Chapter 3

Lecture 1 & 2

Calculus of Variation

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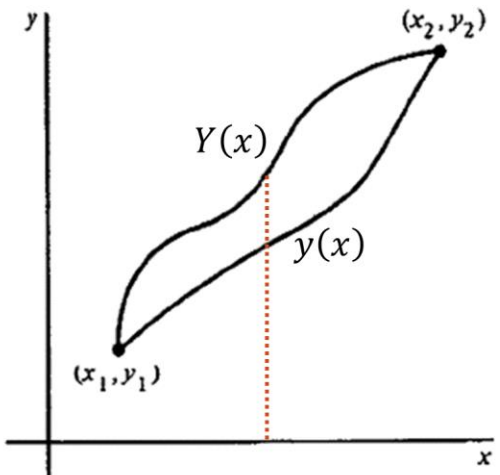
Calculus of Variation: Euler-Lagrange Equation

Let us consider a function defined between two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ by f such that

$$f = f(x, y, y_x) \qquad\qquad \text{where } y_x = \frac{dy}{dx}$$

There must be a path between $P(x_1,y_1)$ and $Q(x_2,y_2)$ along which the value of the integral over function $f(x, y, y_x)$

$$J = \int_{x_1}^{x_2} f(x, y, y_x)dx$$



Has stationary value. (Either maximum of minimum)

Let the value of integral J is stationary along the path y(x),

but the exact path of integration is not known. It can follow any path adjacent to y(x)

Let Y(x) is the adjacent path to y(x) such that $\delta y(x) = Y(x) - y(x)$ is infinitesimal small for all values of x between x_1 and x_2.

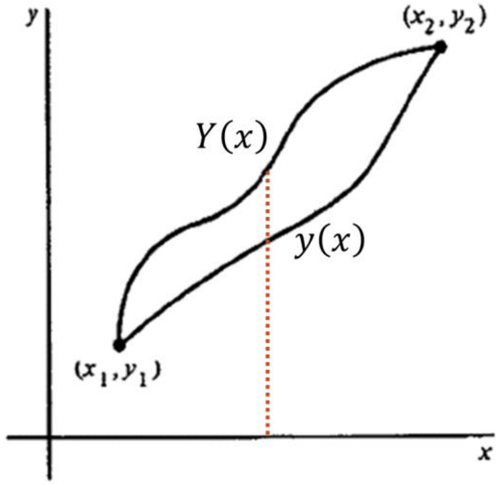
Now $\qquad \delta y(x) = Y(x) - y(x) \qquad \& \qquad \delta f = F(x, y, y_x) - f(x, y, y_x)$

Calculus of Variation: Euler-Lagrange Equation

Now $\delta y(x) = Y(x) - y(x)$ & $\delta f = F(x, y, y_x) - f(x, y, y_x)$

Where $\delta y(x)$ is called variation of y. It represent increase in the quantity "y" from the stationary path to the adjacent path for a given x. this is arbitrary except:

$$\delta y(x_1) = \delta y(x_2) = 0$$



Now $$\delta(y_x) = Y_x - y_x = \delta\left(\frac{dy}{dx}\right) = \frac{dY}{dx} - \frac{dy}{dx}$$

$$\delta(y_x) = \frac{d}{dx}(Y - y)$$

$$\delta(y_x) = \frac{d}{dx}(\delta y)$$

This show that $\frac{d}{dx}$ and δ are commutative

Now $\delta f = F(x, y, y_x) - f(x, y, y_x)$ & $f = f(x, y, y_x)$

Therefore, $\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x$ because $\delta x = 0$

Calculus of Variation: Euler-Lagrange Equation

Therefore $\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x$

Now $\delta J = \delta \int_{x_1}^{x_2}[F(x,y,y_x) - f(x,y,y_x)]dx = \int_{x_1}^{x_2} \delta f(x,y,y_x)dx$

Since J is stationary therefore $\delta J = \int_{x_1}^{x_2} \delta f(x,y,y_x)dx = 0$

$$\delta J = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y_x}\delta y_x\right]dx = 0$$

$$\delta J = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\delta y dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y_x}\delta y_x dx = 0$$

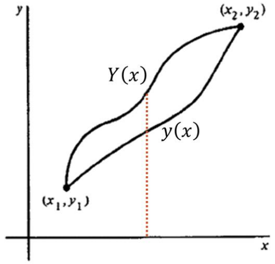
$$\delta J = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\delta y dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y_x}\frac{d}{dx}\delta y dx = 0$$

$$\delta J = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\delta y dx + \underbrace{\left|\frac{\partial f}{\partial y_x}\delta y\right|_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2}\frac{d}{dx}\frac{\partial f}{\partial y_x}\delta y dx = 0 \qquad \delta y(x_2) = \delta y(x_1) = 0$$

$$\delta J = \int_{x_1}^{x_2}\frac{\partial f}{\partial y}\delta y dx - \int_{x_1}^{x_2}\frac{d}{dx}\frac{\partial f}{\partial y_x}\delta y dx = 0$$

Calculus of Variation: Euler-Lagrange Equation

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x}\right] \delta y dx = 0$$



Since $\delta y \neq 0$ through out the path therefore $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

A partial differential eq., know as Euler-Lagrange's eq., associated with variational problem.

Since $f = f(x, y, y_x)$

And $\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y_x}\frac{dy_x}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y_x + \frac{\partial f}{\partial y_x}y_{xx}$

Since $\frac{d}{dx}\left(y_x \frac{\partial f}{\partial y_x}\right) = y_{xx}\frac{\partial f}{\partial y_x} + y_x \frac{d}{dx}\frac{\partial f}{\partial y_x}$

$y_{xx}\frac{\partial f}{\partial y_x} = \frac{d}{dx}\left(y_x \frac{\partial f}{\partial y_x}\right) - y_x \frac{d}{dx}\frac{\partial f}{\partial y_x}$ Putting this in above equation

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y_x + \frac{d}{dx}\left(y_x \frac{\partial f}{\partial y_x}\right) - y_x \frac{d}{dx}\frac{\partial f}{\partial y_x}$$

$$\frac{\partial f}{\partial x} - \frac{df}{dx} + \frac{d}{dx}\left(y_x \frac{\partial f}{\partial y_x}\right) = -y_x\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x}\right] = 0$$

Calculus of Variation: Euler-Lagrange Equation

$$\frac{\partial f}{\partial x} - \frac{df}{dx} + \frac{d}{dx}\left(y_x \frac{\partial f}{\partial y_x}\right) = 0$$

Or $$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$$

This is another form of Euler's Equation. Now if "f" does not depend explicitly on x

$$\frac{\partial f}{\partial x} = 0$$

And $$\frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$$

Or $$f - y_x \frac{\partial f}{\partial y_x} = \text{Constant}$$

Calculus of Variation: Euler-Lagrange Equation

Generalization of Euler-Lagrange Equation

Now generalizing to several dependent variables . We consider the function f as function of independent variables x and several dependent variables $y_1, y_2,\ y_3\ ,\ \ldots..,\ y_n$ and $y_{1x}, y_{2x},$ $y_{3x}\ ,\ \ldots..,\ y_{nx}$. I.e.,

$$f = f(x, y_1, y_2, y_3\ , \ldots.., y_n, y_{1x}, y_{2x}, y_{3x}\ , \ldots.., y_{nx})$$

And $$J = \int_{x_1}^{x_2} f(x, y_1, y_2, y_3\ , \ldots.., y_n, y_{1x}, y_{2x}, y_{3x}\ , \ldots.., y_{nx})dx$$

where J is stationary.

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y_1}\delta y_1 + \frac{\partial f}{\partial y_2}\delta y_2 + \cdots + \frac{\partial f}{\partial y_n}\delta y_n + \frac{\partial f}{\partial y_{1x}}\delta y_{1x} + \frac{\partial f}{\partial y_{2x}}\delta y_{2x} + \cdots + \frac{\partial f}{\partial y_{nx}}\delta y_{nx}$$

$$\delta f = \sum_i^n \frac{\partial f}{\partial y_i}\delta y_i + \sum_i^n \frac{\partial f}{\partial y_{ix}}\delta y_{ix} = \sum_i^n \left[\frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial y_{ix}}\delta y_{ix}\right]$$

Therefore $$\delta J = \int_{x_1}^{x_2} \sum_i^n \left[\frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial y_{ix}}\delta y_{ix}\right] dx$$

Calculus of Variation: Euler-Lagrange Equation

$$\delta J = \int_{x_1}^{x_2} \sum_i^n \left[\frac{\partial f}{\partial y_i}\delta y_i + \frac{\partial f}{\partial y_{ix}}\delta y_{ix}\right] dx$$

$$\delta J = \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_i}\delta y_i dx + \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_{ix}}\delta y_{ix} dx$$

$$\delta J = \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_i}\delta y_i dx + \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_{ix}}\frac{d}{dx}\delta y_i dx$$

$$\delta J = \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_i}\delta y_i dx + \sum_i^n \cancelto{0}{\left|\frac{\partial f}{\partial y_{ix}}\delta y_i\right|_{x_1}^{x_2}} - \sum_i^n \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial f}{\partial y_{ix}}\delta y_i dx$$

$$\delta J = \sum_i^n \int_{x_1}^{x_2} \frac{\partial f}{\partial y_i}\delta y_i dx - \sum_i^n \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial f}{\partial y_{ix}}\delta y_i dx$$

$$\delta J = \sum_i^n \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_{ix}}\right]\delta y_i dx$$

Since $\delta J = 0$ therefore $\sum_i^n \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_{ix}}\right]\delta y_i dx = 0$

$\delta y_i \neq 0$ thought out the path therefore $\sum_i^n \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_{ix}}\right] = 0$ where $i = 1,2,3,\dots,n$

Applications of Calculus of Variation

1. Straight Line (Show that shortest distance between two points in plane is a straight line)

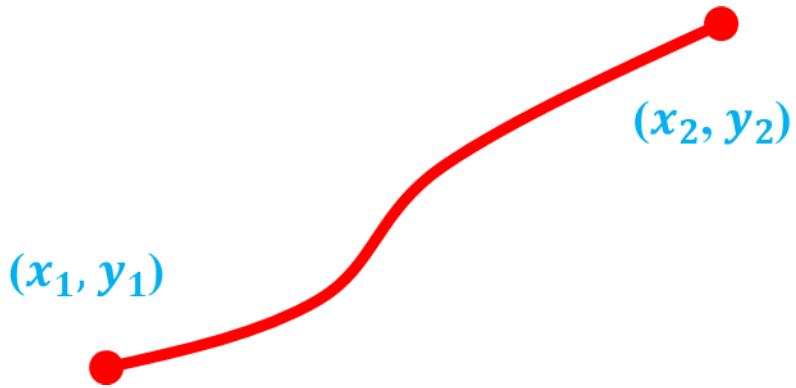
we can apply the calculus of variations to find out the distance between two points in a plane as elements of distance in the xy-plane is given by

$$dS^2 = dx^2 + dy^2$$

$$dS^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right] dx^2$$

$$dS = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$dS = \sqrt{[1 + y_x^2]}dx$$



Now the distance between the two points having coordinates (x_1, y_1) and (x_2, y_2) is given by

$$S = \int_{x_1}^{x_2} dS$$

$$S = \int_{x_1}^{x_2} \sqrt{[1 + y_x^2]}dx$$

Applications of Calculus of Variation

$$S = \int_{x_1}^{x_2} \sqrt{[1 + y_x^2]}dx$$

If S is minimum the Euler's Equation must be satisfied. Now if $f(x, y, y_x) = (1 + y_x^2)^{1/2}$ we can use $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

Since $\frac{\partial f}{\partial y} = 0$ And $\frac{\partial f}{\partial y_x} = \frac{y_x}{(1+y_x^2)^{1/2}}$

Since $\frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

Therefore $\frac{\partial f}{\partial y_x} = \text{constant}$

$$\frac{y_x}{(1+y_x^2)^{1/2}} = constant = c$$

$$y_x^2 = (1 + y_x^2)c^2$$

$$y_x = \frac{c}{(1-c^2)^{1/2}} = a$$

Applications of Calculus of Variation

$$y_x = a$$

$$\frac{dy}{dx} = a$$

$$\mathrm{dy} = adx$$

Integrating above equation

$$\mathrm{y} = ax + b$$

Which is equation of straight line. Thus, the shortest distance between two points in a plane is a straight line.

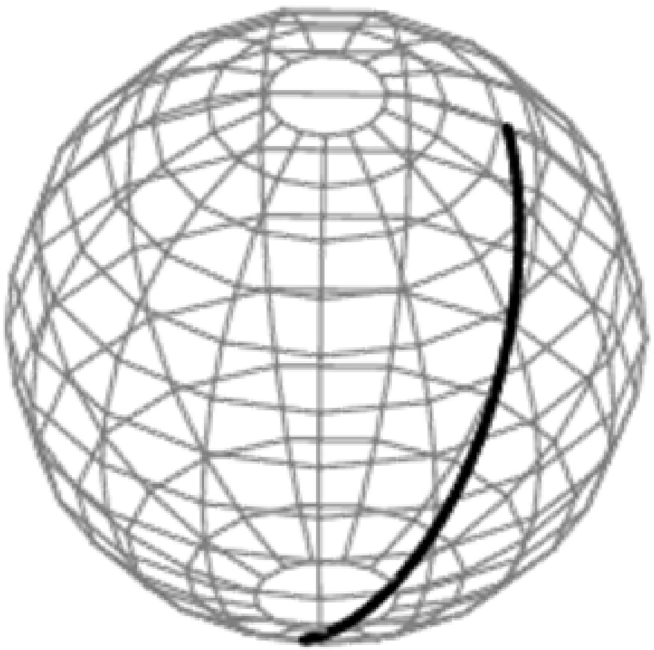
Applications of Calculus of Variation

1. Show that shortest distance between two points on the surface of the sphere is the Arc of great circle. (Great circle or orthodrome or Riemannian Circle)

Solution: Let us consider the element of distance between two points on surface of sphere is

$$dS^2 = dx^2 + dy^2 + dz^2$$

$$dS^2 = a^2[d\theta^2 + sin^2\theta\, d\varphi^2]$$



$$dS = a\sqrt{\left[1 + sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2\right]}d\theta = a\sqrt{[1 + sin^2\theta\varphi_\theta^2]}d\theta$$

Distance between two points having coordinates θ_1 and θ_2 is given

$$S = \int_{\theta_1}^{\theta_2} dS$$

$$S = a\int_{\theta_1}^{\theta_2} \sqrt{[1 + sin^2\theta\varphi_\theta^2]}d\theta$$

Since S is stationary because it must give an Arc of great circle. We can use $\frac{\partial f}{\partial \varphi} - \frac{d}{d\theta}\frac{\partial f}{\partial \varphi_\theta} = 0$

Applications of Calculus of Variation

$$f(\theta,\varphi,\varphi_\theta) = f(\theta,\varphi_\theta) = \sqrt{[1 + sin^2\theta\varphi_\theta^2]}$$

Since $\quad \frac{\partial f}{\partial \varphi} = 0 \quad$ therefore $\quad \frac{d}{d\theta}\frac{\partial f}{\partial \varphi_\theta} = 0$

$$\frac{\partial f}{\partial \varphi_\theta} = constant$$

$And \quad \frac{\partial f}{\partial \varphi_\theta} = \frac{sin^2\theta\varphi_\theta}{\sqrt{[1+sin^2\theta\varphi_\theta^2]}} = c$

$c^2 = \frac{sin^4\theta\varphi_\theta^2}{[1+sin^2\theta\varphi_\theta^2]}$

$[1 + sin^2\theta\varphi_\theta^2]c^2 = sin^4\theta\varphi_\theta^2$

$sin^2\theta\varphi_\theta^2(sin^2\theta - c^2) = c^2$

$\varphi_\theta = \frac{c\ cosec\ \theta}{(sin^2\theta - c^2)^{1/2}}$

Applications of Calculus of Variation

$$f(\theta,\varphi,\varphi_\theta) = f(\theta,\varphi_\theta) = \sqrt{[1+sin^2\theta\varphi_\theta^2]}$$

Since $\quad \dfrac{\partial f}{\partial \varphi} = 0 \quad$ therefore $\quad \dfrac{d}{d\theta}\dfrac{\partial f}{\partial \varphi_\theta} = 0$

$$\frac{\partial f}{\partial \varphi_\theta} = constant$$

$And \quad \dfrac{\partial f}{\partial \varphi_\theta} = \dfrac{sin^2\theta\varphi_\theta}{\sqrt{[1+sin^2\theta\varphi_\theta^2]}} = c$

$c^2 = \dfrac{sin^4\theta\varphi_\theta^2}{[1+sin^2\theta\varphi_\theta^2]}$

$[1+sin^2\theta\varphi_\theta^2]c^2 = sin^4\theta\varphi_\theta^2$

$sin^2\theta\varphi_\theta^2(sin^2\theta - c^2) = c^2$

$\varphi_\theta = \dfrac{c\ cosec\ \theta}{(sin^2\theta - c^2)^{1/2}}$

Applications of Calculus of Variation

$$\varphi_\theta = \frac{c\ cosec\ \theta}{(sin^2\theta - c^2)^{1/2}} = \frac{c\ cosec^2\theta}{(1 - cosec^2\theta c^2)^{1/2}} = \frac{c\ cosec^2\theta}{(1 - c^2 - cot^2\theta c^2)^{1/2}}$$

$$\varphi_\theta = \frac{cosec^2\theta}{\sqrt{\frac{(1-c^2)}{c^2}}(1 - \frac{c^2}{(1-c^2)}cot^2\theta)^{1/2}}$$

$$\varphi_\theta = \frac{d\varphi}{d\theta} = \frac{\left(\frac{c^2}{(1-c^2)}\right)^{1/2} cosec^2\theta}{(1 - \frac{c^2}{(1-c^2)}cot^2\theta)^{1/2}} = \frac{k cosec^2\theta}{(1 - k^2 cot^2\theta)^{1/2}}$$

$$\varphi = \int \frac{k cosec^2\theta}{(1 - k^2 cot^2\theta)^{1/2}} d\theta + \propto$$

Let $x = k \cot\theta$ and $dx = -k cosec^2\theta\ d\theta$

Therefore, $\varphi = \int \frac{-dx}{(1-x^2)^{1/2}} + \propto = -sin^{-1}x + \propto$

$Or\ \ sin^{-1}x = \propto - \varphi$ and $x = sin(\propto - \varphi)$

Applications of Calculus of Variation

$$x = k\cot\theta = sin(\propto -\varphi)$$

$$k\cot\theta = \sin\propto\cos\varphi - \cos\propto\sin\varphi$$

$$k\cos\theta = \sin\theta\sin\propto\cos\varphi - \sin\theta\cos\propto\sin\varphi$$

$$And\quad k\,a\cos\theta = a\sin\theta\sin\propto\cos\varphi - a\sin\theta\cos\propto\sin\varphi$$

$$kz = x\sin\propto - y\cos\propto$$

using cartesian coordinates (x,y,z) and the spherical polar coordinates.

$x = \mathrm{a}\sin\theta\cos\varphi$, $\quad y = \mathrm{a}\sin\theta\, sin\,\varphi$, $\quad z = \mathrm{a}\cos\theta$

This represent a plane passing through the centre of the sphere, which cut the surface of the sphere in a great circle. The shortest distance between the two points on the surface of the sphere is the Arc of the great circle.

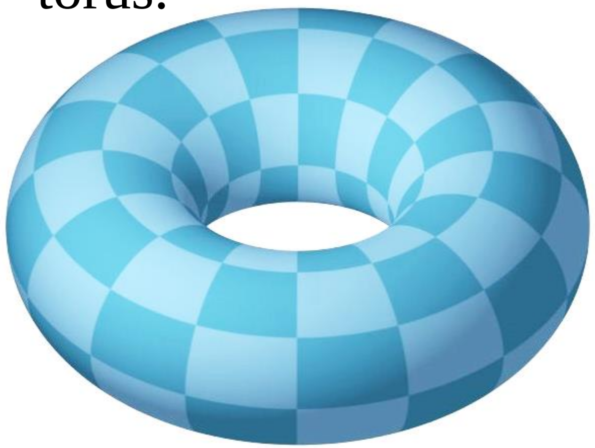
Applications of Calculus of Variation

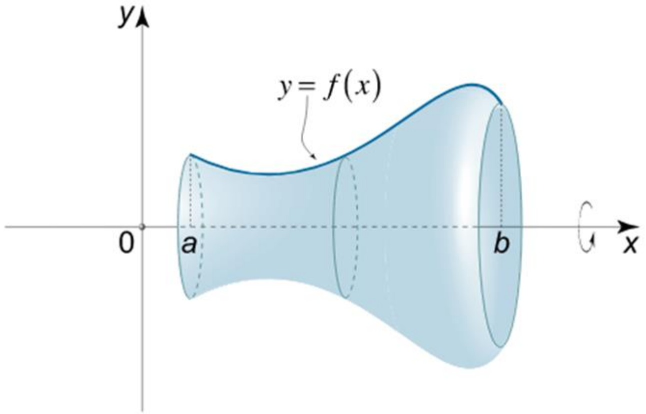
Surface of Revolution:-

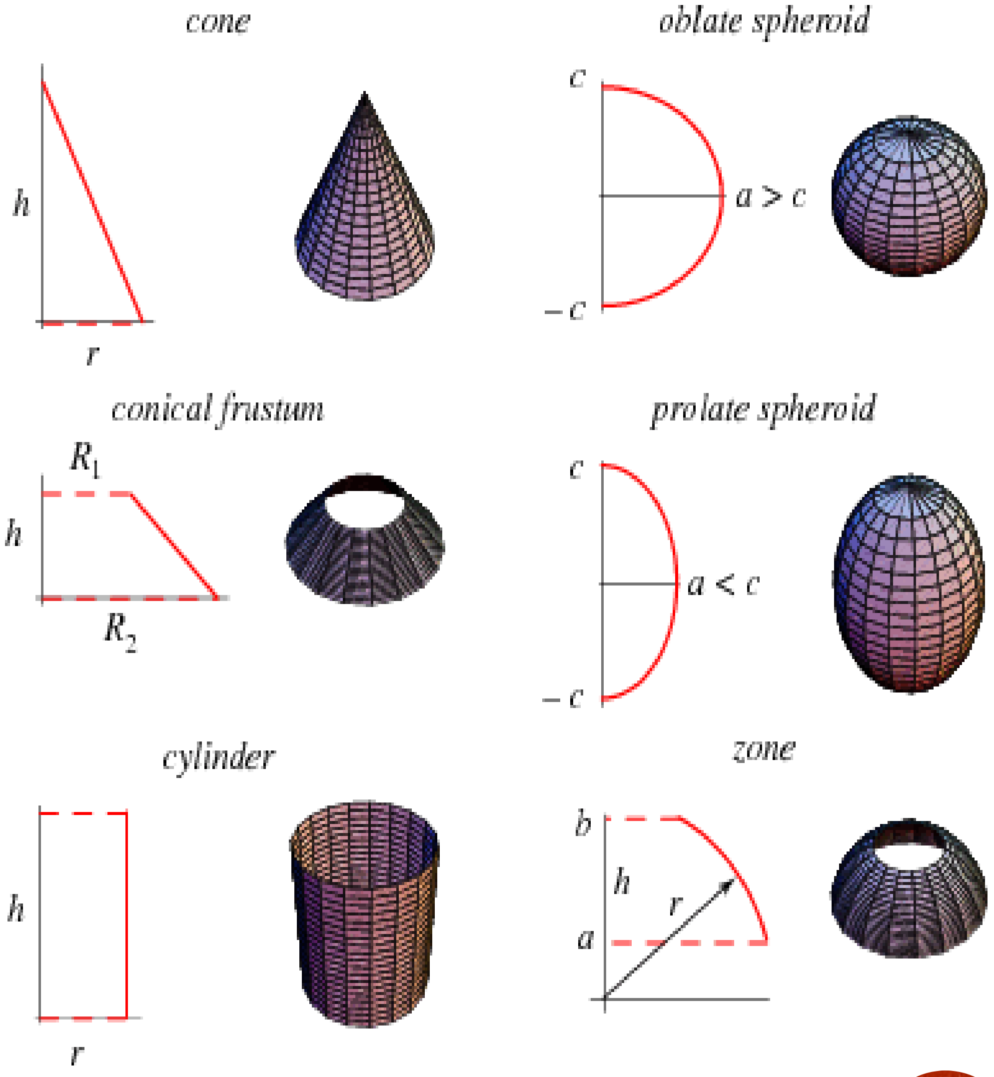
The surface of revolution is a surface created by rotating curve around a straight line in its plane. For example

The surface generated by straight line is cylinder.

Similarly a circle that is rotated about its diameter will generate a sphere and if the circle is rotated about a co-planer axis other then diameter, It generate a torus.



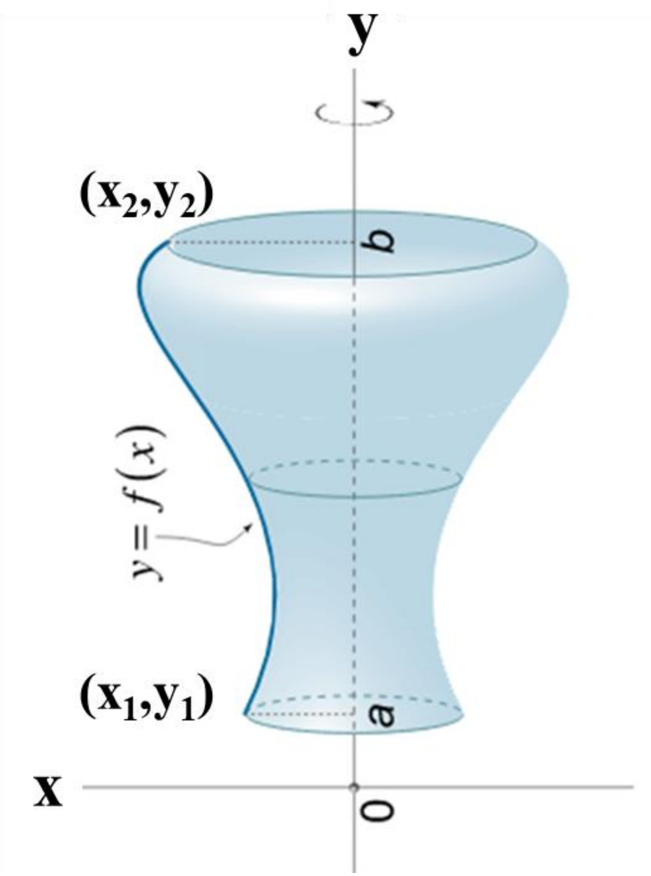




Applications of Calculus of Variation

Fin the curve for which the surface of revolution is minimum.

Solution: Suppose we form a surface of revolution by taking some curve passing between two points (x_1,y_1) and (x_2,y_2) (fixed) and revolution is about y axis. We must find a curve for which the surface area is minimum as shown in figure.



The area of the strip of surface is

$$dA = 2\pi x ds = 2\pi x(dx^2 + dy^2)^{1/2}$$

$$dA = 2\pi x\sqrt{[1 + y_x^2]}dx$$

The total area is $\quad A = 2\pi \int_{x_1}^{x_2} x\sqrt{[1 + y_x^2]}dx$

The extremum value can be found out using $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

Where $f = f(x, y_x) = x(1 + y_x^2)^{1/2}$

Since $\frac{\partial f}{\partial y} = 0 \quad$ therefore $\frac{d}{dx}\frac{\partial f}{\partial y_x} = 0$

Applications of Calculus of Variation

Therefore, $\frac{\partial f}{\partial y_x} = constant = a$

For $f = f(x, y_x) = x(1 + y_x^2)^{1/2}$

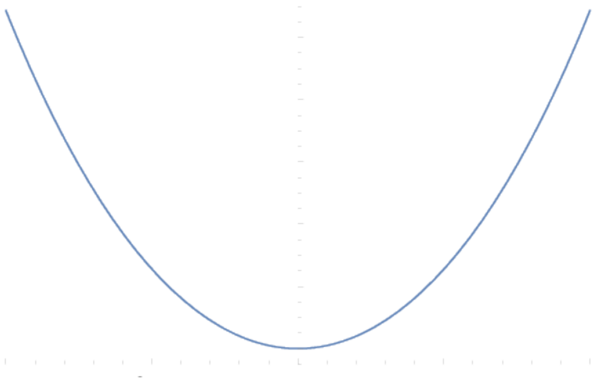
$$\frac{\partial f}{\partial y_x} = \frac{\partial}{\partial y_x} x(1 + y_x^2)^{1/2} = \frac{x y_x}{(1+y_x^2)^{1/2}} = a$$

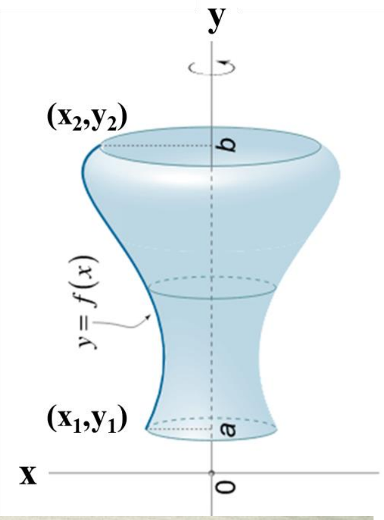
$$y_x = \frac{a}{\sqrt{x^2-a^2}}$$

$$\mathrm{d}y = \frac{a}{\sqrt{x^2-a^2}} dx$$

$$y = a \cosh^{-1}(x/a) + b$$

$$x = a \cosh\left(\frac{y.-b}{a}\right)$$



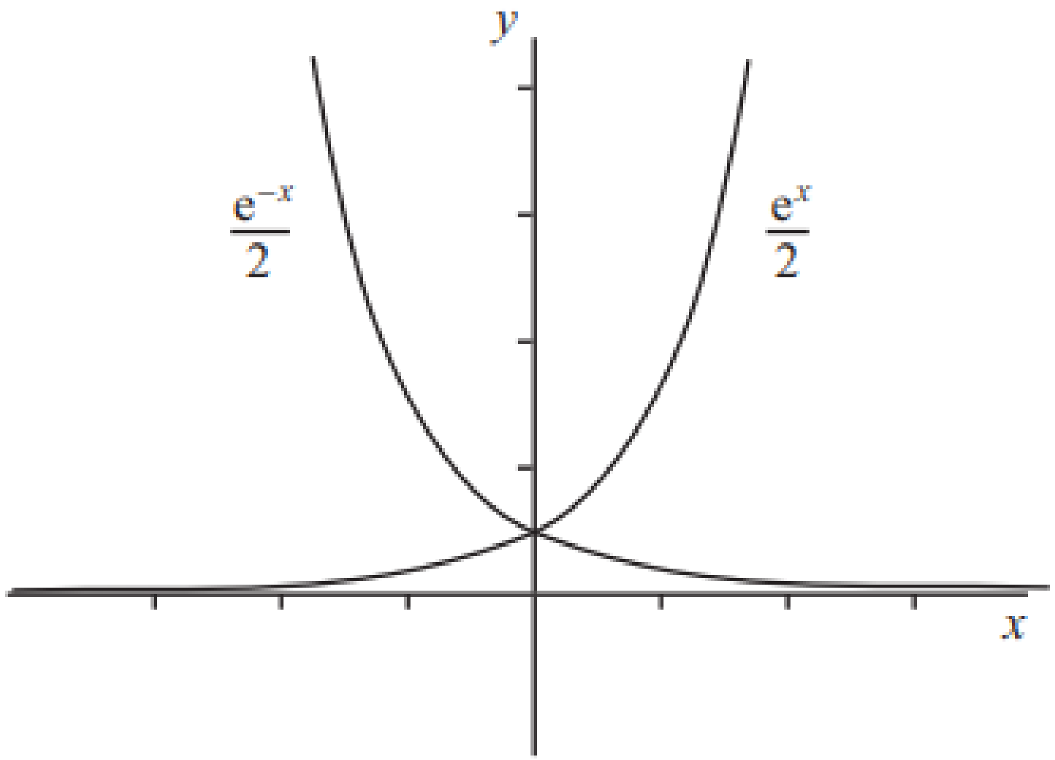


Which is equation of catenary. The shape of $\cosh\left(\frac{y-b}{a}\right)$ if plotted in x,y Plane is shown. The rotation of such curve will give minimum surface of rotation.

Applications of Calculus of Variation

$$\cosh x = \frac{e^x}{2} + \frac{e^{-x}}{2}.$$

To see how this behaves as x gets large, recall the graphs of the two exponential functions.



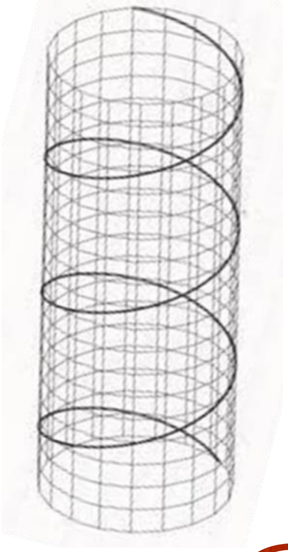
Applications of Calculus of Variation

Helix

A shape like a spiral staircase. It is a type of smooth space curve with tangent line at a constant angle to fixed axis.

A line, thread wire or other structure curved into a shape such as it would assume if wound in a single layer round a cylinder.

Or



A spiral curve lying on a cone or cylinder and cutting the generator at constant angle. The shortest distance on the surface of a sphere or curved surface

Show that the geodesics on the surface of the right circular cylinder is a Helix.

Solution: The element of the distance along the surface is

$$ds = (dx^2 + dy^2 + dz^2)^{1/2}$$

Where $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$$dx = -r\sin\theta\, d\theta, \quad dy = r\cos\theta\, d\theta, \quad dz = dz$$

Applications of Calculus of Variation

$$ds = \sqrt{r^2 \sin^2\theta d\theta^2 + r^2 \cos^2\theta d\theta^2 + dz^2}$$

$$ds = \sqrt{r^2\, d\theta^2 + dz^2}$$

$$ds = \left(\sqrt{r^2 \left(\frac{d\theta}{dz}\right)^2 + 1}\right) dz = \left(\sqrt{r^2\, {\theta_z}^2 + 1}\right) dz$$

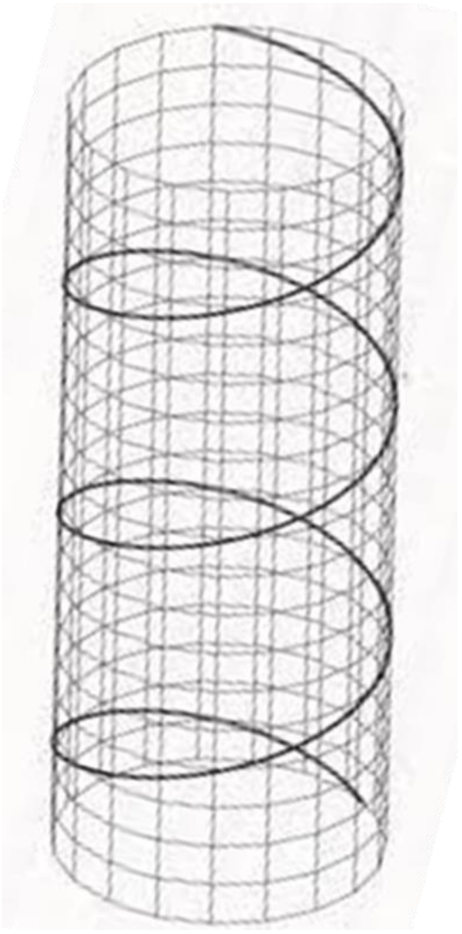
The s to be extremum we have $\frac{\partial f}{\partial \theta} - \frac{d}{dz}\frac{\partial f}{\partial \theta_z} = 0$

Here $f = (r^2\theta_z^2 + 1)^{1/2}$ and $\frac{\partial f}{\partial \theta} = 0$

$$\Rightarrow \frac{d}{dz}\frac{\partial f}{\partial \theta_z} = 0$$

$$\frac{\partial f}{\partial \theta_z} = \frac{\partial}{\partial \theta_z}[r^2\theta_z^2 + 1]^{1/2} = constant = c$$

$$\frac{\partial f}{\partial \theta_z} = \frac{r^2\theta_z}{[r^2\theta_z^2 + 1]^{1/2}} = c$$



Applications of Calculus of Variation

$$\frac{r^4\theta_z^2}{[r^2\theta_z^2+1]} = c^2$$

$$r^4\theta_z^2 = [r^2\theta_z^2 + 1]c^2$$

$$r^2\theta_z^2(r^2 - c^2) = c^2$$

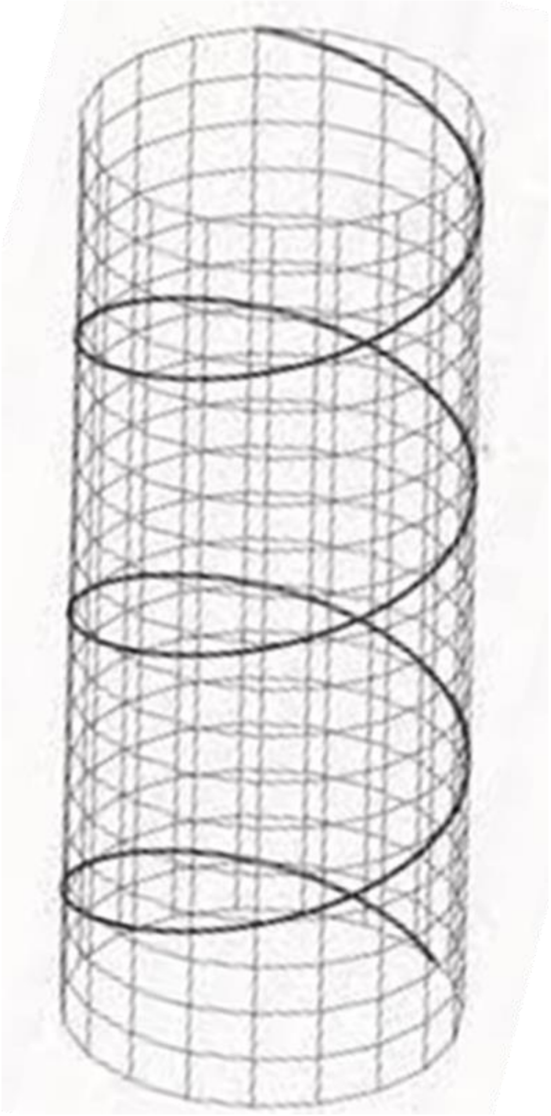
$$r^2\theta_z^2 = \frac{c^2}{(r^2-c^2)}$$

$$r\theta_z = \sqrt{\frac{c^2}{(r^2-c^2)}} = D$$

$$r\frac{d\theta}{dz} = D$$

$$\Rightarrow r\theta = Dz + E$$

where D and E are constants



Applications of Calculus of Variation

BRACHISTOCHRONE or shortest time problem

The Brachistochrone problem is famous in mathematics & solved by Jhon Bernoulli.

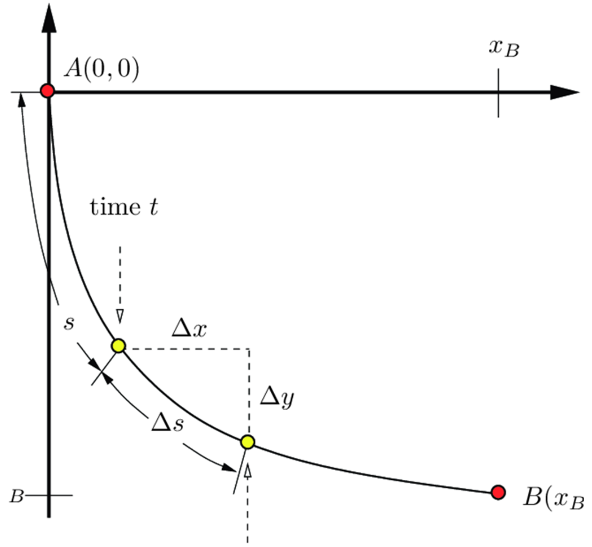
The analysis led to the formal foundation of the calculus of variation.

The problem is about the curve joining two points, along which a particle falling from rest under the influence of gravity, travels from the higher to the lower point in the least time.

If v is the speed along the curve, then the time required to fall on arc length *ds* is *ds/v*

$$t_{12} = \int_1^2 \frac{ds}{v}$$

Where $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + {y_x}^2}dx$



Since the energy of the particle at point 1 is *P.E=mgy.* When particle reaches point 2, its potential energy will become its K.E

$$\frac{1}{2}mv^2 = mgy$$

Applications of Calculus of Variation

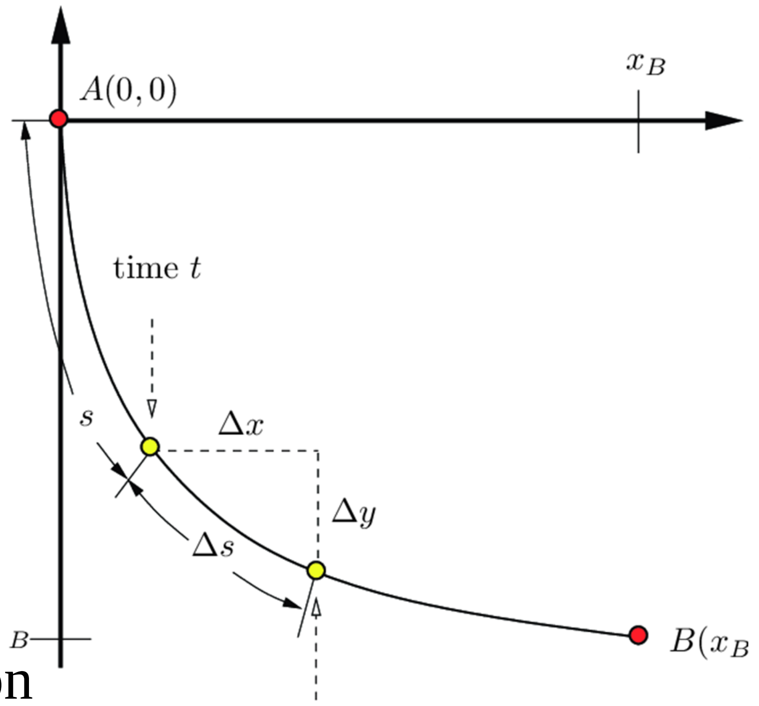
$$v = \sqrt{2gy}$$

Now the expression for t_{12} become

$$t_{12} = \int_1^2 \frac{(1+{y_x}^2)^{1/2}}{\sqrt{2gy}} dx$$

$$f = \frac{(1+{y_x}^2)^{1/2}}{\sqrt{2gy}}$$

$$f(x, y, y_x) = \frac{(1+{y_x}^2)^{1/2}}{\sqrt{2gy}}$$



If the integral has stationary value. We can use Euler equation

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$$

$$\frac{\partial f}{\partial x} = 0 \qquad \Rightarrow \quad f - y_x \frac{\partial f}{\partial y_x} = constant$$

Applications of Calculus of Variation

$$f - y_x \frac{\partial f}{\partial y_x} = c \Rightarrow$$

$$y_x \frac{\partial}{\partial y_x}\left[\frac{(1+y_x{}^2)^{1/2}}{\sqrt{2gy}}\right] = c$$

$$\Rightarrow \frac{(1+y_x{}^2)^{1/2}}{\sqrt{2gy}} - y_x\left[\frac{y_x}{\sqrt{2gy}(1+y_x{}^2)^{1/2}}\right] = c$$

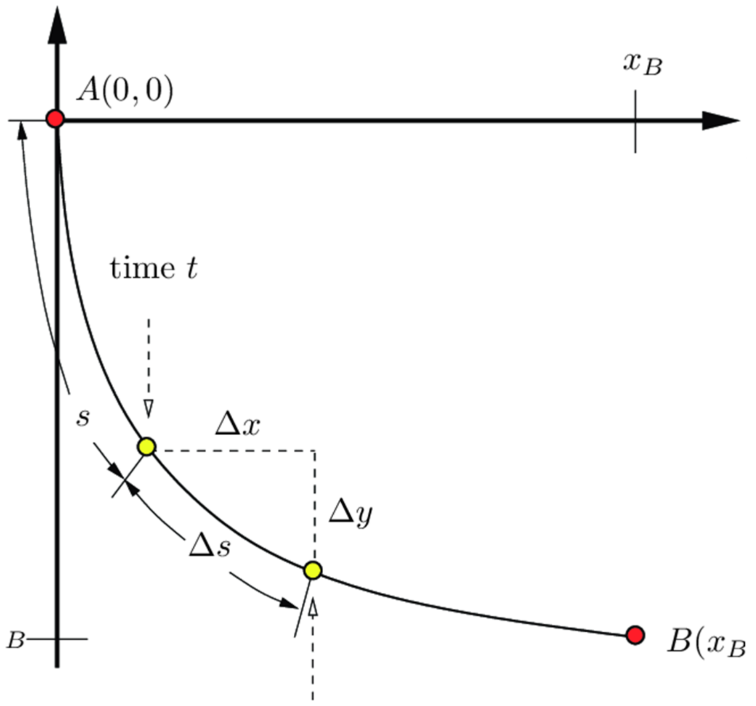
$$\Rightarrow \frac{1}{\sqrt{2gy}}\left[\frac{(1+y_x{}^2)-y_x{}^2}{(1+y_x{}^2)^{1/2}}\right] = c$$

$$\Rightarrow \frac{1}{\sqrt{y}}\left[\frac{1}{(1+y_x{}^2)^{1/2}}\right] = \sqrt{2g}c$$

$$\Rightarrow \sqrt{y(1+y_x{}^2)} = \frac{1}{\sqrt{2g}c}$$

$$\Rightarrow y(1+y_x{}^2) = \frac{1}{2gc^2} = b$$

$$\Rightarrow y(1+y_x{}^2) = b$$



To solve above equation let $y_x = \tan\varphi$ and $y(1+tan^2\varphi) = b$

Applications of Calculus of Variation

$$\Rightarrow y\, sec^2\, \varphi = b$$

$$\Rightarrow y = cos^2\, \varphi\, b = \frac{b}{2}(1 + \cos 2\varphi)$$

and $\quad$ $\mathrm{dy} = (-b \sin 2\varphi) d\varphi$

Now $\quad$ $y_x = \tan\varphi$

$$\Rightarrow \frac{dy}{dx} = \tan\varphi$$

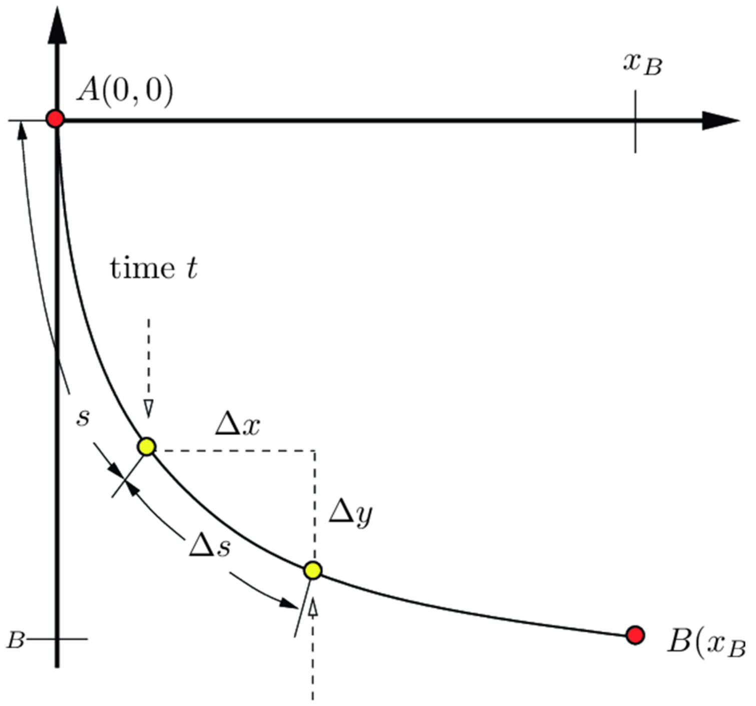
$$\Rightarrow dx = \cot\varphi\, dy = \cot\varphi\, (-b \sin 2\varphi) d\varphi$$

$$\Rightarrow dx = -\,\mathrm{bcot}\,\varphi\, (\sin\varphi \cos\varphi) d\varphi$$

$$\Rightarrow dx = -2b\, cos^2\, \varphi\, d\varphi$$

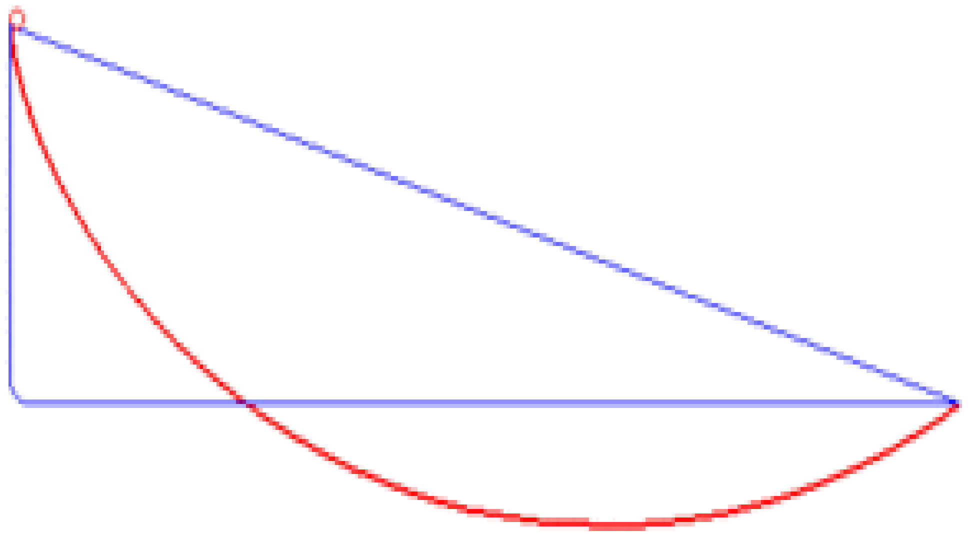
$$\Rightarrow x = a - 2b \int cos^2\, \varphi\, d\varphi = a - 2b \int \frac{1}{2}(1 + cos\, 2\varphi)\, d\varphi$$

$$\Rightarrow x = a - b\, (\varphi + \frac{1}{2} \sin 2\varphi)$$



Applications of Calculus of Variation

$$\Rightarrow x = a - \frac{b}{2}(2\varphi + \sin 2\varphi)$$



The problem can also be solved by assuming

$$y_x = cot\,\varphi$$

And $\quad y = \frac{1}{2}b(1 - cos\,2\varphi)\quad$ and $\quad x = a + \frac{1}{2}b(2\varphi - \sin 2\varphi)$

TRY AT HOME: Homework.

Chapter 3

Lecture 3

Hamilton's Principle & Hamiltonian Mechanics

Dr. Akhlaq Hussain

Hamilton's Principle (Principle of stationary action Or Least action)

The Lagrange's equation was developed from the consideration of the instantaneous state of the system and small virtual displacement about the instantons state.

From "D Almambert principle or differentiable principle"

A virtual displacement is one that take place in time $\delta t = 0$

However, it is also possible to obtain Lagrange's equation for the actual motion of system between the time t_1 and t_2, by considering small virtual variation of the motion from the actual path of the motion

This principle known as integral principle or Hamilton's principle.

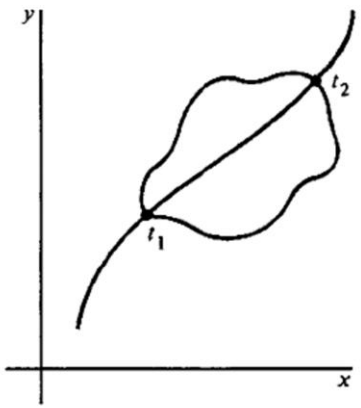
Hamilton's Principle (Principle of stationary action Or Least action)

"Out of all possible paths along which a dynamics system move from one point to another with in a given interval of time (consistent with the force of constraints, if any) the actual path followed is that which gives extremum value to the time integral of Lagrangian"

The principle can alternatively be stated as

"The motion of the system from instant t_1 to instant t_2 is such that the line integral."

$J = \int_{t_1}^{t_2} L\, dt$ is stationary.



Any variation in the value of integral is zero.

$$\delta J = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L\, dt = 0$$

Since zero is the minimum or least value. Therefore, it is also called Hamilton's principle of least action.

Hamilton's Principle (Principle of stationary action Or Least action)

$$L = L(q_i, \dot{q}_i, t)$$

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t$$

Since ethe Lagrangian does not explicitly depends on time therefore $\frac{\partial L}{\partial t}\delta t = 0$

Any variation in the value of integral is zero.

$$\delta L = \sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$$\delta J = \int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \left[\sum_i \frac{\partial L}{\partial q_i}\delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] dt$$

$$\delta J = \int_{t_1}^{t_2} \delta L\, dt = \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] dt$$

Since we know that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$$

Hamilton's Principle (Principle of stationary action Or Least action)

$$\delta J = \int_{t_1}^{t_2} \delta L \, dt = \sum_i \int_{t_1}^{t_2} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i) \right] dt$$

$$\delta J = \sum_i \int_{t_1}^{t_2} \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] dt$$

$$\delta J = \sum_i \left| \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2}$$

Since we know that variation in the generalized coordinate at ends points is zero.

$$\delta q_1(t_1) = \delta q_1(t_2) = 0$$

$$\delta q_2(t_1) = \delta q_2(t_2) = 0$$

$$\vdots \qquad \vdots$$

$$\delta q_n(t_1) = \delta q_n(t_2) = 0$$

Therefore, we can write $\delta J = \int_{t_1}^{t_2} \delta L \, dt = 0$

Derivation of Lagrange's Eq. from Hamilton's principle of least action

Let us consider conservative, holonomic dynamical system whose configuration at any instant is specified by n-generalized coordinates $q_1, q_2, \dots, q_n$

Let the system move in real or configurational space from point "P" to "Q" by two possible paths as shown

Let δ denotes the variation in the Lagrangian function, which does not involve a change in time t. then

$$L = L(q_i, \dot{q}_i)$$

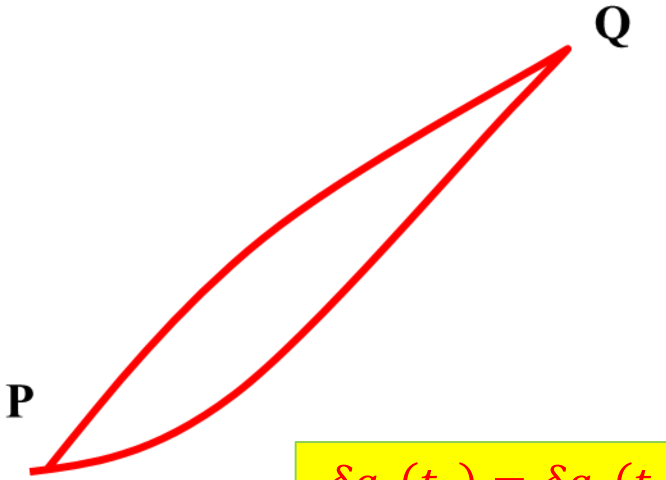
$$\delta J = \delta \int_{t_1}^{t_2} L \, dt$$

$$\delta J = \int_{t_1}^{t_2} \left[\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt$$

$$\delta J = \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i \right] dt$$

$$\delta J = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i dt + \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i dt$$

$$\delta J = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i dt + \sum_i \left| \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right|_{t_1}^{t_2} - \sum_i \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i dt$$



$$\delta q_1(t_1) = \delta q_1(t_2) = 0$$
$$\delta q_2(t_1) = \delta q_2(t_2) = 0$$
$$\vdots \qquad \vdots$$
$$\delta q_n(t_1) = \delta q_n(t_2) = 0$$

Derivation of Lagrange's Eq. from Hamilton's principle of least action

$$\delta J = \sum_i \int_{t_1}^{t_2} \frac{\partial L}{\partial q_i} \delta q_i dt - \sum_i \int_{t_1}^{t_2} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i dt$$

$$\delta J = \sum_i \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i dt$$

Since δq_i is zero only at end points. Except end points the δq_i is nonzero. Therefore $\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right]$ must be zero through out the path.

Therefore,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Which is Lagrange's Equation.

Hamilton's Equation of Motion

H-equation are formulated in 1833 by Irish Mathematician William Rowan Hamilton

1) Hamilton's formulation is a more powerful method of working with physical principles already established.

2) In Lagrangian formulation, the independent variable are q_i and $, \dot{q}_i$ while in Hamilton's formulation the independent variable are generalized coordinates q_i and generalized momenta p_i

Applications;

It helps us to construct more abstract theories in Quantum Mechanics [Probabilities distribution and perturbation theory in phase space] & statistical mechanics [Poisson Algebra]

Hamilton's equations are great in solving problems that involves transformer of energy and momentum.

Provide an easy wat to solve problems that can be hard to solve using Newtonian Mechanics

Derivation of Hamilton's Equation of Motion using Lagrange's Eq.

Let a mechanical system be represented at any instant by n generalized coordinates

$$q_1, q_2, \dots, q_n$$

The Lagrange's equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{where } i =, 1,2,3,4, \dots.., n$$

Where $L = L(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t)$

And $\mathrm{d}L = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$ 1

For conservative system i.e $\frac{\partial V}{\partial \dot{q}_i} = 0$

Then $\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial (T-V)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \sum \frac{1}{2} m_i \dot{q}_i^2 = m_i \dot{q}_i = p_i \qquad \text{a}$$

Derivation of Hamilton's Equation of Motion using Lagrange's Eq.

And $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}p_i = \dot{p}_i$

Since $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}p_i = \dot{p}_i$ b

Putting equation, a & b in equation 1

And $\mathrm{d}L = \sum_i \dot{p}_i dq_i + \sum_i p_i\, d\dot{q}_i + \frac{\partial L}{\partial t}dt$ 2

Consider the term $\sum_i d(p_i\, \dot{q}_i) = \sum_i p_i\, d\dot{q}_i + \sum_i \dot{q}_i dp_i$

$$\Rightarrow \sum_i p_i\, d\dot{q}_i = \sum_i d(p_i\, \dot{q}_i) - \sum_i \dot{q}_i dp_i$$

Putting in equation 2 $\mathrm{d}L = \sum_i \dot{p}_i dq_i + \sum_i d(p_i\, \dot{q}_i) - \sum_i \dot{q}_i dp_i + \frac{\partial L}{\partial t}dt$

$$\Rightarrow \mathrm{d}[L - \sum_i p_i\, \dot{q}_i] = \sum_i \dot{p}_i dq_i - \sum_i \dot{q}_i dp_i + \frac{\partial L}{\partial t}dt$$

$$\Rightarrow -\mathrm{d}[\sum_i p_i\, \dot{q}_i - L] = \sum_i \dot{p}_i dq_i - \sum_i \dot{q}_i dp_i + \frac{\partial L}{\partial t}dt$$

Derivation of Hamilton's Equation of Motion using Lagrange's Eq.

Where $H = \sum_i p_i \dot{q}_i - L$

$$\Rightarrow -\mathrm{dH} = \sum_i \dot{p}_i dq_i - \sum_i \dot{q}_i dp_i + \frac{\partial L}{\partial t} dt$$

Or $$\Rightarrow \mathrm{dH} = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt \qquad 3$$

From equation 3 we can conclude that $H = H(q_i, p_i, t)$

And $$\mathrm{d}H = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \qquad 4$$

Comparing Equation 3 and 4

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad c$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad d$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad e$$

Equation c & d are called Hamilton's equations of motion.

Derivation of Hamilton's Equation of Motion using Lagrange's Eq.

Special Cases

If H is not an explicit function of time, then H is a constant of motion.

Since H is independent of time $H = H(q_i, p_i)$

$$\Rightarrow \frac{dH}{dt} = \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial H}{\partial p_i}\dot{p}_i$$

since $\frac{\partial H}{\partial q_i} = -\dot{p}_i$ & $\frac{\partial H}{\partial p_i} = \dot{q}_i$

therefore $\Rightarrow \frac{dH}{dt} = -\sum_i \dot{p}_i\dot{q}_i + \sum_i \dot{q}_i\dot{p}_i = 0$

$$\Rightarrow H = \sum_i p_i\, \dot{q}_i - L = Constant$$

If the equations of transformation do not depend on time and if the potential energy is velocity independent, then H is the total energy of the system

$$\sum_i p_i\dot{q}_i = \sum_i \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i = \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i}\sum_j^N \frac{1}{2} m_j \dot{r}_j^2$$

Derivation of Hamilton's Equation of Motion using Lagrange's Eq.

$$\sum_i p_i \dot{q}_i = \sum_i \dot{q}_i \sum_j^n m_j \dot{r}_j \; \frac{\partial \dot{r}_j}{\partial \dot{q}_i}$$

$$\sum_i p_i \dot{q}_i = \sum_j^n m_j \dot{r}_j \sum_i \frac{\partial \dot{r}_j}{\partial \dot{q}_i} \dot{q}_i$$

$$\sum_i p_i \dot{q}_i = \sum_j^n m_j \dot{r}_j \sum_i \frac{\partial r_j}{\partial q_i} \dot{q}_i$$

$$\sum_i p_i \dot{q}_i = \sum_j^n m_j \dot{r}_j \cdot \dot{r}_j$$

$$\sum_i p_i \dot{q}_i = 2 \sum_j^n \frac{1}{2} m_j \dot{r}_j^2 = 2T$$

Therefore $\quad H = \sum_i p_i \, \dot{q}_i - L = 2T - L$

$$H = 2T - T + V$$

Therefore $\quad H = T + V = E$

So, we conclude that of Hamiltonian does not depend on time it represents the total energy of the system.

Cyclic or Ignorable Coordinate

If a Lagrangian $L = T - V$ of the dynamical system does not contain a coordinate explicitly, then that coordinate is called cyclic or ignorable coordinate.

Thus, if q_i is an ignorable coordinate then $\frac{\partial L}{\partial q_i} = 0$

Where $\frac{\partial L}{\partial \dot{q}_i}$ may not be zero.

From Lagrange's equation of motion. i.e.,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_i} = constant$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}_i} = p_i = constant$$

Where p_i is the conjugate momentum for q_i, so if Lagrangian of a dynamical system does not contain a coordinate q_i Then corresponding conjugate momentum p_i is conserved.

Applications of Hamilton's Equation of Motion

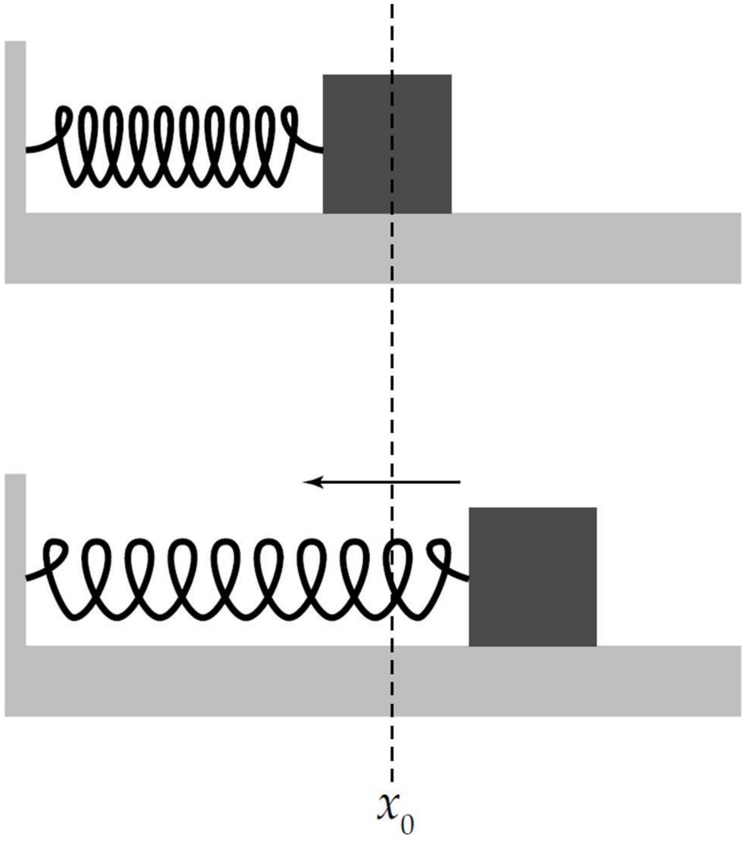
Derive Equation of motion for one dimensional hormonic Oscillator using of Hamilton's Equation of Motion

$$T = \frac{1}{2} m\dot{x}^2$$

$$V = \frac{1}{2} kx^2$$

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$



Now $$H = p_x\dot{x} - L = m\dot{x} \cdot \dot{x} - \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2\right]$$

$$H = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2$$

$$H = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{p_x^2}{2m} + \frac{1}{2} kx^2$$

Now using Hamilton's equation of motion

Applications of Hamilton's Equation of Motion

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}\left[\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right]$$

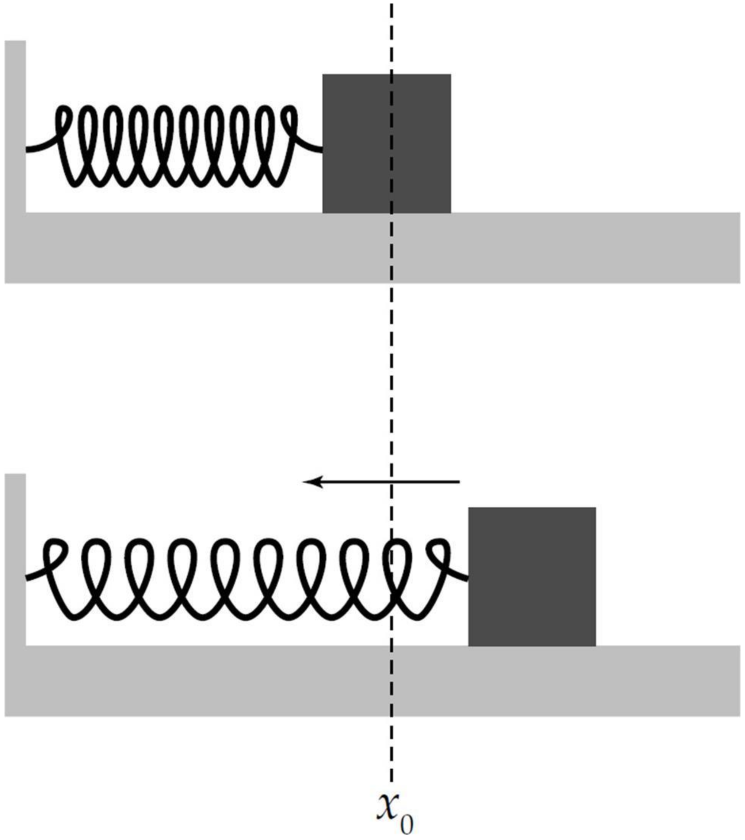
$$\dot{p}_x = -kx$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{\partial}{\partial p_x}\left[\frac{p_x^2}{2m} + \frac{1}{2}kx^2\right]$$

$$\dot{x} = \frac{p_x}{m}$$

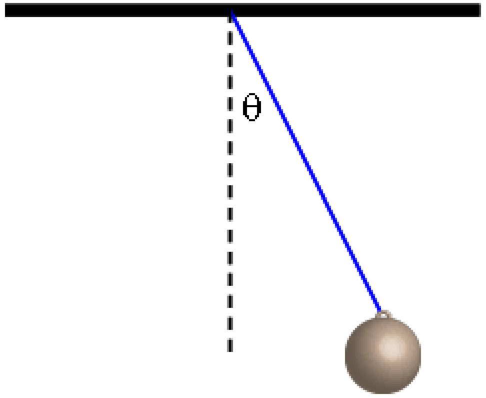
$$\Rightarrow p_x = m\dot{x}$$

$$\Rightarrow \dot{p}_x = m\ddot{x} = -kx$$



Applications of Hamilton's Equation of Motion

Derive Hamilton's Equation of motion for simple pendulum.



$$\mathrm{T} = \frac{1}{2} m l^2 \dot{\theta}^2 \qquad \& \qquad \mathrm{V} = -mgy = -mgl\cos\theta$$

$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{m l^2}$$

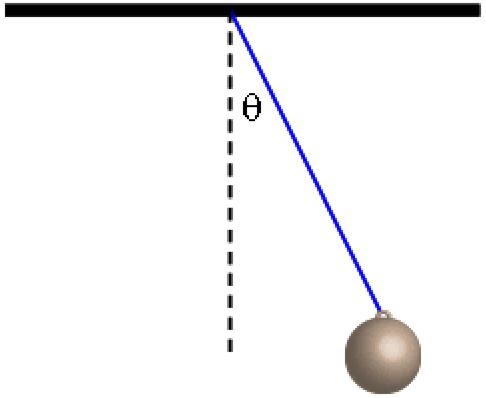
Now $$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{m l^2} - \left[\frac{p_\theta^2}{2 m l^2} + mgl\cos\theta\right]$$

$$H = \frac{p_\theta^2}{2 m l^2} - mgl\cos\theta$$

Now using Hamilton's equation of motion

Applications of Hamilton's Equation of Motion

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{\partial}{\partial p_\theta}\left[\frac{p_\theta^2}{2ml^2} - mgl\cos\theta\right]$$



$$\Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial}{\partial \theta}\left[\frac{p_\theta^2}{2ml^2} - mgl\cos\theta\right]$$

$$\dot{p}_\theta = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta$$

$$\dot{p}_\theta = ml^2\ddot{\theta} = -mgl\sin\theta$$

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

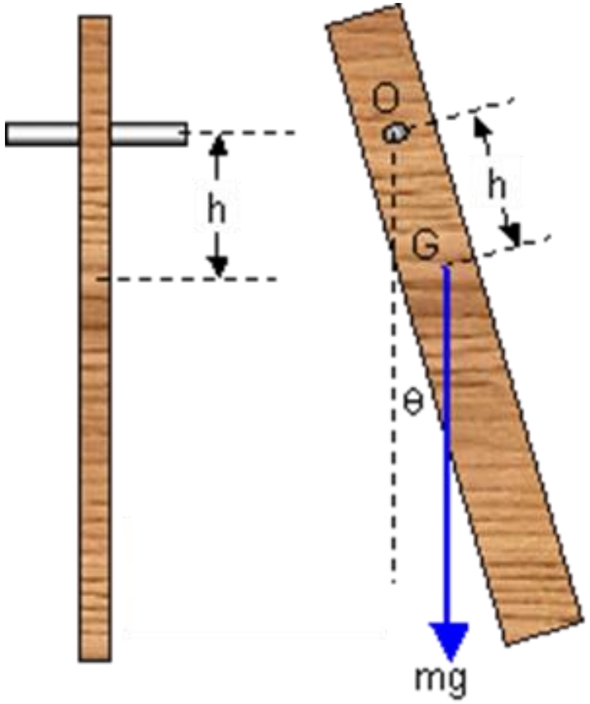
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

Or $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Applications of Hamilton's Equation of Motion

Derive Hamilton's Equation of motion for compound pendulum.

$$\mathrm{T} = \frac{1}{2} I\dot{\theta}^2 \qquad \& \qquad \mathrm{V} = -mgy = -mgh\cos\theta$$



$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{1}{2} I\dot{\theta}^2 + mgh\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{I}$$

Now $\quad H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{I} - \left[\frac{p_\theta^2}{2I} + mgh\cos\theta\right]$

$$H = \frac{p_\theta^2}{2I} - mgh\cos\theta$$

Now using Hamilton's equation of motion

Applications of Hamilton's Equation of Motion

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{\partial}{\partial p_\theta}\left[\frac{p_\theta^2}{2I} - mgh\cos\theta\right]$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta}{I}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial}{\partial \theta}\left[\frac{p_\theta^2}{2I} - mgh\cos\theta\right]$$

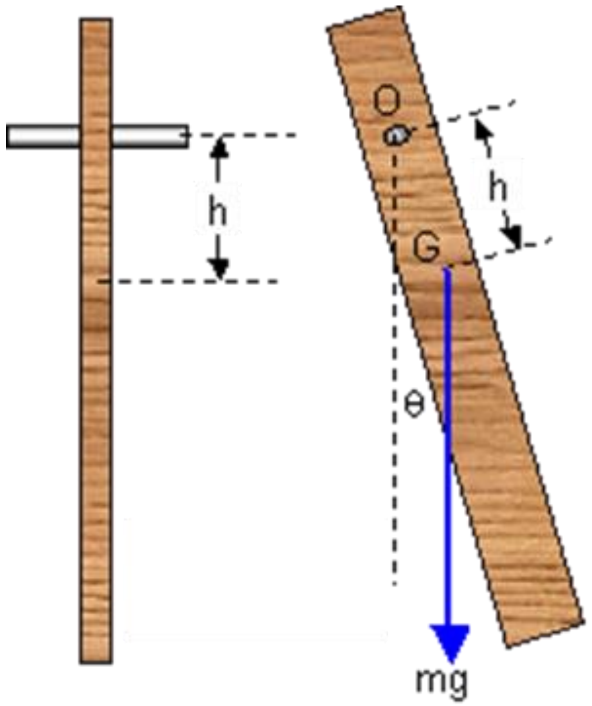
$$\dot{p}_\theta = \frac{d}{dt}(I\dot{\theta}) = -mgh\sin\theta$$

$$\dot{p}_\theta = I\ddot{\theta} = -mgh\sin\theta$$

$$\mathrm{I}\ddot{\theta} = -mgh\sin\theta$$

$$\ddot{\theta} = -\frac{mgh}{I}\sin\theta$$

Or $$\ddot{\theta} + \frac{mgh}{I}\sin\theta = 0$$



Chapter 3

Lecture 4

Hamilton's Principle & Hamiltonian Mechanics

Dr. Akhlaq Hussain

Hamilton's Canonical Equation in Spherical Coordinates $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$

Hamilton's canonical equations in spherical coordinates

$$H = \sum_i p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - L$$

$$T = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2\right) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2 \sin^2\theta} \quad \text{And} \quad V = V(r, \theta, \varphi)$$

$$H = p_r \dot{r} + p_\theta \dot{\theta} + p_\varphi \dot{\varphi} - L$$

$$p_r = \frac{\partial T}{\partial \dot{r}}, \; p_\theta = \frac{\partial T}{\partial \dot{\theta}}, \; p_\varphi = \frac{\partial T}{\partial \dot{\varphi}}$$

$$H = p_r \frac{p_r}{m} + p_\theta \frac{p_\theta}{mr^2} + p_\varphi \frac{p_\varphi}{mr^2 \sin^2\theta} - \left[\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2 \sin^2\theta} - V(r, \theta, \varphi)\right]$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2 \sin^2\theta} + V(r, \theta, \varphi)$$

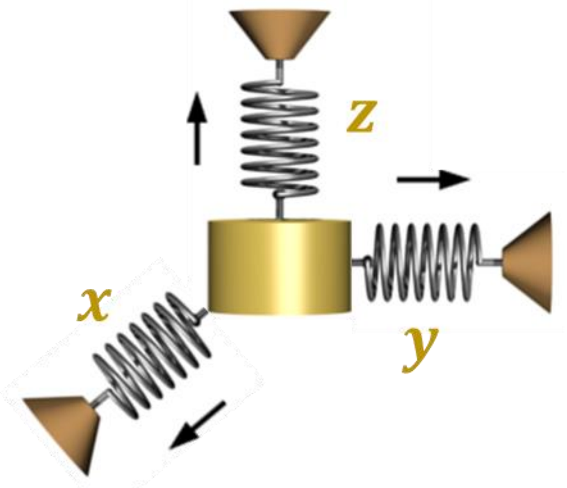
Applying Hamilton's canonical Equations

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + \frac{p_\varphi^2}{mr^3 \sin^2\theta} - \frac{\partial V(r,\theta,\varphi)}{\partial r},$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\varphi^2 \cos\theta}{mr^2 \sin^3\theta} - \frac{\partial V(r,\theta,\varphi)}{\partial \theta},$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2 \sin^2\theta} \qquad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial V(r,\theta,\varphi)}{\partial \varphi}$$

Hamilton's Canonical Equation For three dimensional Oscillator



$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V = \frac{1}{2}k(x^2 + y^2 + z^2)$$

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}k(x^2 + y^2 + z^2)$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$H = \sum_i p_i\, \dot{q}_i - L = p_x\,\dot{x} + p_y\dot{y} + p_z\,\dot{z} - L = \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} - \frac{1}{2}k(x^2 + y^2 + z^2)\right]$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}k\left(x^2 + y^2 + z^2\right)$$

Applying Hamilton's canonical Equations

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx, \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -ky \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = -kz$$

Hamilton's Canonical Equation For particle falling under gravity

$$\mathrm{T} = \frac{1}{2} m\dot{y}^2$$

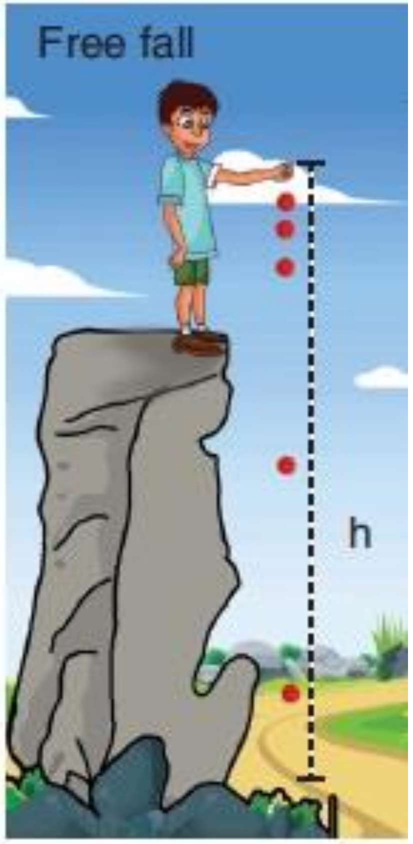
$$\mathrm{V} = mgy$$

$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{1}{2} m\dot{y}^2 - mgy$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$H = \sum_i p_i \dot{q}_i - L = p_y \dot{y} - L = \frac{p_y^2}{m} - \left[\frac{p_y^2}{2m} - mgy\right]$$

$$H = \frac{p_y^2}{2m} + mgy$$



Applying Hamilton's canonical Equations

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -mg$$

Since $\dot{p}_y = m\ddot{y} = -mg$ or $\ddot{y} = -g$

Hamilton's Canonical Equation For particle under Central Force

Derive the Hamilton's equations and Hamiltonian in polar coordinates for a particle of mass m, which is under the influence of central potential Force (-*k*/*r*²)

Since $\mathrm{F} = -\nabla \mathrm{V} \Rightarrow \mathrm{V} = -\int F dr = -\int -\frac{k}{r^2} dr = -\frac{k}{r}$

and $T = \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2}$

$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{k}{r}$$

$$H = \sum_i p_i\, \dot{q}_i - L = p_r\, \dot{r} + p_\theta \dot{\theta} - L = \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \left[\frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + \frac{k}{r}\right] = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} - \frac{k}{r}$$

Applying Hamilton's canonical Equations

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \Rightarrow p_r = m\dot{r}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2} \Rightarrow m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{k}{r^2},$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \Rightarrow p_\theta = mr^2\dot{\theta} \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow p_\theta = \text{constant}$$

Hamilton's Canonical Equation For Projectile motion

Derive Hamilton's eq.s and Hamiltonian for projectile motion of a particle of mass m, in space

Sol: Let (x, y, z) be he coordinates of projectile in space at time "t"

Therefore $T = \frac{1}{2}m[\dot{x}^2 + \dot{y}^2 + \dot{z}^2] = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m}$ and $V = mgz$

$$\mathrm{L = T - V} = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} - mgz$$

$$H = \sum_i p_i\,\dot{q}_i - L = p_x\,\dot{x} + p_y\dot{y} + p_z\,\dot{z} - L = \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} - mgz\right]$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz$$

Applying Hamilton's canonical Equations

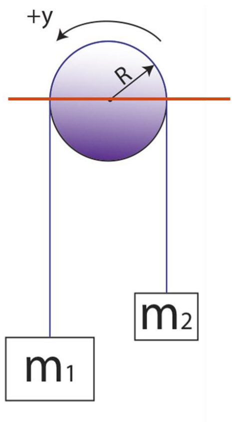
$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$, $\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$, $\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$

$\dot{p}_x = -\frac{\partial H}{\partial x} = 0$, $\dot{p}_y = -\frac{\partial H}{\partial y} = 0$ $\dot{p}_z = -\frac{\partial H}{\partial z} = -mg$ or $\ddot{z} = -g$

Hamilton's Canonical Equation For Atwood machine

Derive the Hamilton's equation and Hamiltonian for Atwood Machine with mass less support.

Atwood machine is a simple machine where two masses can move over a frictional less pully.



$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}(m_1 + m_2)\dot{y}^2 \quad \&$$

$$V = -m_1 gy - m_2 g(l - y) = -gy(m_1 - m_2) - m_2 gl$$

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + gy(m_1 - m_2) + m_2 gl$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = (m_1 + m_2)\dot{y} \qquad or \qquad \dot{y} = \frac{p_y}{(m_1+m_2)}$$

$$H = \sum_i p_i\,\dot{q}_i - L = p_y\dot{y} - L = p_y\frac{p_y}{(m_1+m_2)} - \left[\frac{1}{2}\frac{p_y^2}{(m_1+m_2)} + gy(m_1 - m_2) + m_2 gl\right]$$

$$\Rightarrow H = \frac{1}{2}\frac{p_y^2}{(m_1+m_2)} - gy(m_1 - m_2) - m_2 gl$$

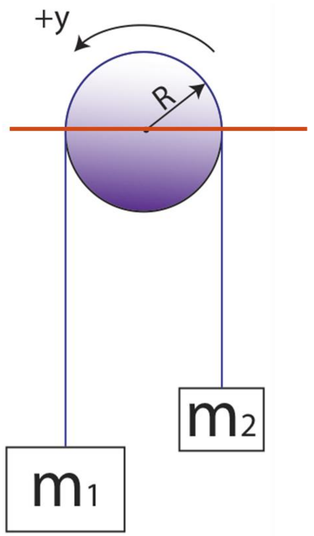
Using Hamilton's Equations $\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{(m_1+m_2)}$ & $\dot{p}_y = -\frac{\partial H}{\partial y} = g(m_1 - m_2)$

$$\Rightarrow \ddot{y} = g\,\frac{(m_1-m_2)}{(m_1+m_2)}$$

Hamilton's Canonical Equation For Atwood machine

Derive the Hamilton's equation and Hamiltonian for Atwood Machine for pully of moment of inertia I and Radius R.

Atwood machine is a simple machine where two masses can move over a frictional less pully.



$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(m_1+m_2)\dot{y}^2 + \frac{1}{2}I\frac{\dot{y}^2}{R^2} \quad \text{where } \dot{\theta} = \frac{\dot{y}}{R}$$

$$T = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y}^2 + \qquad \&$$

$$V = -m_1 gy - m_2 g(l - \pi R - y) = -gy(m_1 - m_2) - m_2 gl - m_2 g\pi R$$

$$L = T - V = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y}^2 + gy(m_1 - m_2) + m_2 gl - m_2 g\pi R$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y} \qquad or \qquad \dot{y} = \frac{p_y}{\left(m_1+m_2+\frac{I}{R^2}\right)}$$

$$H = \sum_i p_i\,\dot{q}_i - L = p_y\dot{y} - L = p_y\frac{p_y}{\left(m_1+m_2+\frac{I}{R^2}\right)} - \left[\frac{1}{2}\frac{p_y^2}{\left(m_1+m_2+\frac{I}{R^2}\right)} + gy(m_1 - m_2) + m_2 gl - m_2 g\pi R\right]$$

Hamilton's Canonical Equation For Atwood machine

$$\Rightarrow H = \frac{1}{2}\frac{p_y^2}{\left(m_1+m_2+\frac{I}{R^2}\right)} - gy(m_1 - m_2) - m_2 gl + m_2 g\pi R$$

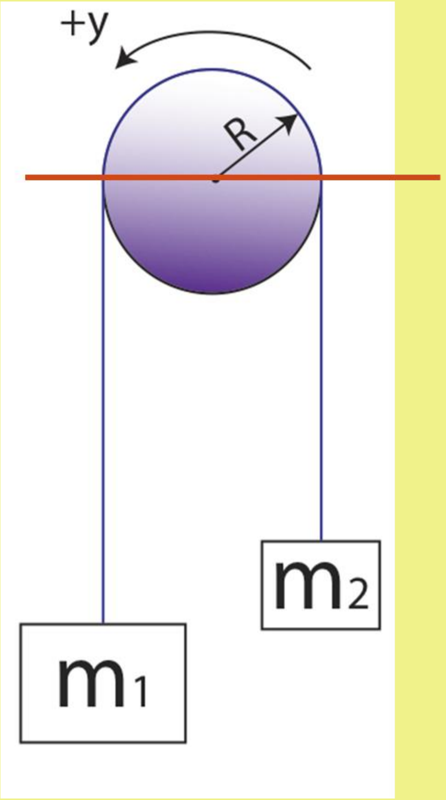
Using Hamilton's Equations

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{\left(m_1+m_2+\frac{I}{R^2}\right)}$$

$$\& \quad p_y = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{y}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = g(m_1 - m_2)$$

$$\Rightarrow \ddot{y} = g\frac{(m_1-m_2)}{\left(m_1+m_2+\frac{I}{R^2}\right)}$$



Derive Hamilton's Canonical Equation From Hamilton's Principle

Hamiltonian of a system is given as: $H = \sum_i p_i \, \dot{q}_i - L$

$\Rightarrow \quad L = \sum_i p_i \, \dot{q}_i - H$

Consider the system move from initial point to final position in time interval $\Delta t = t_2 - t_1$ by any two possible path. If δL is the variation in the Lagrangian of the System

$\Rightarrow \quad \delta L = \sum_i \delta p_i \, \dot{q}_i + \sum_i p_i \, \delta \dot{q}_i - \delta H$

Where $H = H(q_i, p_i)$

$$\delta H = \sum_i \frac{\partial H}{\partial q_i} \delta q_i + \sum_i \frac{\partial H}{\partial p_i} \delta p_i$$

$$\Rightarrow \quad \delta L = \sum_i \delta p_i \, \dot{q}_i + \sum_i p_i \, \delta \dot{q}_i - \delta H = \sum_i \delta p_i \, \dot{q}_i + \sum_i p_i \, \delta \dot{q}_i - \sum_i \frac{\partial H}{\partial q_i} \delta q_i - \sum_i \frac{\partial H}{\partial p_i} \delta p_i$$

Taking integral over both sides

$$\int_{t_1}^{t_2} \delta L dt = \sum_i \int_{t_1}^{t_2} \left(\delta p_i \, \dot{q}_i + p_i \, \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

Considering the term $\int_{t_1}^{t_2} p_i \, \delta \dot{q}_i dt = \int_{t_1}^{t_2} p_i \, \frac{d}{dt} \delta q_i dt = \cancelto{0}{|p_i \, \delta q_i|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{dp_i}{dt} \delta q_i dt = -\int_{t_1}^{t_2} \dot{p}_i \, \delta q_i dt$

Derive Hamilton's Canonical Equation From Hamilton's Principle

Therefore

$$\int_{t_1}^{t_2} \delta L dt = \sum_i \int_{t_1}^{t_2} \left(\dot{q}_i \delta p_i - \dot{p}_i \, \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

Since $\int_{t_1}^{t_2} \delta L dt = 0$ Hamilton's Principle

$$\sum_i \int_{t_1}^{t_2} \left(\delta p_i \, \dot{q}_i - \dot{p}_i \, \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) dt = 0$$

$$\Rightarrow \sum_i \int_{t_1}^{t_2} \left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right) dt = 0$$

Since δq_i and δp_i are not zero throughout the path. Therefore

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \quad \text{and} \quad \dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$

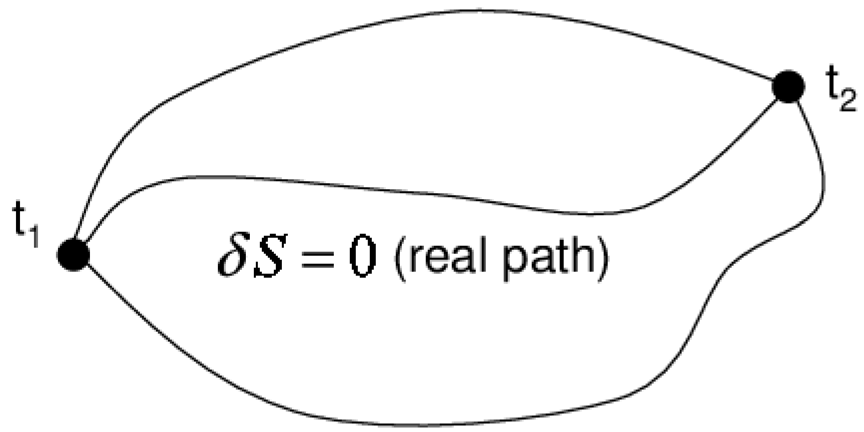
OR

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Derive Hamilton's principle using Newton's Law

Consider the case of a single particle. Let the particle move from r (t_1) to r (t_2) representing the position of particle at instants t_1 and t_2. Consider another path connected the same end points. Since the end points are the same for the paths

$$\delta r(t_1) = \delta r(t_2) = 0$$



Newton's equation of motion at any instant is

$$\mathrm{F} = m\ddot{r}$$

Let δW be the work done on a particle, then

$$\delta W = \bar{F} \cdot \delta\bar{r}$$

$$\delta W = m\ddot{\bar{r}} \cdot \delta\bar{r}$$

The total force acting on a particle will be

$\bar{F} = \bar{F}_a + \bar{F}_c$ and $\bar{F}_c \cdot \delta\bar{r} = 0$ work done by constraint force

Derive Hamilton's principle using Newton's Law

And the equation become

$$\delta W = \bar{F} \cdot \delta\bar{r} = \bar{F}_a \cdot \delta\bar{r}$$

If the applied force $\bar{F}_a$ is a conservative force and hence is derivable from a potential energy function V, Then

$$\delta W = \bar{F}_a \cdot \delta\bar{r} = -\bar{\nabla} V \cdot \delta\bar{r} = -\delta V$$

$$\delta W = -\delta V = m\ddot{\bar{r}} \cdot \delta\bar{r}$$

Consider a term

$$\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r}) = \dot{\bar{r}} \cdot \delta\dot{\bar{r}} + \ddot{\bar{r}} \cdot \delta\bar{r} = \delta\left(\frac{1}{2}\dot{r}^2\right) + \ddot{\bar{r}} \cdot \delta\bar{r}$$

Or $$\ddot{\bar{r}} \cdot \delta\bar{r} = \frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r}) - \delta\left(\frac{1}{2}\dot{\bar{r}}^2\right)$$

Therefore $$-\delta V = m\ddot{\bar{r}} \cdot \delta\bar{r} = m\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r}) - \delta\left(\frac{1}{2}m\dot{r}^2\right) = m\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r}) - \delta T$$

$$\delta T - \delta V = m\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r})$$

Derive Hamilton's principle using Newton's Law

$$\delta T - \delta V = m\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r})$$

$$\delta L = \delta T - \delta V = m\frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r})$$

Integrating with respect to time from t_1 to t_2

$$\int_{t_1}^{t_2} \delta L dt = m\int_{t_1}^{t_2} \frac{d}{dt}(\dot{\bar{r}} \cdot \delta\bar{r})dt = m|\dot{\bar{r}} \cdot \delta\bar{r}|_{t_1}^{t_2} = 0$$

because the variation in $\delta\bar{r}$ are zero at end points

Therefore

$$\delta \int_{t_1}^{t_2} L dt = 0$$

Which is Hamilton's principle

Derive Newton's Law from Hamilton's principle

Let us consider a particle of mass "m" at position $\vec{r} = \vec{r}(x, y, z, t)$ is moving under the action of a force F.

If force is conservative

$$\bar{F} = -\bar{\nabla} V$$

The virtual work done on particle by force F

$$\delta W = \bar{F} \cdot \delta\bar{r} = -\bar{\nabla} V \cdot \delta\bar{r}$$

Where the K.E of the particle will be $\frac{1}{2} m\dot{r}^2$

If the time integral of Lagrangian of a system is stationary (Hamilton's Principle)

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \delta(T - V) dt = \int_{t_1}^{t_2} \delta\left(\frac{1}{2} m\dot{r}^2 - V\right) dt = 0$$

$$\int_{t_1}^{t_2} (m\dot{\bar{r}} \cdot \delta\dot{\bar{r}} - \delta V) dt = 0$$

Considering the First term $\int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \delta\dot{\bar{r}} dt = \int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \frac{d}{dt} \delta\bar{r} dt$

Derive Newton's Law from Hamilton's principle

Considering the First term $\int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \delta\dot{\bar{r}}dt = \int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \frac{d}{dt}\delta\bar{r}dt$

$$\int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \delta\dot{\bar{r}}dt = m|\dot{\bar{r}} \cdot \delta\bar{r}|_{t_1}^{t_2} - m\int_{t_1}^{t_2} \frac{d}{dt}\dot{\bar{r}} \cdot \delta\bar{r}dt$$

Since the variation in $\delta\bar{r}$ are zero at end points

$$\int_{t_1}^{t_2} m\dot{\bar{r}} \cdot \delta\dot{\bar{r}}dt = -m\int_{t_1}^{t_2} \ddot{\bar{r}} \cdot \delta\bar{r}dt = -\int_{t_1}^{t_2} m\ddot{\bar{r}} \cdot \delta\bar{r}dt$$

Putting in Hamilton's Principle

$$\int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2}(-m\ddot{\bar{r}} \cdot \delta\bar{r} + \bar{F} \cdot \delta\bar{r})dt = \int_{t_1}^{t_2}(-m\ddot{\bar{r}} + \bar{F}) \cdot \delta\bar{r}dt = 0$$

Since variation $\delta\bar{r}$ is zero only at the end points but the variation $\delta\bar{r}$ is not zero through out the path and the above equation is zero through out the path.

Therefore $-m\ddot{\bar{r}} + \bar{F} = 0$

Or $\qquad \bar{F} = m\ddot{\bar{r}}$

Chapter 3

Lecture 5

Hamilton's Principle

Dr. Akhlaq Hussain

Application of Hamilton's Principle

Using Hamilton's principle to find the equation of motion of a particle of unit mass moving ion a plane in a conservative field

Solution: Let us consider a particle of mass (m=1) moving in xy-plane

And $\quad \mathrm{K.E} = \mathrm{T} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$

And $\quad F_x = -\frac{\partial V}{\partial x}$ and $F_y = -\frac{\partial V}{\partial y}$

Lagrangian of the system

$$L = T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V$$

Using Hamilton's Principle

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (\delta T - \delta V) dt$$

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - \delta V) dt$$

Application of Hamilton's Principle

Since

$$V = V(x, y)$$

$$\delta V = \frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y$$

Therefore, $\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} - \frac{\partial V}{\partial x}\delta x - \frac{\partial V}{\partial y}\delta y\right) dt$

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left(\dot{x}\delta\dot{x} - \frac{\partial V}{\partial x}\delta x\right) dt + \int_{t_1}^{t_2} \left(\dot{y}\delta\dot{y} - \frac{\partial V}{\partial y}\delta y\right) dt$$

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left(-\ddot{x}\delta x - \frac{\partial V}{\partial x}\delta x\right) dt + \int_{t_1}^{t_2} \left(-\ddot{y}\delta y - \frac{\partial V}{\partial y}\delta y\right) dt$$

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \left(-\ddot{x} - \frac{\partial V}{\partial x}\right) \delta x dt + \int_{t_1}^{t_2} \left(-\ddot{y} - \frac{\partial V}{\partial y}\right) \delta y dt = 0$$

Only if $-\ddot{x} - \frac{\partial V}{\partial x} = 0$ and $-\ddot{y} - \frac{\partial V}{\partial y} = 0$

Or $\ddot{x} = -\frac{\partial V}{\partial x}$ and $\ddot{y} = -\frac{\partial V}{\partial y}$

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Show that the equation of motion remain unchanged when a time derivative of some function is added to the Lagrangian.

Sol: Let f be a function dependent of q's and t and df/dt is added to a Lagrangian L, then the new Lagrangian L'

$$L'(q,\dot{q},t) = L(q,\dot{q},t) + \frac{df}{dt} \qquad \text{where } f = f(q,t)$$

Now the new Hamilton's function J'

$$J' = \int_{t_1}^{t_2} L'(q,\dot{q},t)dt = \int_{t_1}^{t_2} \left[L(q,\dot{q},t) + \frac{df}{dt}\right] dt$$

$$J' = \int_{t_1}^{t_2} \left[L(q,\dot{q},t) + \frac{df}{dt}\right] dt = J + \int_{t_1}^{t_2} \frac{df}{dt} dt$$

Now variation in J'

$$\delta J' = \delta J + \delta \int_{t_1}^{t_2} \frac{df}{dt} dt = \delta J + |\delta f|_{t_1}^{t_2}$$

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Now variation in J'

$$\delta J' = \delta J + \delta \int_{t_1}^{t_2} \frac{df}{dt} dt = \delta J + |\delta f|_{t_1}^{t_2}$$

Since $\quad f = f(q,t)$

$$\delta f = \sum \frac{\partial f}{\partial q} \delta q$$

Therefore,

$$\delta J' = \delta J + \delta \int_{t_1}^{t_2} \frac{df}{dt} dt = \delta J + \left| \sum \frac{\partial f}{\partial q} \delta q \right|_{t_1}^{t_2}$$

Since variation at the end points is zero i.e $\quad \delta q(t_1) = \delta q(t_2) = 0$

$$\Rightarrow \left| \sum \frac{\partial f}{\partial q} \delta q \right|_{t_1}^{t_2} = 0$$

Hence $\quad \delta J' = \delta J = 0$

Exercise 2 Goldstein Page 65

Suppose it is known experimentally that a particle fell a given distance y_o in a time $t_o = \sqrt{2y_o/g}$. The time of fall for the distance other than y_o are not known.

Suppose further that the Lagrangian for the problem is known but that instead of solving the equation of motion for y as a function of time t, it is guessed that the functional form is $y = at + bt^2$

If the constant a and b are adjusted always so that the time to fall y_o is correctly given by t_o. Show directly that the integral $\int_{t_1}^{t_2} L dt$ Is an extremum for real values of the coefficients only when a = 0 and b = g/2

Sol: The Lagrangian of the system or mass falling

$$L = \frac{1}{2} m\dot{y}^2 + mgy$$

Exercise 2 Goldstein Page 65

Since $y = at + bt^2$ and $\dot{y} = a + 2bt$

and $L = \frac{1}{2}m\dot{y}^2 + mgy = \frac{1}{2}m(a^2 + 4b^2t^2 + 4abt) + mg(at + bt^2)$

$$\int_0^t Ldt = \int_0^t \left[\frac{1}{2}m(a^2 + 4b^2t^2 + 4abt) + mg(at + bt^2)\right]dt$$

$$\int_0^t Ldt = \frac{1}{2}m\left(a^2t + \frac{4}{3}b^2t^3 + 2abt^2\right) + mg(\frac{1}{2}at^2 + \frac{1}{3}bt^3)$$

$$\int_0^t Ldt = \frac{1}{2}ma^2t + \frac{2}{3}mb^2t^3 + mabt^2 + \frac{1}{2}mgat^2 + \frac{1}{3}mgbt^3$$

Since $y_o = at_o + bt_o{}^2 \Rightarrow a = \frac{y_o - bt_o{}^2}{t_o}$

Putting in above equation

$$\int_{t_1}^{t_2} Ldt = \frac{1}{2}m\left(\frac{y_o - bt_o{}^2}{t_o}\right)^2 t_o + \frac{2}{3}mb^2t_o^3 + m\left(\frac{y_o - bt_o{}^2}{t_o}\right)bt_o^2 + \frac{1}{2}mg\left(\frac{y_o - bt_o{}^2}{t_o}\right)t_o^2 + \frac{1}{3}mgb\,t_o^3$$

For minimum value $\frac{d}{db}\int_{t_1}^{t_2} Ldt = 0$

Exercise 2 Goldstein Page 65

$$\int_{t_1}^{t_2} L dt = \frac{1}{2} m \left(\frac{y_o - bt_o{}^2}{t_o}\right)^2 t_o + \frac{2}{3} m b^2 t_o^3 + m\left(\frac{y_o - bt_o{}^2}{t_o}\right) b t_o^2 + \frac{1}{2} m g \left(\frac{y_o - bt_o{}^2}{t_o}\right) t_o^2 + \frac{1}{3} m g b\, t_o^3$$

$For\ minimum\ value$ $\qquad \frac{d}{db}\int_{t_1}^{t_2} L dt = 0$

$$\frac{d}{db}\int_{t_1}^{t_2} L dt = m\left(\frac{y_o - bt_o{}^2}{t_o}\right) t_o \left(\frac{-t_o{}^2}{t_o}\right) + \frac{4}{3} m b t_o^3 + m\left(\frac{y_o - bt_o{}^2}{t_o}\right) t_o^2 + m b t_o^2 \left(\frac{-t_o{}^2}{t_o}\right) + \frac{1}{2} m g t_o^2(-t_o) + \frac{1}{3} m g\, t_o^3$$

$$\frac{d}{db}\int_{t_1}^{t_2} L dt = -m(y_o - bt_o{}^2)t_o + \frac{4}{3} m b t_o^3 + m(y_o - bt_o{}^2)t_o - m b t_o^3 + \frac{1}{2} m g t_o^2(-t_o) + \frac{1}{3} m g\, t_o^3$$

$$\frac{d}{db}\int_{t_1}^{t_2} L dt = \cancel{-m y_o t_o} + \cancel{m b t_o{}^3} + \frac{4}{3} m b t_o^3 + \cancel{m y_o t_o} - \cancel{m b t_o{}^3} - m b t_o^3 - \frac{1}{2} m g t_o^3 + \frac{1}{3} m g\, t_o^3$$

$$\frac{d}{db}\int_{t_1}^{t_2} L dt = \frac{1}{3} m b t_o^3 - \frac{1}{6} m g t_o^3 = m t_o^3\left(\frac{1}{3} b - \frac{1}{6} g\right) = 0$$

$$\Rightarrow \left(\frac{1}{3} b - \frac{1}{6} g\right) = 0$$

$$\Rightarrow \left(b - \frac{1}{2} g\right) = 0 \qquad\qquad \Rightarrow b = \frac{1}{2} g$$

Exercise 2 Goldstein Page 65

Putting $t_o = \sqrt{{}^{2y_o}/_{g}}$ & $b = \frac{1}{2} g$

in $a = \frac{y_o - b t_o{}^2}{t_o}$

$$a = \frac{y_o - \left(\frac{g}{2}\right)\left(\frac{2y_o}{g}\right)}{t_o} = 0$$

Now using another technique

Since $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = m\ddot{y} - mg = 0 \Rightarrow \ddot{y} = g$

$\ddot{y} = g$

Since $y = at + bt^2$

Exercise 2 Goldstein Page 65

Differentiating above equation

$$\dot{y} = a + 2bt$$

$$\ddot{y} = 2b = g$$

Or $$b = \frac{g}{2}$$

Now at time t_o the equation will be

$$y_o = at_o + bt_o^2$$

$$y_o = a\sqrt{2y_o/g} + b(2y_o/g)$$

$$1 = a\sqrt{2/y_o g} + b(2/g) = a\sqrt{2/y_o g} + \frac{g}{2}(2/g)$$

$$1 = a\sqrt{2/y_o g} + 1 \Rightarrow a = 0$$

Hamiltonian for charge Particle in E.M Field

Consider a charge particle of mass m with charge q moving with a velocity v in an electromagnetic field. The Lagrangian of charge particle is

$$L = \frac{1}{2}mv^2 - q\varphi + q(\boldsymbol{v}.\boldsymbol{A}) \qquad or \qquad L = \frac{1}{2}mv^2 - q\varphi + \frac{q}{c}(\boldsymbol{v}.\boldsymbol{A})$$

And Hamiltonian of system

$$H = \boldsymbol{p}.\boldsymbol{v} - L$$

Now $\quad \boldsymbol{p} = \frac{\partial L}{\partial v} = m\boldsymbol{v} + q\boldsymbol{A}$

And $\quad \boldsymbol{v} = \frac{1}{m}(\boldsymbol{p} - q\boldsymbol{A})$

$$H = (m\boldsymbol{v} + q\boldsymbol{A}) \cdot \boldsymbol{v} - \frac{1}{2}mv^2 + q\varphi - q(\boldsymbol{v}.\boldsymbol{A})$$

$$H = mv^2 + q(\boldsymbol{v}.\boldsymbol{A}) - \frac{1}{2}mv^2 + q\varphi - q(\boldsymbol{v}.\boldsymbol{A})$$

$$H = \frac{1}{2}mv^2 + q\varphi = \frac{1}{2m}[\boldsymbol{p} - q\boldsymbol{A}]^2 + q\varphi$$

Hamiltonian for charge Particle in E.M Field

And $$H = \frac{1}{2}mv^2 + q\varphi = \frac{1}{2m}[p - q\boldsymbol{A}]^2 + q\varphi$$

Now $$\dot{p} = \frac{dp}{dt} = \frac{d}{dt}[m\boldsymbol{v} + q\boldsymbol{A}] = m\boldsymbol{a} + q\frac{d\boldsymbol{A}}{dt}$$

And $$\dot{p} = -\frac{\partial H}{\partial r} = -\overline{\nabla} H = -\boldsymbol{\nabla}\left[\frac{1}{2m}[p - q\boldsymbol{A}]^2 + q\varphi\right] = -\left[-\frac{q}{m}[p - q\boldsymbol{A}]\cdot\boldsymbol{\nabla}\boldsymbol{A} + q\boldsymbol{\nabla}\varphi\right]$$

$$m\boldsymbol{a} + q\frac{d\boldsymbol{A}}{dt} = -[-q(\boldsymbol{v}\cdot\overline{\nabla})\boldsymbol{A} + q\overline{\nabla}\varphi] \quad \text{Because } \frac{1}{m}[p - q\boldsymbol{A}] = \boldsymbol{v}$$

$$m\boldsymbol{a} = -q\frac{d\boldsymbol{A}}{dt} - [-q(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + q\boldsymbol{\nabla}\varphi] = q\left[-\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t}\right] + q\left[(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \frac{\partial \boldsymbol{A}}{\partial t} - \frac{d\boldsymbol{A}}{dt}\right]$$

$$m\boldsymbol{a} = q\boldsymbol{E} + q\left(\boldsymbol{v}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\right) = q\boldsymbol{E} + q\,(\boldsymbol{v}\times\boldsymbol{B})$$

Hamilton's Principle for conservative system

Show that $\delta \int_A^B \sum_i p_i \; dq_i \; = 0$

Since $\int \delta L dt = 0$

$$\int \delta L dt \; = \delta \int (\sum_i p_i \, \dot{q}_i - H \,) dt = 0$$

$$\delta \int \left(\sum_i p_i \, \frac{dq_i}{dt} - H \right) dt = \delta \int \sum_i p_i \, \frac{dq_i}{dt} dt - \int \delta H dt = 0$$

$$\delta \int \sum_i p_i \, \frac{dq_i}{dt} dt - \int \delta H dt = 0$$

Since variation in H is zero, therefore the second term will vanish. Or the Hamiltonian H has the same constant value in both varied and actual motion. Therefore

$$\boxed{\int_{t_1}^{t_2} \delta L dt = \delta \int_A^B \sum_i p_i \; dq_i = 0}$$

$$\int_{t_1}^{t_2} \delta L dt = \sum_i \int_A^B \delta p_i \, dq_i + \sum_i \int_A^B p_i \, d\delta q_i$$

Hamilton's Principle for conservative system

Show that $\delta \int_A^B \sum_i p_i \; dq_i \; = 0$

$$\delta \int_A^B \sum_i p_i \; dq_i = \sum_i \int_A^B \delta p_i \, dq_i + \sum_i \int_A^B p_i \, d\delta q_i$$

Integrating Second term

$$\sum_i \int_A^B \delta p_i \, dq_i + \sum_i \int_A^B p_i \, d\delta q_i = \sum_i \int_A^B \delta p_i \, dq_i + \overset{0}{\sum_i |p_i \, \delta q_i|_A^B} - \sum_i \int_A^B dp_i \, \delta q_i$$

$$\sum_i \int_A^B \delta p_i \, dq_i + \sum_i \int_A^B dp_i \, \delta q_i = \sum_i \int_A^B \delta p_i \, dq_i - \sum_i \int_A^B dp_i \, \delta q_i$$

Since $\qquad dq_i = \frac{\partial H}{\partial p_i} dt \qquad$ and $\qquad dp_i = -\frac{\partial H}{\partial q_i} dt$

Therefore, $\quad \delta \int_A^B \sum_i p_i \; dq_i = \sum_i \int_A^B \delta p_i \, dq_i - \sum_i \int_A^B dp_i \, \delta q_i$

$$\delta \int_A^B \sum_i p_i \; dq_i \; = \sum_i \int_A^B \frac{\partial H}{\partial p_i} \delta p_i \, dt + \sum_i \int_A^B \frac{\partial H}{\partial q_i} \delta q_i dt = 0$$

$$\delta \int_A^B \sum_i p_i \; dq_i \; = \int_A^B \delta H dt = 0$$

Hamilton's Principle for conservative system

Show that $\delta \int T dt = 0$ & $\delta \int_A^B [m(E-V)]^{1/2} dl = 0$

Since we know that $\delta \int_A^B \sum_i p_i \, dq_i = 0$

and $H = T + V$ for conservative and holonomic system.

Where V is independent of velocity therefore

$$\frac{\partial H}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = p_i$$

$$\Rightarrow \delta \int_A^B \sum_i \frac{\partial T}{\partial \dot{q}_i} dq_i = 0$$

$$\Rightarrow \delta \int_A^B \sum_i \frac{\partial T}{\partial \dot{q}_i} \frac{dq_i}{dt} dt = 0$$

$$\Rightarrow \delta \int_A^B \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} dt = 0$$

Since $\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial}{\partial \dot{q}_i} \left(\sum_j \frac{1}{2} m_j \dot{r}_j^2 \right)$

Hamilton's Principle for conservative system

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i \left(\sum_j m_j \dot{r}_j \frac{\partial \dot{r}_j}{\partial \dot{q}_i}\right) = \sum_j m_j \dot{r}_j \sum_i \frac{\partial \dot{r}_j}{\partial \dot{q}_i} \dot{q}_i$$

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_j m_j \dot{r}_j \sum_i \frac{\partial r_j}{\partial q_i} \dot{q}_i$$

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_j m_j \dot{r}_j \sum_i \frac{\partial r_j}{\partial q_i} \dot{q}_i = \sum_j m_j \dot{r}_j \cdot \dot{r}_j = 2T$$

Therefore, $\delta \int_A^B \sum_i p_i \, dq_i = \delta \int_A^B \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} dt = 2\delta \int_{t_1}^{t_2} T dt = 0$

$$\Rightarrow \delta \int_{t_1}^{t_2} T dt = 0$$

It is another form of Hamilton's Principle.

Since $\quad E = T + V$

Or $\quad T = E - V$

$$\Rightarrow v = \left[\frac{2(E-V)}{m}\right]^{1/2}$$

Hamilton's Principle for conservative system

$$\Rightarrow \frac{dl}{dt} = \left[\frac{2(E-V)}{m}\right]^{1/2}$$

$$\Rightarrow dt = \left[\frac{m}{2(E-V)}\right]^{1/2} dl$$

Putting in Previous equation.

$$\delta \int_A^B T \left[\frac{m}{2(E-V)}\right]^{1/2} dl = 0$$

$$\Rightarrow \delta \int_A^B (E-V) \left[\frac{m}{2(E-V)}\right]^{1/2} dl = 0$$

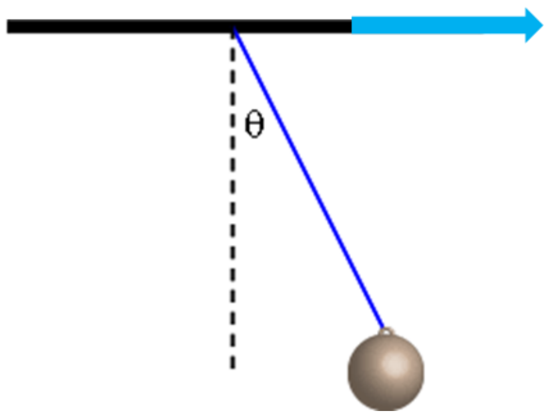
$$\Rightarrow \delta \int_A^B \left[\frac{m(E-V)}{2}\right]^{1/2} dl = 0$$

$$\Rightarrow \delta \int_A^B [m(E-V)]^{1/2} dl = 0$$

Applications of Hamilton's Equation of Motion

Derive Hamiltonian and Hamilton's Equation of motion for simple pendulum constraint to move along horizontal straight line.

Consider a simple pendulum which move along horizontal x-ais and vibrate along vertical y-axis



$$x' = x + l\sin\theta \Rightarrow \dot{x}' = \dot{x} + \dot{\theta} l\, cos\,\theta$$

$$y' = l\cos\theta \qquad \Rightarrow \dot{y}' = -\dot{\theta} l\, sin\,\theta$$

$$\mathrm{T} = \frac{1}{2}m\left(\dot{x}'^2 + \dot{y}'^2\right) = \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta\right)$$

& $\quad \mathrm{V} = -mgy = -mgl\cos\theta$

$$\mathrm{L} = \mathrm{T} - \mathrm{V} = \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x}\dot{\theta}l\cos\theta\right) + mgl\cos\theta$$

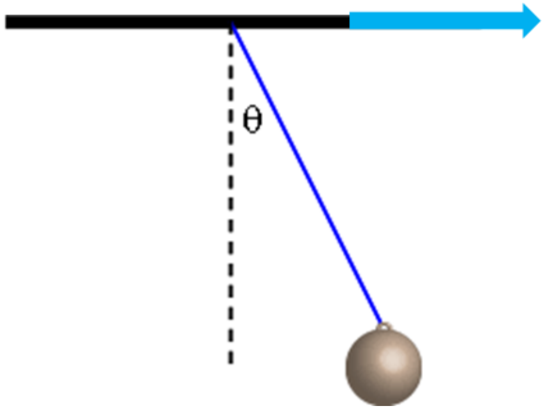
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\left(\dot{x} + \dot{\theta}l\cos\theta\right) \qquad \Rightarrow \frac{p_x}{m} = \dot{x} + \dot{\theta}l\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\left(l^2\dot{\theta} + \dot{x}l\cos\theta\right) \quad \Rightarrow \frac{p_\theta}{ml} = \left(l\dot{\theta} + \dot{x}\cos\theta\right)$$

Applications of Hamilton's Equation of Motion

$$\frac{p_\theta}{ml}\cos\theta = l\dot{\theta}\cos\theta + \dot{x}\cos^2\theta$$

And $$\frac{p_\theta}{ml}\cos\theta - \frac{p_x}{m} = l\dot{\theta}\cos\theta + \dot{x}\cos^2\theta - \dot{x} - \dot{\theta}l\cos\theta$$



$$\frac{p_\theta}{ml}\cos\theta - \frac{p_x}{m} = \dot{x}\cos^2\theta - \dot{x}$$

$$\frac{p_\theta}{ml}\cos\theta - \frac{p_x}{m} = -\dot{x}(1-\cos^2\theta)$$

$$\frac{p_\theta}{ml}\cos\theta - \frac{p_x}{m} = -\dot{x}\,sin^2\theta$$

$$\dot{x} = \frac{1}{m\,sin^2\theta}\left(p_x - \frac{p_\theta}{l}\cos\theta\right)$$

And from same equation

$$l\dot{\theta} = \frac{p_\theta}{ml} - \dot{x}\cos\theta = \frac{p_\theta}{ml} - \frac{\cos\theta}{m\,sin^2\theta}\left(p_x - \frac{p_\theta}{l}\cos\theta\right)$$

$$l\dot{\theta} = \frac{1}{m\,sin^2\theta}\left(\frac{p_\theta}{l} - p_x\cos\theta\right)$$