHANICS
CHAPTER 1 LECTURE 1

Dr. Akhlaq Hussain

CLASSICAL MECHANICS

Mechanics is a scheme for studying the motion of all bodies.

OR

Mechanics is the study of time evolution of the state of system.

Mechanics are of three types

Classical Mechanics large object, slow speed

Quantum Mechanics very small particles,

Relativistic Mechanics very small particles, very high speed

CLASSICAL MECHANICS

Before the development of Quantum mechanics and relativity, Classical Mechanics was known as Mechanics only.

Mechanics were having three categories, statics, dynamics, kinematics.

Statics Bodies at rest.

Dynamics bodies in motion under a known force (force is essential part to be known)

Kinematics Bodies in motion without concerning the force acting on body.

Dynamical system A system of particles is called dynamical system.

Configuration The set of positions of all the particles is known as configuration of dynamical system.

Coordinate system

Coordinate system in the way to describe space quantitively or any event in space.

Space is three dimensional, therefore point in space is labeled by three coordinates.

Cartesian coordinate system

Origin is a fixed point and three Axis which are mutually perpendicular are labeled as (x, y, z)

The position vector of point $P = (x, y, z)$ is

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors and

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

CARTESIAN COORDINATE SYSTEM

For 1-dimenssional

$\vec{x}$ is the position vector and its time derivative gives velocity.

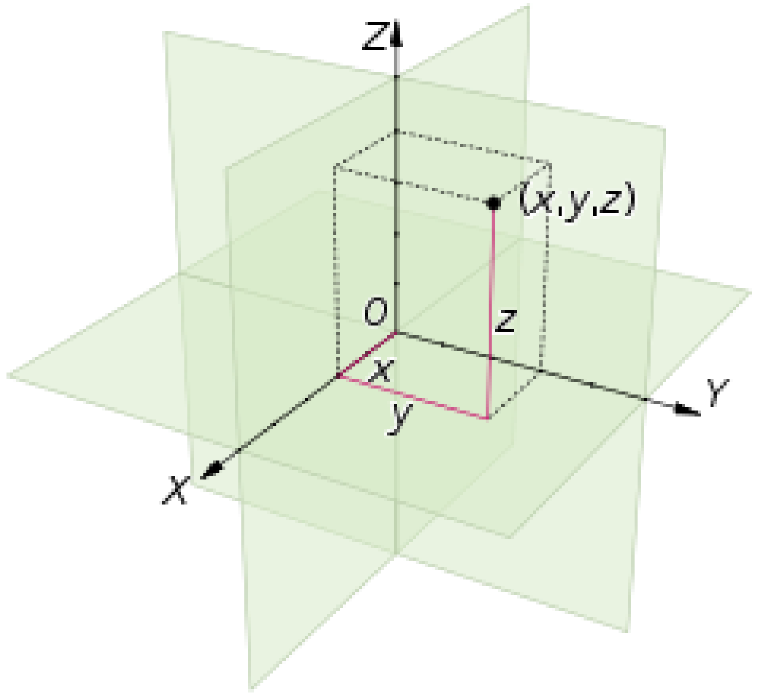
$$\vec{v} = \frac{d\vec{x}}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$\vec{v} = \dot{\vec{x}}$ Dot is used instead of writing $\frac{d}{dt}$ again and again

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\dot{\vec{x}}}{dt}$$

Now $\vec{v} = \dot{\vec{x}}$

$$a = \ddot{\vec{x}}$$



For three dimensions

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

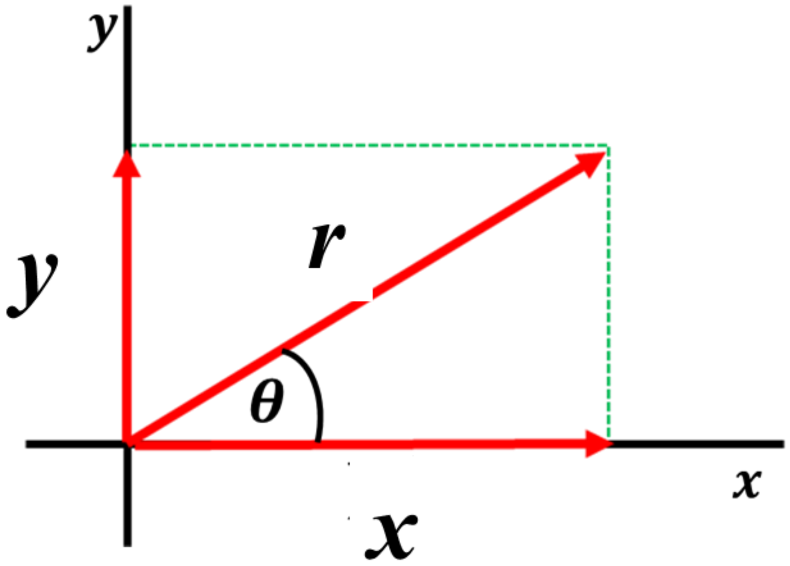
$$\vec{v} = \dot{r} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\,\hat{k}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

PLANE POLAR COORDINATES

$x = r\cos\theta$

$y = r\sin\theta$



Squaring and adding $\quad x^2 + y^2 = r^2(cos^2\theta + sin^2\theta)$

$\Rightarrow r = \sqrt{x^2 + y^2}$

And dividing y by x

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\Rightarrow \tan\theta = \frac{y}{x}$

$\theta = tan^{-1}\left(\frac{y}{x}\right)$

$\vec{r} = x\hat{\imath} + y\hat{\imath}$

$\vec{r} = r\hat{r} = r(\hat{\imath}\cos\theta + j\sin\theta)$

$r = 0 \to \infty$

$x = 0 \to \infty$

$y = 0 \to \infty$

PLANE POLAR COORDINATES

$\Rightarrow \hat{r} = \hat{\imath}\cos\theta + \hat{\jmath} sin\theta$

$\hat{\theta}$ is a unit vector perpendicular to $\hat{r}$ in direction of increasing θ

$$\hat{\theta} = \hat{\imath}\, cos(90+\theta) + \hat{\jmath}\, sin(90+\theta)$$

$$\hat{\theta} = -\hat{\imath}\, sin\,\theta + \hat{\jmath}\cos\theta$$

Now $$\frac{d\hat{r}}{d\theta} = \frac{d}{d\theta}(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta)$$

$$= -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}$$

Similarly, $$\Rightarrow \frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

And $$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{dt}\frac{d\theta}{d\theta} = \frac{d\theta}{dt}\frac{d\hat{r}}{d\theta} = \dot{\theta}\hat{\theta}, \qquad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

PLANE POLAR COORDINATES

Now $\quad \vec{r} = r\hat{r}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r})$$

$$= \dot{r}\hat{r} + r\frac{d}{dt}\hat{r} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Now for acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right)$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

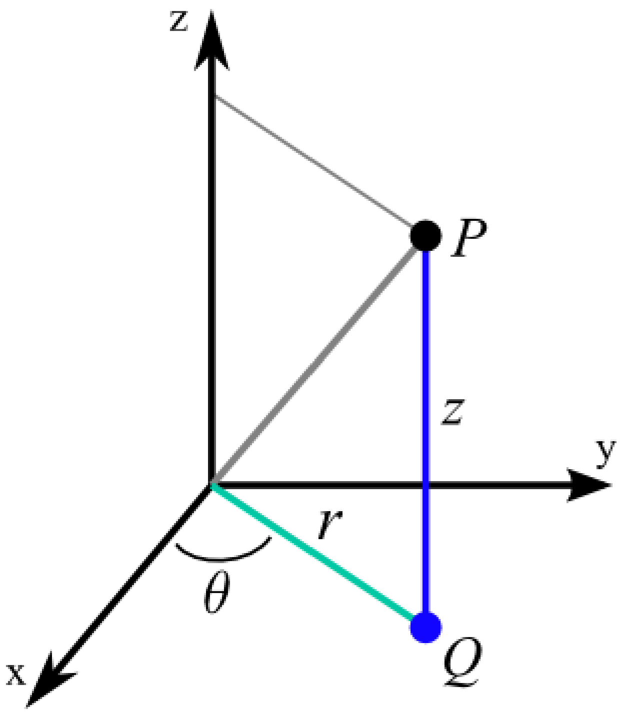
$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

Radial part $\quad$ Angular part

Transformation equation $x = r\cos\varphi$ $y = r\sin\varphi$ $z = z$

$$\left.\begin{array}{l} r=\sqrt{x^2+y^2} \\ \varphi=tan^{-1}\left(\frac{y}{x}\right) \end{array}\right\}\text{Inverse transformation equation}$$



$$\bar{R} = \overline{OP}$$

$$= r\hat{r} + z\hat{k}$$

Velocity $\vec{v} = \frac{d\bar{R}}{dt} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k}$

Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k}\right)$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi} + \dot{r}\dot{\varphi}\hat{\varphi} + r\ddot{\varphi}\hat{\varphi} + r\dot{\varphi}(-\dot{\varphi}\hat{r}) + \ddot{z}\hat{k}$$

$$= [\ddot{r} - r\dot{\varphi}^2]\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{k}$$

$$= [\ddot{r} - r\dot{\varphi}^2]\hat{r} + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi})\right]\hat{\varphi} + \ddot{z}\hat{k}$$

SPHERICAL POLAR COORDINATES

$$\vec{r} = \overrightarrow{OP}$$
$$= \overrightarrow{OA} + \overrightarrow{AP}$$

Or $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$x = r\,sin\,\theta\,cos\,\varphi$$

$$y = r\,sin\,\theta\,sin\,\varphi$$

$$z = r\,cos\,\theta$$

$$\vec{r} = r(\sin\theta\cos\varphi\hat{\imath} + \sin\theta\sin\varphi\hat{\jmath} + \cos\theta\,\hat{k}) = r\hat{r}$$

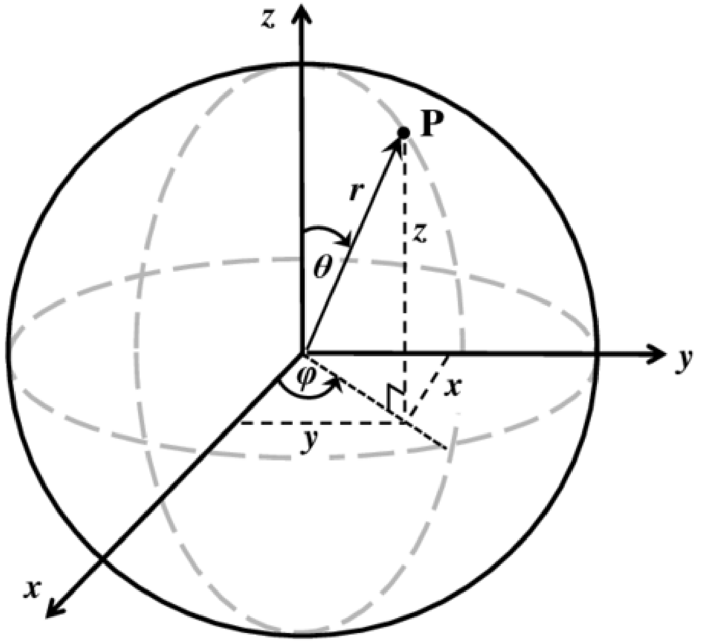
$$\vec{r} = \sin\theta\cos\varphi\hat{\imath} + \sin\theta\sin\varphi\hat{\jmath} + \cos\theta\,\hat{k}$$

For simplification we take

$$x\hat{\imath} + y\hat{\jmath} = \vec{\rho}$$

And $\vec{\rho} = r\sin\theta\hat{\rho}$

$$\hat{\rho} = \cos\varphi\hat{\imath} + \sin\varphi\hat{\jmath}$$



$x = r\,sin\,\theta\,cos\,\varphi$

$y = r\,sin\,\theta\,sin\,\varphi$

$z = r\,cos\,\theta$

$r = \sqrt{x^2 + y^2 + z^2}$

$\theta = cos^{-1}(z/r)$

$\varphi = tan^{-1}(y/x)$

Limits

$$0 < r < \infty$$

$$0 < \theta < \pi$$

$$0 < \varphi < 2\pi$$

SPHERICAL POLAR COORDINATES

And $$\hat{\varphi} = \hat{\imath}\cos\left(\varphi + \frac{\pi}{2}\right) + \hat{\jmath}\sin\left(\varphi + \frac{\pi}{2}\right)$$

$$= -\sin\varphi\hat{\imath} + \cos\varphi\hat{\jmath}$$

And $$\frac{d\hat{\rho}}{dt} = \dot{\varphi}\hat{\varphi}$$

$$\frac{d\hat{r}}{dt} = \dot{\theta}(-\sin\theta\hat{k} + \cos\theta\hat{\rho}) + \sin\theta\frac{d\hat{\rho}}{dt}$$

And $$\frac{d\hat{\varphi}}{dt} = -\dot{\varphi}\hat{\rho}$$

Now $$\vec{r} = r\left(\cos\theta\hat{k} + \sin\theta\hat{\rho}\right)$$

$$\hat{r} = \cos\theta\hat{k} + \sin\theta\hat{\rho} \quad \Rightarrow \quad \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}$$

And $$\hat{\theta} = -\sin\theta\hat{k} + \cos\theta\hat{\rho} \quad \Rightarrow \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} + \dot{\varphi}\cos\theta\,\hat{\varphi}$$

Note $$\hat{r}\sin\theta = \sin\theta\cos\theta\,\hat{k} + sin^2\theta\hat{\rho}$$

$$\hat{\theta}\cos\theta = -\sin\theta\cos\theta\,k + cos^2\theta\hat{\rho}$$

$$\Rightarrow \hat{\rho} = \left(\sin\theta\hat{r} + \cos\theta\hat{\theta}\right)$$

SPHERICAL POLAR COORDINATES

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \vec{v} = \frac{dr\hat{r}}{dt} = \dot{r}\hat{r} + r\left(\dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}\right)$$

$$\vec{a} = \frac{d\hat{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\left(\dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}\right) + \dot{r}\left(\dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}\right) + r\left(\ddot{\theta}\hat{\theta} + \dot{\theta}\frac{d\hat{\theta}}{dt} + \ddot{\varphi}\sin\theta\,\hat{\varphi} + \dot{\varphi}\dot{\theta}\cos\theta\,\hat{\varphi} + \dot{\varphi}\sin\theta\frac{d}{dt}\hat{\varphi}\right)$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\varphi}\sin\theta\hat{\varphi} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\varphi}\sin\theta\,\hat{\varphi} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\left(-\dot{\theta}\hat{r} + \dot{\varphi}\cos\theta\,\hat{\varphi}\right) + r\ddot{\varphi}\sin\hat{\theta}\,\hat{\varphi} + r\dot{\varphi}\dot{\theta}\cos\theta\,\hat{\varphi} - r\dot{\varphi}^2\sin\theta\hat{\rho}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} + \left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right)\hat{\varphi} - r\dot{\varphi}^2\sin\theta\left(\sin\theta\hat{r} - \cos\theta\hat{\theta}\right)$$

$$= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2\right)\hat{\theta} + \left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta\right)\hat{\varphi}$$

If φ is constant in above equation is reduced to

$$\breve{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$$

REVIEW OF MECHANICS

Dr. Akhlaq Hussain

NEWTON'S LAWS

Background

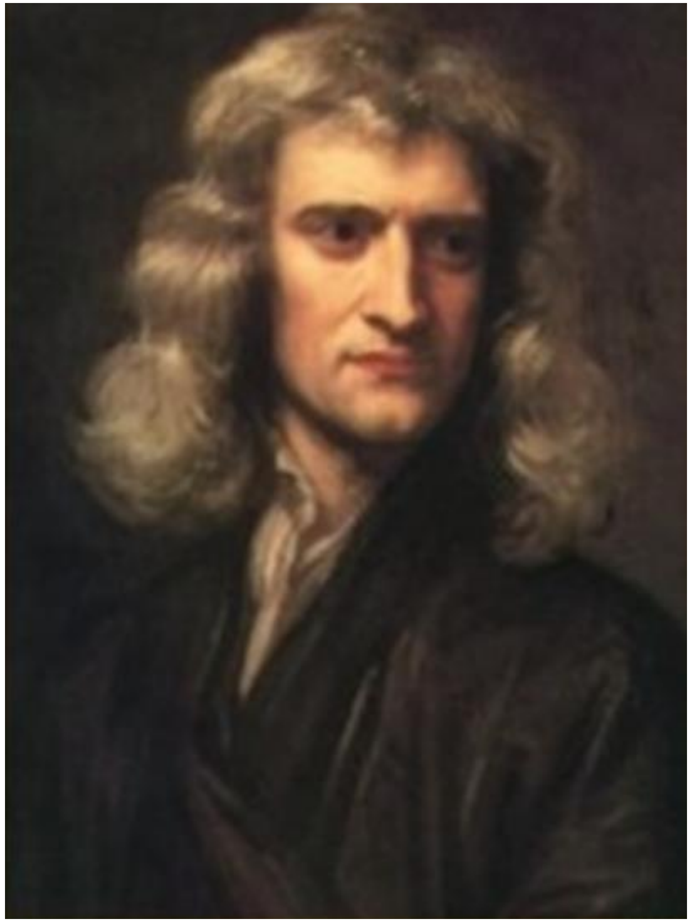
Sir Isaac Newtons(1643-1727)

English mathematician, Physicist, astronomer, theologian and an author.

Famous for his laws of motion and discovery of the law of gravity.

Book

Philosophiae Naturalis Principia Mathematica (Mathematical Principles of Natural Philosophy) published in 1687, laid the foundations of Classical Mechanics.



Force

An agent which produce motion or change in motion, in an object and is directly related to the change in state of motion

$$\boldsymbol{F} = m\boldsymbol{a} = m\ddot{\boldsymbol{r}}$$

Momentum

The product of mass and velocity of the body. ($\boldsymbol{P} = m\boldsymbol{v} = m\dot{\boldsymbol{r}}$)

Mass

Inertial mass $m = \frac{F}{a}$

Gravitational Masses $m = \frac{F_g}{g}$

NEWTON'S LAWS

Newton's First Law

Objects at rest tends to stay at rest and an object in motion tends to stay in motion until and unless acted upon an unbalanced force.

Newton's Second Law

The force acting on a body produces acceleration in a body which is directly proportional to the force and inversely proportional to the mass of the body.

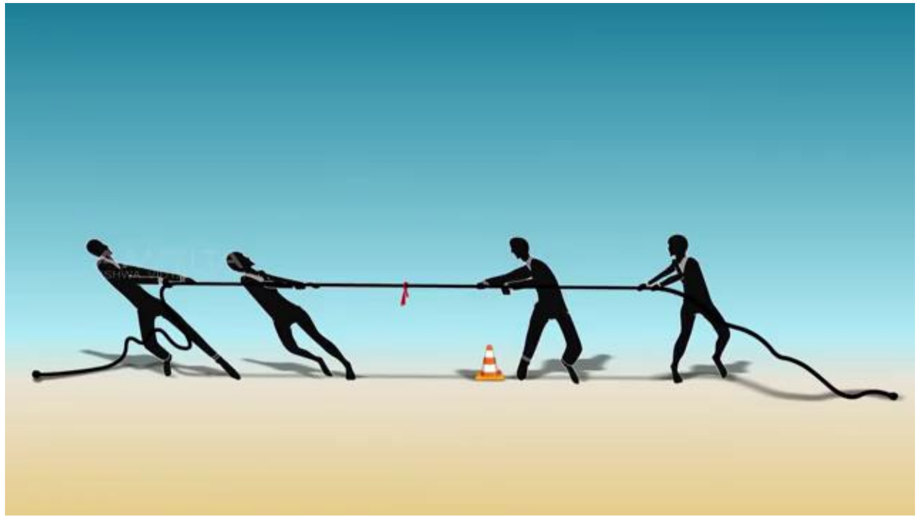
Newton's Third Law

For every action force there is an equal and opposite reaction force.

NEWTON'S LAWS

Newton's First Law

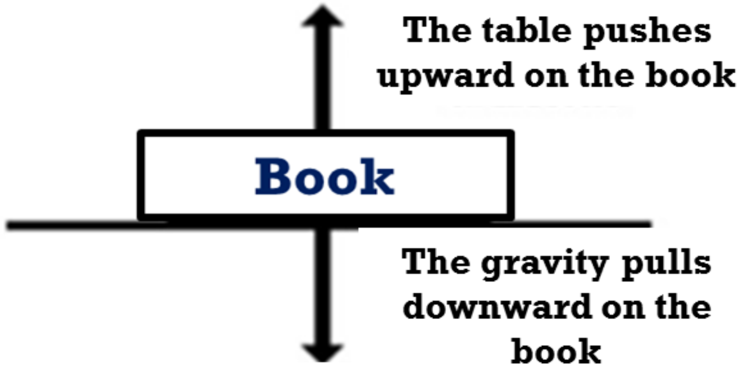
Objects at rest tends to stay at rest and an object in motion tends to stay in motion until and unless acted upon an unbalanced force.

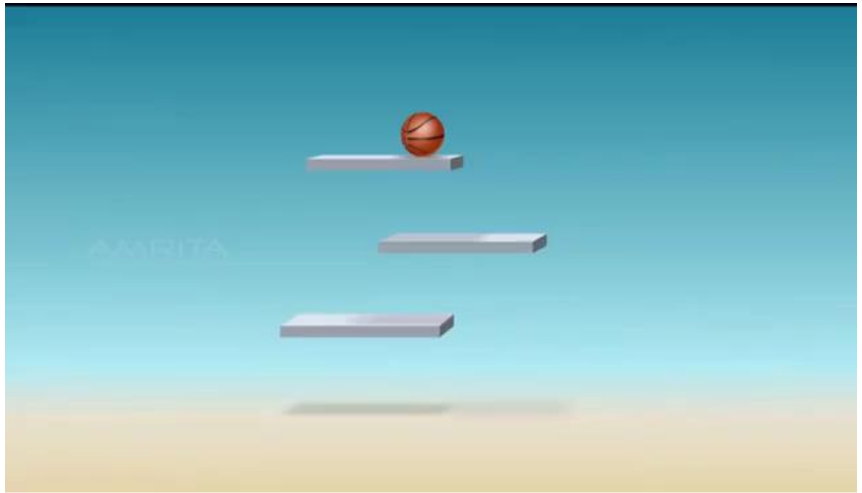


For example
A Soccer ball is sitting at rest. It takes an unbalance force of a kick to change its motion

What is meant by Unbalance force?

- If the forces on an object are equal and opposite, they are said to be balanced,
- The object experience no change in the motion.
- If they are not equal and opposite, then the forces are unbalanced and the motion of the object changes.





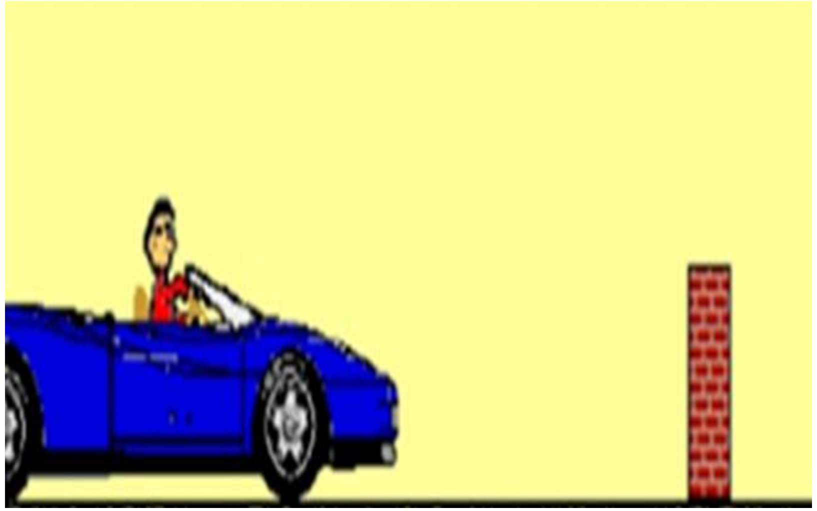
NEWTON'S LAWS

Newton's First Law is also called the Law of Inertia

Inertia: the tendency of an object to resist changes in its state of motion
The firs Law states that all objects have inertia. The more mass an object has the more inertial it has (and the harder it is to change its motion.)

Don't let this to be you.

If a car moving with 80km/hr is stopped suddenly or hit something, the driver will keep moving with 80km/hr and can hit the front glass.



Therefore, wear a seat belt because of inertia your body will resist to stop along with car.

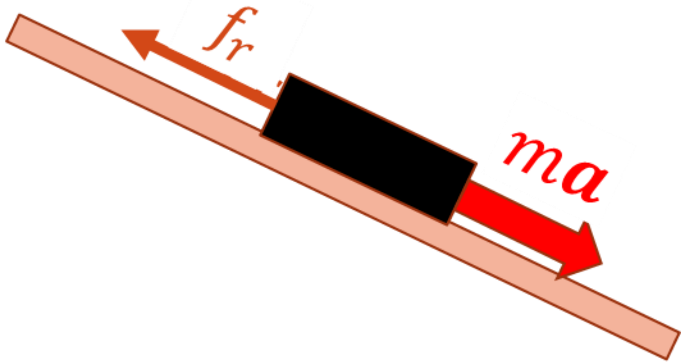
NEWTON'S LAWS

Newton's First Law is also called the Law of Inertia

[If the object in motion tends to stay in motion, then why don't moving object keep moving forever?]

Things don't keep moving forever because there's almost always an unbalanced force acting upon it.

A book sliding across a table slows down and stop because of the force of friction.



The soccer ball kicked by the player will eventually fall down because of air friction and gravity

Inertia in Space

In outer space, away from gravity and any source of friction, a rocket ship launched with a certain speed and direction would keep going in that same direction and at that same speed forever

Newton's Second Law

The force acting on a body produce acceleration in a body which is directly proportional to the force and inversely proportional to the mass of the body.

$$\ddot{\boldsymbol{r}} \propto F$$

$$\ddot{\boldsymbol{r}} \propto {}^{1}/_{m}$$

$$\ddot{\boldsymbol{r}} = {}^{F}/_{m}$$

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = m\boldsymbol{a}$$

Newton's Second Law

$F = ma$ basically means that the force of an object comes from its mass and its acceleration.

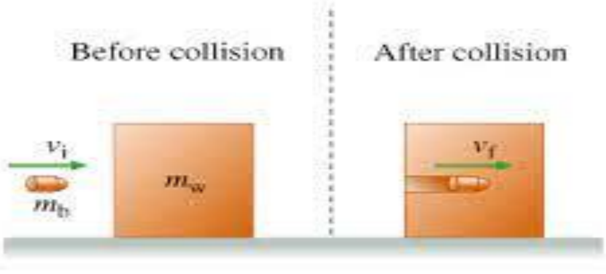
If we double the mass, double will be the force and vice verse.

Doubling the mass and the acceleration quadruples the force

$$(2m)(2\ddot{r}) = 4m\ddot{r} = 4F$$

Something very massive can produce very large force even for very small acceleration,

& something very small can produce very large force if the acceleration is very large. Such as bullet can produce very large force if stopped very quickly in a body.



NEWTON'S LAWS

Newton's Second Law

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = m\frac{d\dot{\boldsymbol{r}}}{dt}$$

$$\boldsymbol{F} = m\left(\frac{\dot{\boldsymbol{r}}_f - \dot{\boldsymbol{r}}_i}{t_2 - t_1}\right) = \frac{m\dot{\boldsymbol{r}}_f - m\dot{\boldsymbol{r}}_i}{\Delta t}$$

$$\Rightarrow \boldsymbol{F} = \frac{\boldsymbol{P}_f - \boldsymbol{P}_i}{\Delta t} = \frac{\Delta \boldsymbol{P}}{\Delta t}$$

$$\Rightarrow \boldsymbol{F} = \lim_{\Delta t} \frac{\Delta \boldsymbol{P}}{\Delta t}$$

$$\Rightarrow \boldsymbol{F} = m\ddot{\boldsymbol{r}} = \frac{d\boldsymbol{P}}{dt}$$

Newton's Third Law

For every action force there is an equal
and opposite reaction force.

- Gravity is pulling you down in your seat.
- seat is pushing you up against with equal force.
-
- As a result, you are not moving.
- There is a balanced force acting on you.
- Gravity pulling down, your seat pushing up

NEWTON'S LAWS

Newton's Third Law.

Newton's Third Law

A tank of mass 80,000kg is dropped from a height of 2200 meters as a result of explosion in a tank carrier aircraft. If the soldiers in the tank, try to minimize the impact with ground by diving into a lake situated at 800 meters from the dropping point.

The tank is capable to fire the cannon shells of mass 10 kg with a velocity of 400 meters per seconds. If we neglect the air resistance:

a) Find the number of shells fired by the soldiers to achieve this distance.
b) Find the number of shells to minimize the impact with ground.
c) Find the minim shells fires per second to achieve this.
d) If the tank can fire only 3.5 shells per second, determine the impact velocity of the tank during collision with lake water?

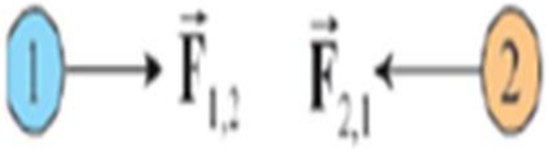
NEWTON'S LAWS

Newton's Third Law.

Consider two bodies engaged in a mutual interaction.

$\vec{F}_{12}$ be the force on body 1 due to the interaction of body 2

$\vec{F}_{21}$ be the force on body 2 due to interaction with body 1



$$\vec{F}_{1,2} = -\vec{F}_{2,1}$$

Gravitational Force $\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{r}_{12}$ $\hat{r}_{12} = -\hat{r}_{21}$

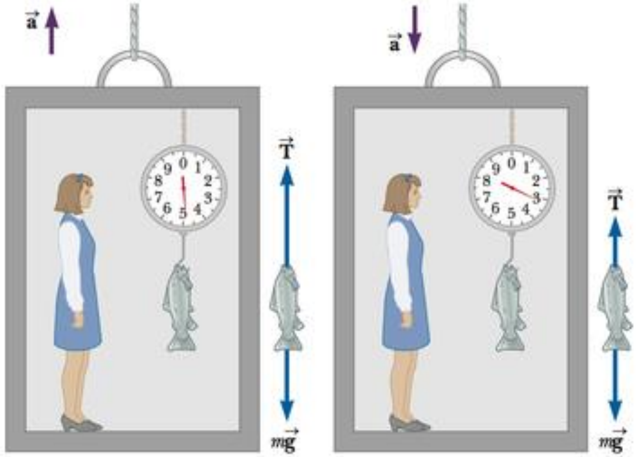
Coulomb Force $\vec{F}_{12} = k\frac{q_1 q_2}{r^2}\hat{r}_{12}$ $\vec{F}_{12} = -\vec{F}_{21}$

Inertial and Non-inertial Frame of references

Frame of reference (observing system)

A frame of reference in which newton's laws are valid are Inertial frame of reference

A frame of reference at rest or moving with constant velocity are inertial frame of reference.



Non-inertial frame of reference are accelerated frame of reference.

Newtons laws are not valid in the non-inertial frame of reference.

Non-inertial frame of reference always gives a false force.

NEWTON'S LAW OF UNIVERSAL GRAVITATION

The Universal Law of Gravitation (Newton's law of gravity):

Famous apple story

NEWTON'S LAW OF UNIVERSAL GRAVITATION

The Universal Law of Gravitation (Newton's law of gravity):

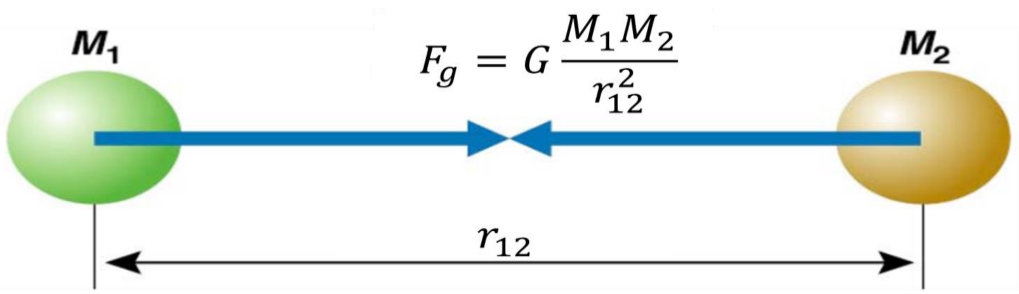
1. Every mass attracts every other mass.
2. Attraction is *directly* proportional to the product of their masses.
3. Attraction is *inversely* proportional to the *square* of the distance b/w their centres.

$$F_g \propto M_1 M_2$$

$$F_g \propto \frac{1}{r_{12}^2}$$

$$F_g \propto \frac{M_1 M_2}{r_{12}^2}$$

$$F_g = G\frac{M_1 M_2}{r_{12}^2}$$



$$G = 6.673 \times 10^{-11} Nm^2kg^{-2}$$

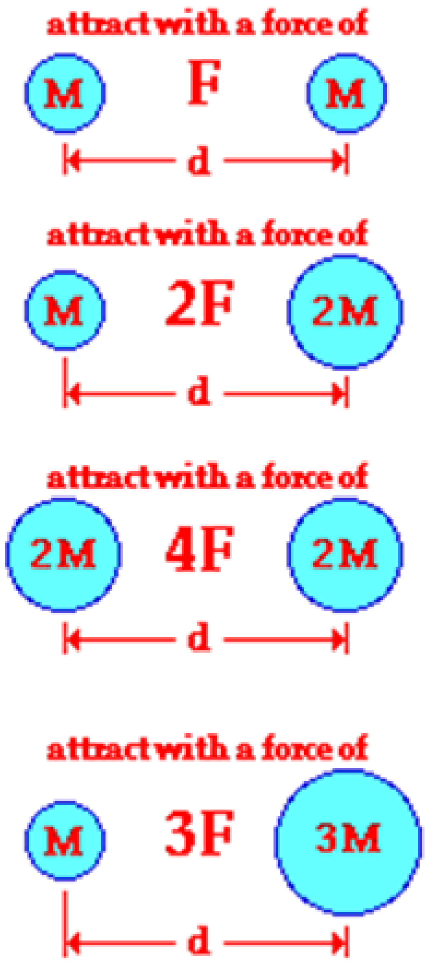
G is universal constant → same value for any pair

NEWTON'S LAW OF UNIVERSAL GRAVITATION

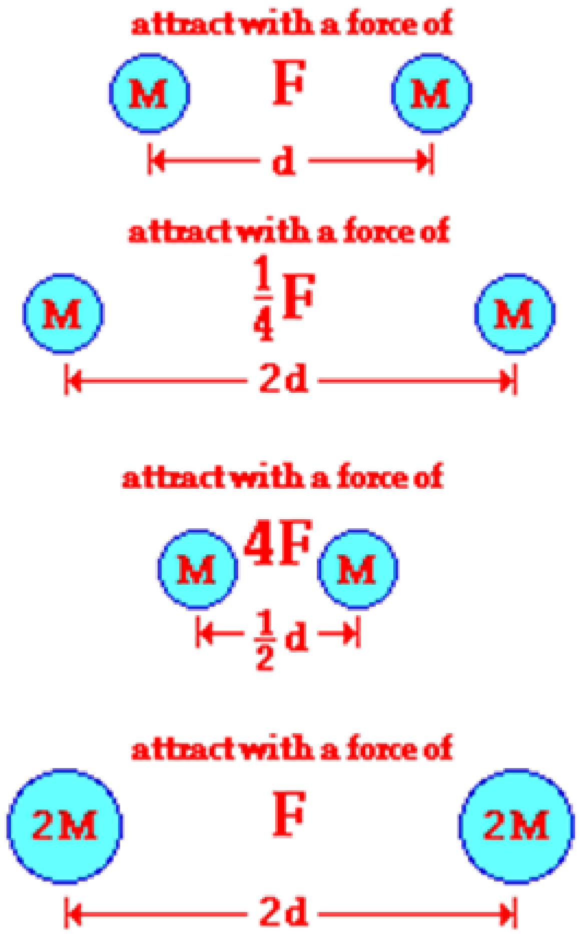
1. Gravitational force is always attractive.
2. Form an action reaction pair
3. Central Force
4. Independent of the presence of other bodies and properties of the intervening medium
5. Weakest force
6. Provides necessary centripetal force for moon & satellites to revolve around the earth
7. Formation of tides in ocean
8. Make bodies fall from height

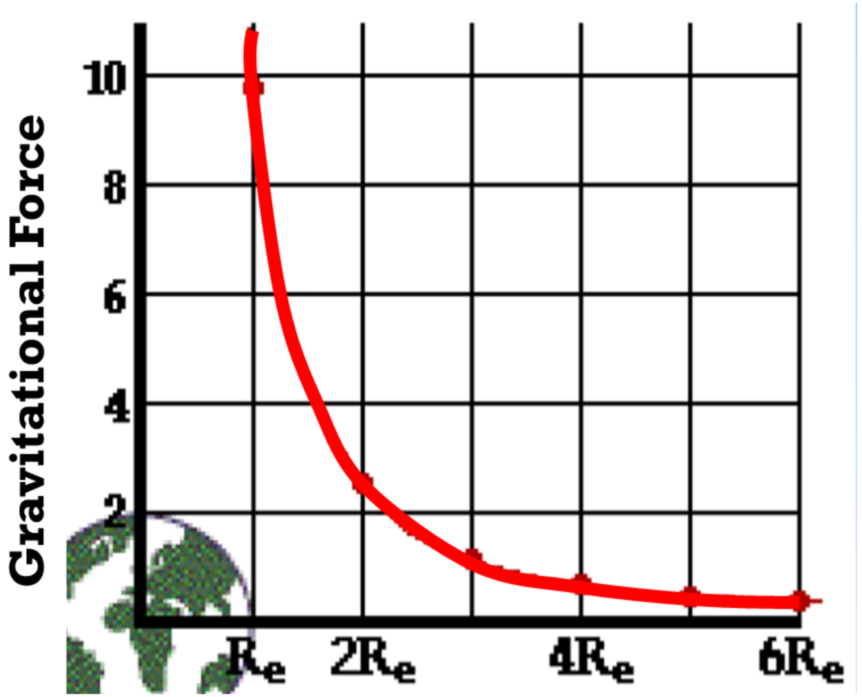
NEWTON'S LAW OF UNIVERSAL GRAVITATION

Effect of Mass on F_{grav}



Effect of Distance on F_{grav}





Distance from the center of Earth

Gravitational Mass and Inertial Mass

Gravitational force $F_g = G\frac{M_e m}{r^2} = mg$ $\therefore g = G\frac{M_e m}{r^2}$

Gravitational mass $m = \frac{F_g}{g}$

So if we put ab object in a gravitation field, it responds with its gravitational mass

Gravitational Mass measures the gravitational force exerted and experience by an object

Newton's 2nd Law $F = ma$

Inertial mass $m = \frac{F}{a}$

If we push an object, it responds to that push with its inertial mass.

Inertial mass measure's an objects resistance to the acceleration by a force

Gravitational Mass and Inertial Mass

Question

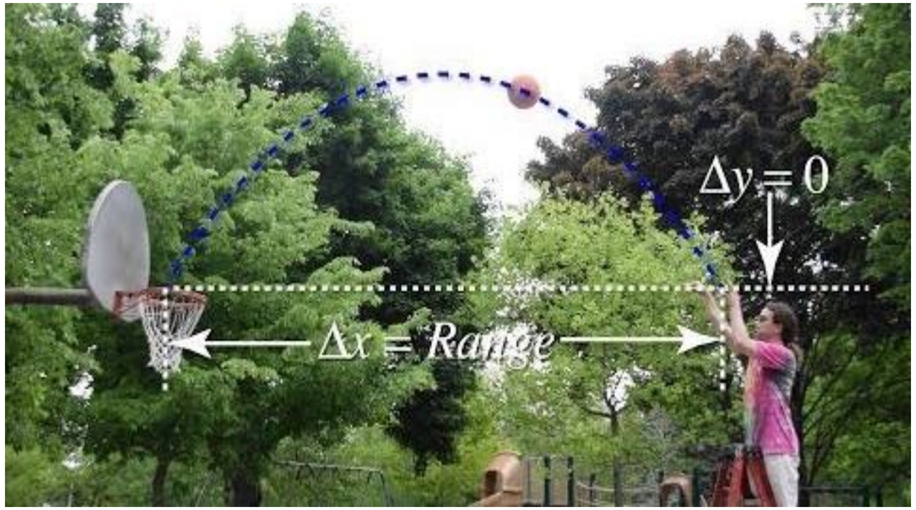
Are these masses different??

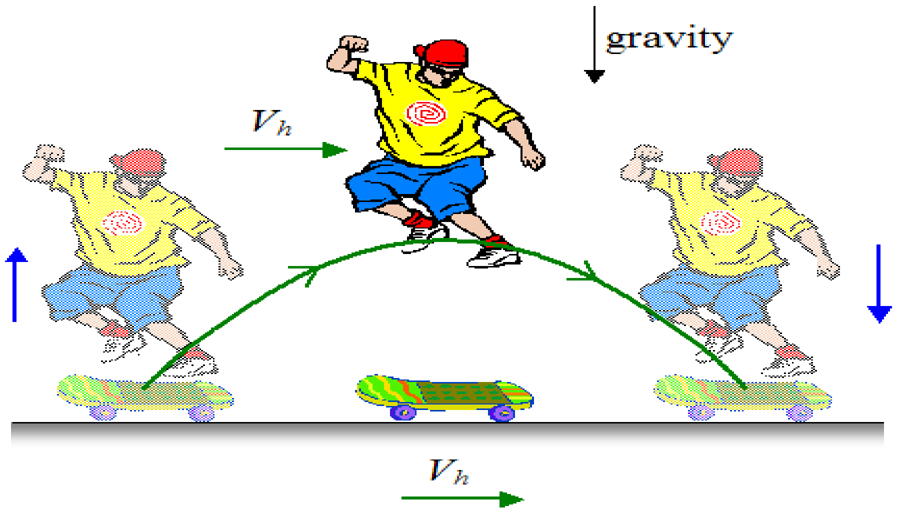
No these masses give same value in many cases

PROJECTILE MOTION

PROJECTILE MOTION

Projectile motion is a form of motion experienced by an object or particle (a projectile) that is projected near the Earth's surface and moves along a curved path under the action of gravity.





Assumptions of Projectile Motion

1) The free-fall acceleration is constant over the range of motion

2) It is directed downward

3) It is reasonable as long as the range is small compared to the radius of the Earth

4) The effect of air friction is negligible

5) With these assumptions, an object in projectile motion will follow a parabolic path ! This path is called the trajectory

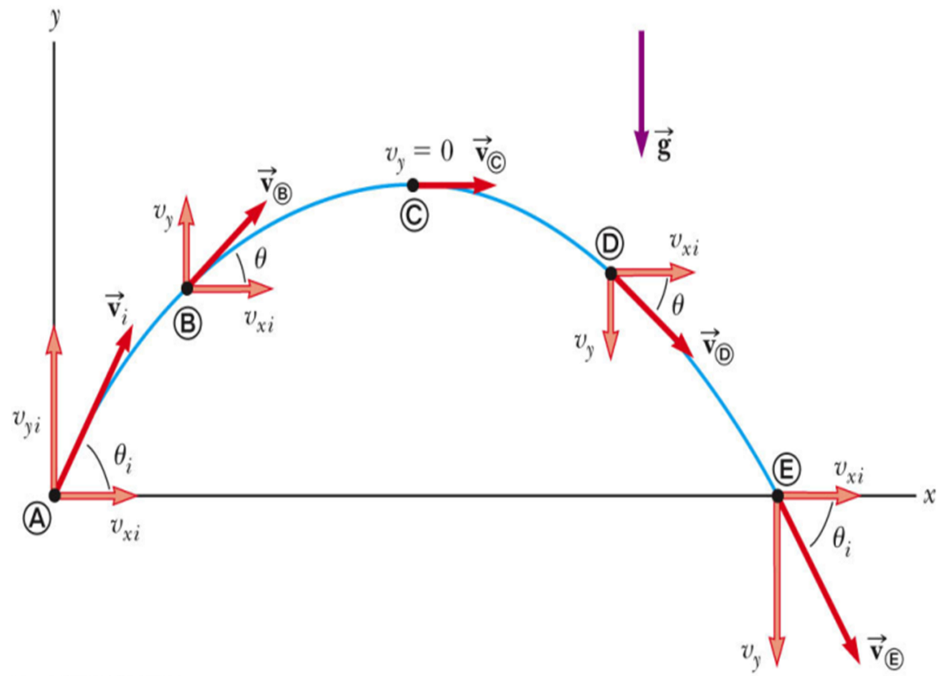
Equation of Motion we will be using:

$$v_f = v_i + at$$

$$S = v_i t + {}^1\!/_2\, at^2$$

$$2aS = v_f^2 - v_i^2$$

- Let a projectile is projected at an initial angle "θ" with initial velocity "v_i" at instant "t=0"



At t=0

$$x = 0$$
$$v_{xi} = v_i \cos\theta$$
$$a_{xi} = 0$$

$$y = 0$$
$$v_{yi} = v_i \sin\theta$$
$$a_{yi} = -g$$

At t=t

$$x(t) = v_{xi}t = v_i \cos\theta\, t$$
$$v_{x(t)} = v_i \cos\theta = v_{xi}$$
$$a_{x(t)} = 0$$

$$y(t) = v_{yi}t + \frac{1}{2}a_{yi}t^2 = v_i \sin\theta t - \frac{1}{2}gt^2$$
$$v_{y(t)} = v_{yi} + a_{yi}t = v_i \sin\theta - gt$$
$$a_{y(t)} = -g$$

PROJECTILE MOTION

At t=t

$$x = v_{xi}t = v_i \cos\theta \, t \quad \Rightarrow \quad t = \frac{x}{v_i \cos\theta}$$

Putting in equation of y at time t

$$y = v_i \sin\theta t - \frac{1}{2}gt^2$$

$$y = v_i \sin\theta \left(\frac{x}{v_i \cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_i \cos\theta}\right)^2$$

$$y = x\,tan\,\theta - \frac{1}{2}\left(\frac{g}{v_i^2 \cos^2\theta}\right)x^2$$

$$y = x\,tan\,\theta - \frac{1}{2}\left(\frac{g}{v_i^2 \cos^2\theta}\right)x^2$$

$$y = ax - bx^2$$

This is equation of trajectory and it represents the parabolic path

PROJECTILE MOTION

When projectile hit the ground at time T (time of flight), y=0

$$y = v_i \sin\theta \, T - \frac{1}{2} g T^2 = 0$$

$$T = \frac{2 v_i \sin\theta}{g}$$

Putting the value of "T" the x=R range of the projectile

$$x = v_i \cos\theta \, T$$

$$R = x = v_i \cos\theta \left(\frac{2 v_i \sin\theta}{g} \right)$$

$$R = \frac{2 {v_i}^2 \cos\theta \sin\theta}{g} = \frac{{v_i}^2 \sin 2\theta}{g}$$

PROJECTILE MOTION

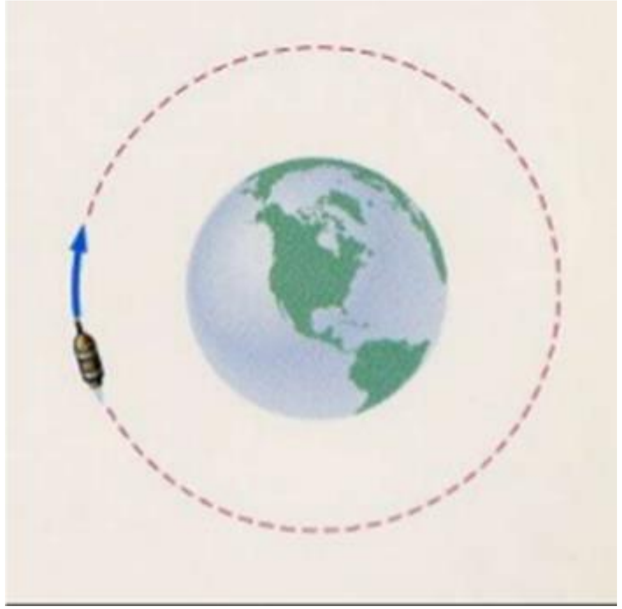
Maximum height "h" is achieved at time $t = \frac{T}{2} = \frac{v_i \sin\theta}{g}$

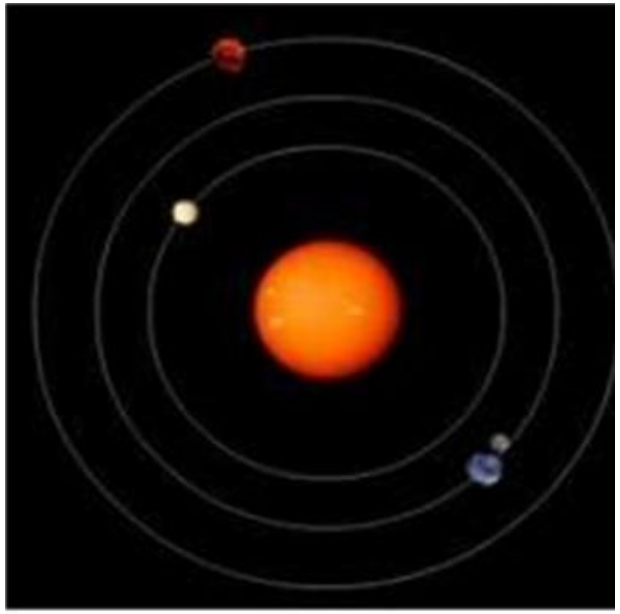
$$\mathrm{h} = \mathrm{y} = v_i \sin\theta \left(\frac{v_i \sin\theta}{g}\right) - \frac{1}{2} g \left(\frac{v_i \sin\theta}{g}\right)^2$$

$$\mathrm{h} = \frac{{v_i}^2 \sin^2\theta}{g} - \frac{1}{2}\frac{{v_i}^2 \sin^2\theta}{g}$$

$$\mathrm{h} = \frac{{v_i}^2 \sin^2\theta}{2g}$$

CIRCULAR MOTION





Circular Motion

Angular Displacement

Angular Speed

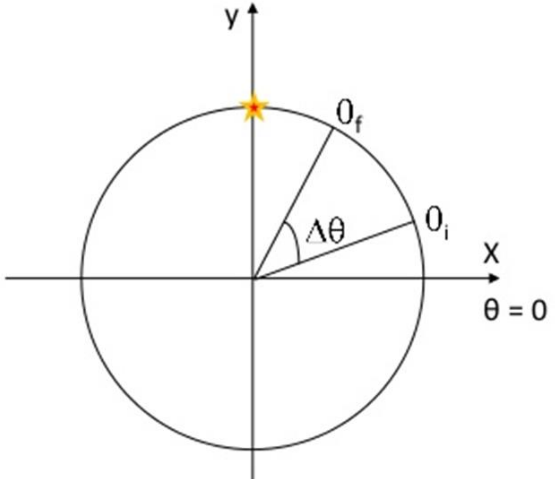
Angular Acceleration

Circular Motion

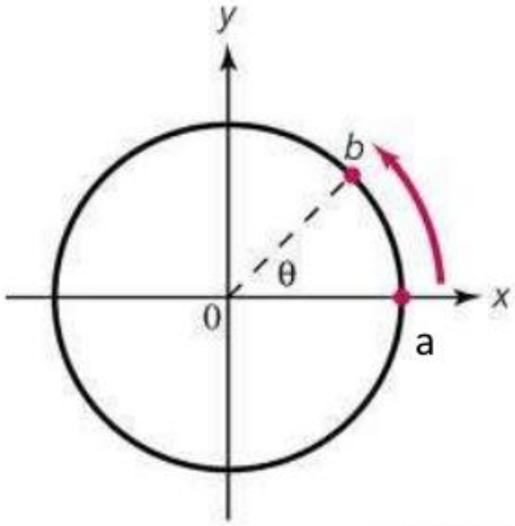
If an object is moving along a circular path

Uniform Circular Motion:

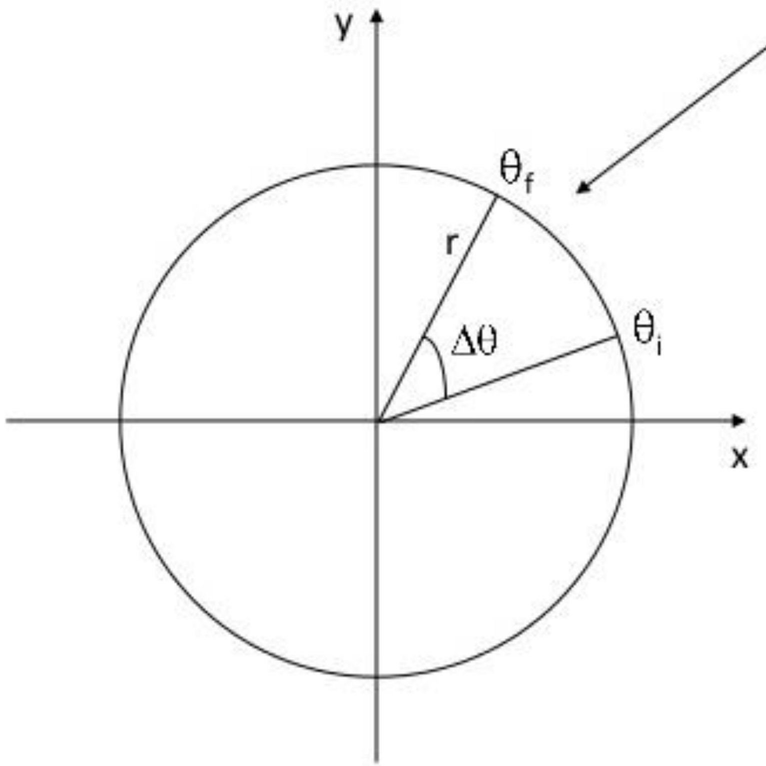
A body moving with uniform speed along a circular path



- The angle through which the radius vector representing the position of a particle rotates is called angular displacement
- The change in position of the particle in a circular path with respect to its centre is called angular displacement.
- The angular displacement of a body with respect to a reference line is denoted as θ.



arclength = s = $r\Delta\theta$



$$\Delta\theta = \frac{s}{r}$$

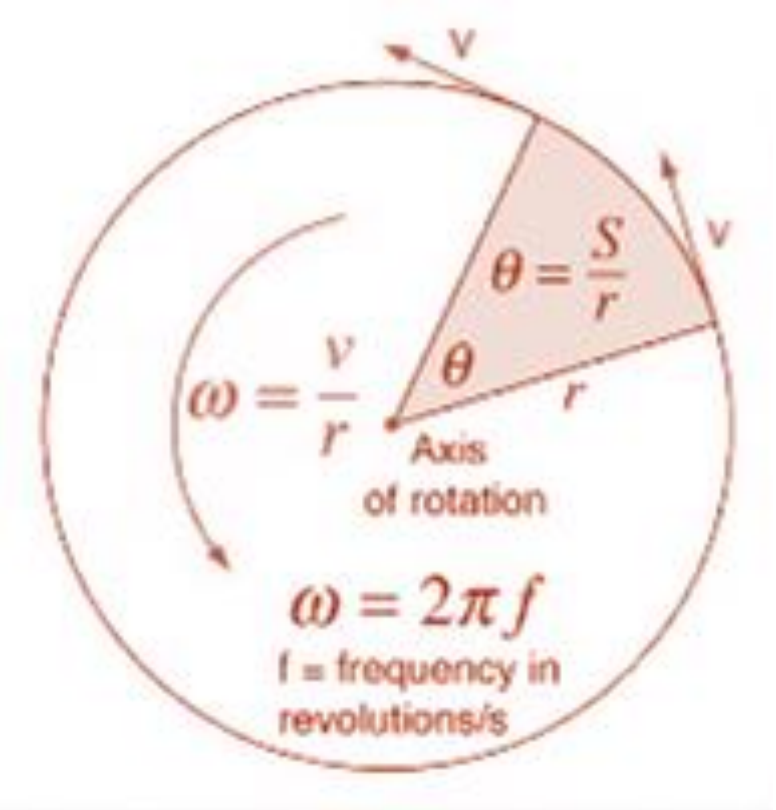
$\Delta\theta$ is a ratio of two lengths; it is a dimensionless ratio! This is a radian measure of angle

If we go all the way round s $=2\pi r$ and $\Delta\theta = 2\pi$

Angular displacement is measured in degree radian or cycles

CIRCULAR MOTION

Angular Velocity, also called rotational velocity is the measure of rotation per unit time.



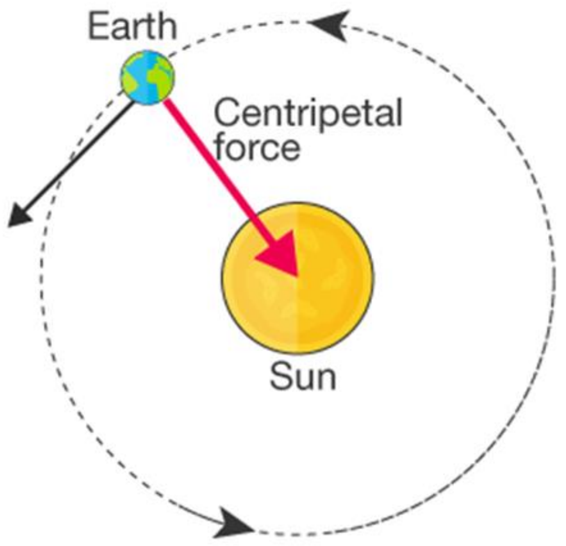
CENTRIPETAL FORCE

- The force acting on earth due to sun is the gravitational force.

- What if the force disappears?

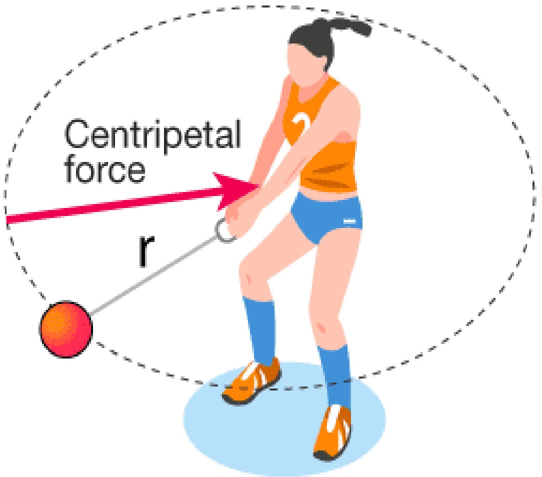
- The earth will no longer circulate around the sun

- This sun gravitational force is providing the necessary force to keep the earth in its orbit



- A lady is trying to swing the ball in circular orbit.

- The tension in the string provide the force necessary for orbital motion



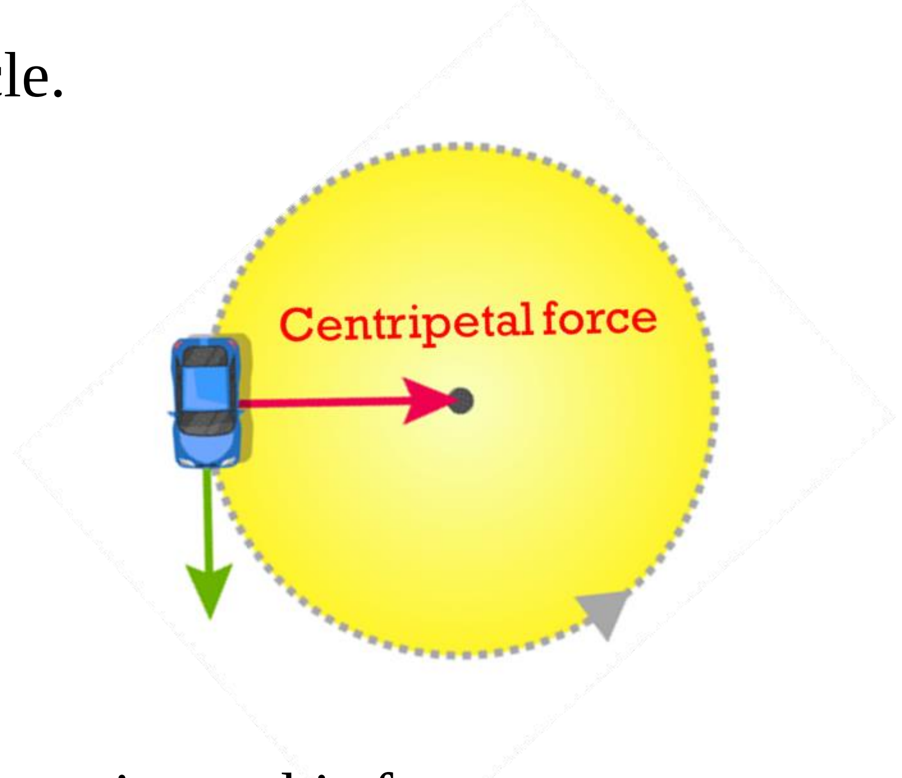
CENTRIPETAL FORCE

Centripetal force : the force necessary to keep a body in circular motion,

- The force is directed towards the center of the circle.
- The magnitude of the force is given as

$$F_c = \frac{mv^2}{r}$$

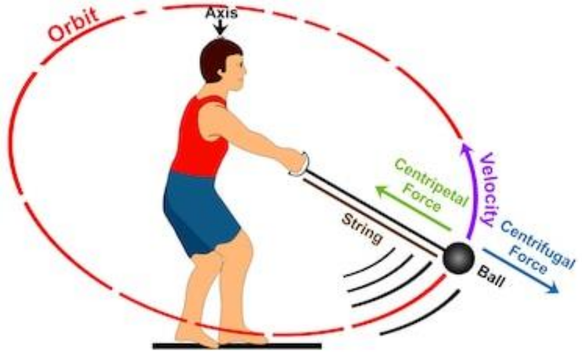
$$\vec{F_c} = -\frac{mv^2}{r}\hat{r}$$



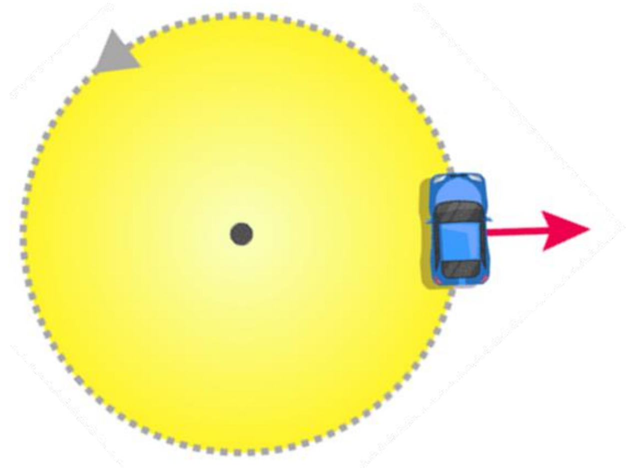
- All the objects moving along a circular path will experience this force.
- This is a kind of reaction force for a force keeping an object in circular motion

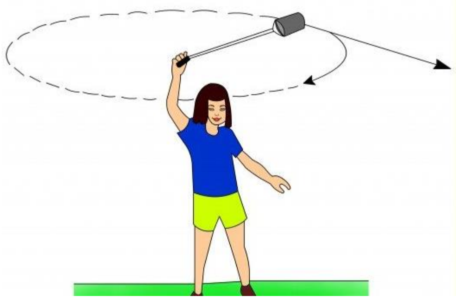
CENTRIFUGAL FORCE

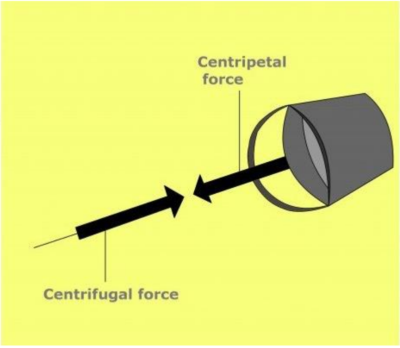
Centrifugal force : outwards inertial force experienced by the objects in circular motion,



- If a car take a sudden turn the passengers will experience an outer word force due to inertia.
- The inertia will tend to retain the motion in original direction.
- The force is directed away from the center.
- Its does not arise from interaction of particles therefore its not a true force.







- What acceleration will result when a 12N net force applied to a 3 kg object? & a 6kg object.

- A net force of 16 N causes a mass to accelerate at a rate of 5 m/s^2. Determine the mass.

- How much force is need to accelerate a 66 kg Skier at 1m/s^2?

Chapter 1

Lecture 3

Classical Mechanics

Dr. Akhlaq Hussain

Simple Pendulum

Simple pendulum is a mass attached with string which can vibrate in vertical plane.

$$F = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Length "l" is constant and $\dot{r} = \dot{l} = 0 \Rightarrow F = -m(l\dot{\theta}^2\hat{r} + l\ddot{\theta}\hat{\theta})$

For r dependent part: $-ml\dot{\theta}^2 = mg\cos\theta - T$

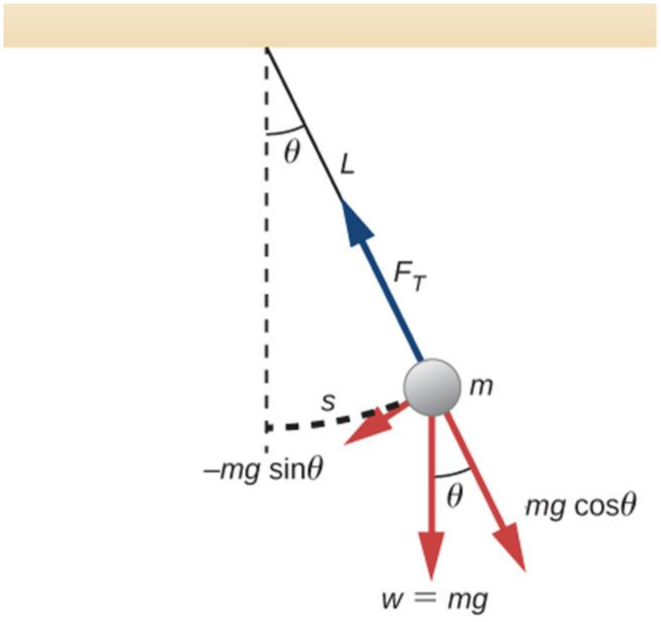
For θ dependent part: $ml\ddot{\theta} = -mg\sin\theta$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\ddot{\theta} + \omega^2\sin\theta = 0 \qquad \text{for } \omega = \frac{g}{l}$$

$$\ddot{\theta} + \omega^2\theta = 0 \qquad \text{if } \theta \text{ is very small}$$



Simple Pendulum

$$\ddot{\theta} + \omega^2\theta = 0 \qquad \text{if } \theta \text{ is very small}$$

This is a second order differential equation and solution of this equation is as following

$$\theta = A\cos\omega t + B\sin\omega t$$

Where A and B are constants

The time period of the pendulum is

$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

Simple Pendulum

Simple Pendulum Case II (solution)

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Integrating the above equation; $\int \ddot{\theta} d\theta = \int \frac{d\dot{\theta}}{dt} d\theta = -\frac{g}{l}\int \sin\theta \, d\theta$

If the pendulum move from maximum value "α" to "θ" and the velocity changes from zero to "$\dot{\theta}$"

$$\int_o^{\dot{\theta}} \frac{d\theta}{dt} d\dot{\theta} = \int_o^{\dot{\theta}} \dot{\theta} d\dot{\theta} = -\frac{g}{l}\int_\alpha^{\theta} \sin\theta \, d\theta$$

$$\frac{\dot{\theta}^2}{2} = \frac{g}{l}(\cos\theta - \cos\alpha)$$

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\alpha) \quad \text{---------------------------(1)}$$

We define a new term; $\sin\varphi = \frac{\sin{}^{\theta}/_{2}}{\sin{}^{\alpha}/_{2}} \Rightarrow \sin{}^{\theta}/_{2} = \sin\varphi \sin{}^{\alpha}/_{2}$

Simple Pendulum

We define a new term; $sin\,\varphi = \frac{sin^{\theta}/_2}{sin^{\alpha}/_2} \Rightarrow sin\,{}^{\theta}/_2 = sin\,\varphi\, sin\,{}^{\alpha}/_2$

Differentiating above equation ⇒ $ {}^1/_2\, cos\,{}^{\theta}/_2\,\dot{\theta} = sin\,{}^{\alpha}/_2 \cos\varphi\,\dot{\varphi}$---------- (2)

Not: squaring equation (2)

$$sin^2\,{}^{\alpha}/_2 \cos^2\varphi\,\dot{\varphi}^2 = \frac{1}{4}\cos^2{}^{\theta}/_2\,\dot{\theta}^2$$

and putting value of $\dot{\theta}$ from equation 1

$$sin^2\,{}^{\alpha}/_2 \cos^2\varphi\,\dot{\varphi}^2 = \frac{1}{4}\cos^2{}^{\theta}/_2\left[\frac{2g}{l}(\cos\theta - \cos\alpha)\right]$$

$$= \frac{g}{2l}\cos^2{}^{\theta}/_2\,(\cos\theta - \cos\alpha)$$

$$= \frac{g}{2l}\cos^2{}^{\theta}/_2\,(1 - 2sin^2\,{}^{\theta}/_2 - 1 + 2\,sin^2\,{}^{\alpha}/_2)$$

$$sin^2\,{}^{\alpha}/_2 \cos^2\varphi\,\dot{\varphi}^2 = \frac{g}{l}\cos^2{}^{\theta}/_2\, sin^2\,{}^{\alpha}/_2\left(1 - \frac{sin^2\theta/_2}{sin^2\alpha/_2}\right)$$

Simple Pendulum

$$\cos^2\varphi\,\dot{\varphi}^2 = \frac{g}{l}\cos^2{}^{\theta}/_2\,(1 - sin^2\varphi)$$

$$\cos^2\varphi\,\dot{\varphi}^2 = \frac{g}{l}\cos^2{}^{\theta}/_2\,cos^2\varphi$$

$$\dot{\varphi}^2 = \frac{g}{l}\cos^2{}^{\theta}/_2$$

$$\dot{\varphi}^2 = \frac{g}{l}(1\ - sin^2{}^{\theta}/_2) = \frac{g}{l}(1\ - sin^2{}^{\alpha}/_2\,sin^2\varphi)$$

$$\dot{\varphi} = \sqrt{\frac{g}{l}}(1\ - sin^2{}^{\alpha}/_2\,sin^2\varphi)^{1/2}$$

$$\frac{d\varphi}{(1 - sin^2{}^{\alpha}/_2\,sin^2\varphi)^{1/2}} = \sqrt{\frac{g}{l}}\,dt$$

To obtain the time period we integrate both sides.

$$\sqrt{\frac{g}{l}}\int_0^t dt = \int_0^{\pi/2}(1\ - k^2\,sin^2\varphi)^{-1/2}\,d\varphi$$

Simple Pendulum

$$t = \sqrt{\frac{l}{g}} \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-1/2} \, d\varphi$$

since the time period for complete cycle (2π) is τ. Therefore for $\frac{\pi}{2}$ the time must be $\frac{1}{4}\tau$

$$\tau = 4t = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-1/2} \, d\varphi$$

$$\tau = 4t = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left[1 + \frac{1}{2} k^2 \sin^2 \varphi + \left(\frac{1\times3}{2\times4}\right) k^4 \sin^4 \varphi + \cdots \right] d\varphi$$

And $\int_0^{\pi/2} d\varphi = \pi/2$

$$\int_0^{\pi/2} \sin^2 \varphi \, d\varphi = \frac{\pi}{2} \cdot \frac{1}{2}$$

$$\int_0^{\pi/2} \sin^4 \varphi \, d\varphi = \frac{\pi}{2} \cdot \frac{3}{4} \cdot \frac{1}{2}$$

and so on…

Simple Pendulum

$$\tau = 4\sqrt{\frac{l}{g}}\frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\times3}{2\times4}\right)^2 k^4 + \cdots + \frac{(2n-1)!!}{(2n)!!}k^{2n}\right]$$

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\times3}{2\times4}\right)^2 k^4 + \cdots + \frac{(2n-1)!!}{(2n)!!}k^{2n}\right]$$

If α is very small $\Rightarrow sin\,{}^{\alpha}/_{2}$ will be very small therefore k^2 will be very small….

Ignoring all higher power terms of k

$$\tau_o = 2\pi\sqrt{\frac{l}{g}}$$

For second approximation

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2\right] = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 sin^2\left(\frac{\alpha}{2}\right)\right]$$

Simple Pendulum

$$\tau = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \left(\frac{\alpha}{2}\right)^2\right]$$

$\therefore$ for $\alpha \ll \Rightarrow sin\left(\frac{\alpha}{2}\right) \approx \frac{\alpha}{2}$

$$\tau = \tau_o\left[1 + \frac{\alpha^2}{16}\right] = \tau_o + \frac{\alpha^2}{16}\tau_o$$

$$\tau - \tau_o = \frac{\alpha^2}{16}\tau_o$$

$$\frac{\tau - \tau_o}{\tau_o} = \frac{\nabla\tau}{\tau_o} = \frac{\alpha^2}{16}$$

If $\alpha = 30^o$ $\rightarrow$ $\frac{\nabla\tau}{\tau_o} = \frac{1}{16}\left(\frac{30\times\pi}{180}\right)^2 = 0.017\ or\ 1.7\%$

And using $k = sin^2\left(\frac{\alpha}{2}\right)$ the difference is only 0.0167 or 1.67%

Therefore, the good approximation for time period is $\tau = 2\pi\sqrt{\frac{l}{g}}$

Drag Force

The motion of falling objects is usually described with constant acceleration.

This is only approximately true.

The motion in air will only be described by the drag force in air resistance or friction force.

which increase with speed of the body and directed in direction opposite to the motion and force of gravity.

In case of motion through any fluid a resistive force will always be there.

At small speeds fluid resistance is linear

$\boldsymbol{F}_{resistive} = -b\boldsymbol{v}$ (Low speed) Viscus drag

Where b has unit {Ns/m} or {kg/sec}

Drag Force

For larger speeds fluid resistance is better described by quadratic drag

$\boldsymbol{F}_{resistive} = -c\boldsymbol{v}^2$ (higher speed) Quadratic drag

C unit is {Ns^2/m^2} or {kg/m}

$$\boldsymbol{F}_{resistive} = -b\boldsymbol{v} - c\boldsymbol{v}^2$$

If $\boldsymbol{v}$ is very small the second term dropped and if $\boldsymbol{v}$ is very large, first term will drop.

The speeds of everyday objects are large enough that the effects of fluid resistance are entirely due to quadratic drag.

Sky driver are slow down by quadratic drag and Hollywood heroes evading gunfire by swimming under water are also due to 2nd term of the equation.

Viscus Drag (Linear Dependence on speed)

It appear one of two situation; with driving force usually gravity and on its own.

Jello case (A bullet entering a Jello medium)

Let an object has initial velocity “v_o” and mass “m” encounter a resistive force

$$\boldsymbol{F}_{\boldsymbol{resistiv}} = \boldsymbol{F}_{net} = -b\boldsymbol{v}$$

$$\Rightarrow m\boldsymbol{a} = -b\boldsymbol{v}$$

$$\Rightarrow \frac{d\boldsymbol{v}}{dt} = -\frac{b}{m}\boldsymbol{v}$$

$$\Rightarrow \frac{d\boldsymbol{v}}{dt} = -\frac{b}{m}\boldsymbol{v}$$

$$\Rightarrow \frac{d\boldsymbol{v}}{\boldsymbol{v}} = -\frac{b}{m}dt$$

$$\Rightarrow \int_{v_o}^{v} \frac{d\boldsymbol{v}}{\boldsymbol{v}} = -\frac{b}{m}\int_{o}^{t} dt$$

Viscus Drag (Linear Dependence on speed)

$$\Rightarrow \ln\left(\frac{v}{v_o}\right) = -\frac{b}{m}t$$

$$\Rightarrow \boldsymbol{v} = v_o e^{-bt/m}$$

Now the distance traveled x

$$x = \int_0^t v_o e^{-bt/m} dt$$

$$\Rightarrow x = v_o \int_0^t e^{-bt/m} dt = v_o \left(\frac{-m}{b}\right) \left| e^{-bt/m} \right|_o^t$$

$$\Rightarrow x = v_o \left(\frac{-m}{b}\right)\left(e^{-bt/m} - e^0\right)$$

$$\Rightarrow x = \left(\frac{mv_o}{b}\right)\left(1 - e^{-bt/m}\right)$$

Viscus Drag (Linear Dependence on speed)

If t→∞, the exponential part $e^{-bt/m}$→0

$$\Rightarrow x = \left(\frac{mv_o}{b}\right)$$

For very small time t ≪, the term $e^{-bt/m}$ can be expended as

$$e^{-bt/m} = 1 - \frac{bt}{m} + H.P.T$$

and neglecting higher power term

$$\Rightarrow x = \left(\frac{mv_o}{b}\right)\left(1 - 1 + \frac{bt}{m}\right)$$

$$\Rightarrow x = \left(\frac{mv_o}{b}\right)\left(\frac{bt}{m}\right) = v_o t$$

Similar to $x = v_o t + \frac{1}{2} a t^2$

Free Fall

$$\boldsymbol{F}_{net} = mg - b\boldsymbol{v}$$

$$\Rightarrow m\boldsymbol{a} = mg - b\boldsymbol{v}$$

$$\Rightarrow \frac{d\boldsymbol{v}}{dt} = g - \frac{b}{m}\boldsymbol{v}$$

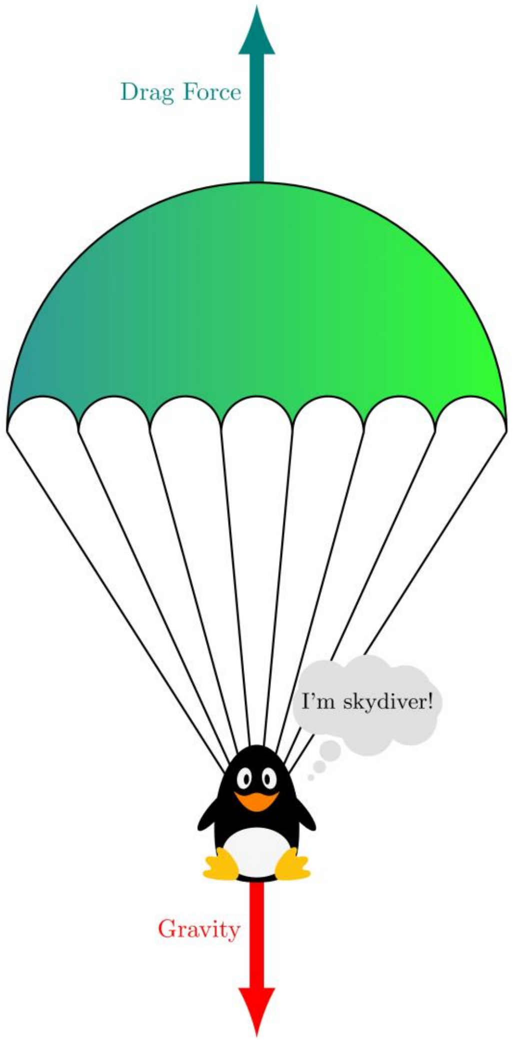
$$\Rightarrow \frac{d\boldsymbol{v}}{dt} = -\frac{b}{m}\left(\boldsymbol{v} - \frac{mg}{b}\right)$$

$$\Rightarrow \frac{d\boldsymbol{v}}{\left(\boldsymbol{v} - \frac{mg}{b}\right)} = -\frac{b}{m}dt$$

$$\Rightarrow \int_{v_o}^{v} \frac{d\boldsymbol{v}}{\left(\boldsymbol{v} - \frac{mg}{b}\right)} = -\frac{b}{m}\int_{o}^{t} dt$$

$$\Rightarrow \ln \frac{\left(\boldsymbol{v} - \frac{mg}{b}\right)}{\left(v_o - \frac{mg}{b}\right)} = -\frac{b}{m}t$$

$$\Rightarrow \boldsymbol{v} = \frac{mg}{b} + v_o e^{-bt/m} - \frac{mg}{b}e^{-bt/m}$$



Free Fall

$$\Rightarrow \boldsymbol{v}(t) = \frac{mg}{b} + v_o e^{-bt/m} - \frac{mg}{b} e^{-bt/m}$$

$$\Rightarrow \boldsymbol{v}(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right) + v_o e^{-bt/m}$$

If $t \longrightarrow \infty$ the velocity $\boldsymbol{v}(t) = V_T$

$V_T = \frac{mg}{b}$ $\therefore e^{-bt/m} \rightarrow 0$ for $t \rightarrow \infty$

$t \ll$ $\Rightarrow \boldsymbol{v}(t) = \frac{mg}{b}(1 - 1 + {}^{bt}/_{m} + \cdots) + v_o(1 - {}^{bt}/_{m})$

$\Rightarrow \boldsymbol{v}(t) = gt + v_o - v_o\, {}^{bt}/_{m}$ Ignoring higher power terms

Free Fall

For very small drag force $\quad {}^{bt}/_{m} \Rightarrow 0$

$$\Rightarrow \boldsymbol{v}(t) = gt + v_o - v_o\, {}^{bt}/_{m}$$

$$\Rightarrow \boldsymbol{v}(t) = v_o + gt$$

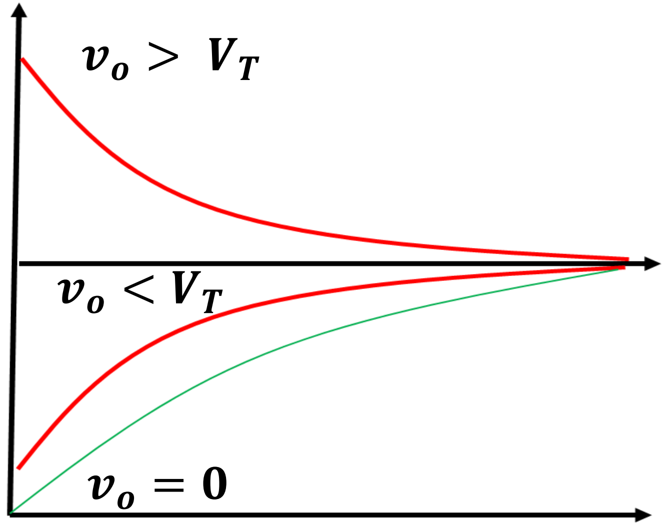
For the solution $\quad \boldsymbol{v}(t) = \frac{mg}{b}\left(1 - e^{-bt/m}\right) + v_o e^{-bt/m}$

Now if the body has initial velocity

$$v_o > V_T$$

$$v_o < V_T$$

$$v_o = 0$$



Free Fall

Now the distance traveled by the body can be calculated as following for body starting from rest, i.e. $v_o = 0$

$$x = \int_0^t v\, dt$$

$$x = \int_0^t \frac{mg}{b}\left(1 - e^{-bt/m}\right)dt = \frac{mgt}{b} - \frac{mg}{b}\left(-\frac{m}{b}\right)\left|e^{-bt/m}\right|_o^t$$

$$x = \frac{mgt}{b} - \frac{m^2 g}{b^2}\left(1 - e^{-bt/m}\right)$$

For very small time i.e., $t \ll$

The term $e^{-bt/m}$ can be expended as $e^{-bt/m} = 1 - \frac{bt}{m} + \frac{b^2t^2}{2m^2} - \cdots$

Therefore, the distance traveled

$$x = \frac{mgt}{b} - \frac{m^2 g}{b^2}\left(1 - e^{-bt/m}\right) = \frac{mgt}{b} - \frac{m^2 g}{b^2}\left(1 - 1 + {bt}/{m} - \frac{b^2t^2}{2m^2} + \cdots\right)$$

$$x = \frac{mgt}{b} - \frac{mgt}{b} + \frac{gt}{2} = \frac{1}{2}gt^2$$