Chapter 4
Lecture 1
Two body central Force Problem

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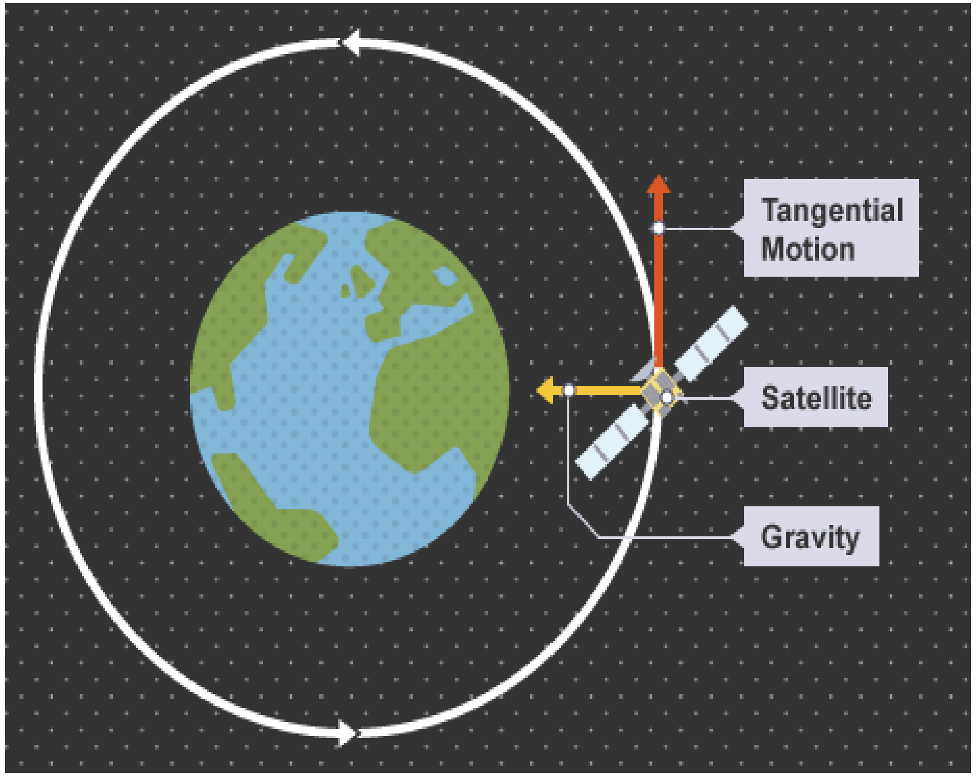
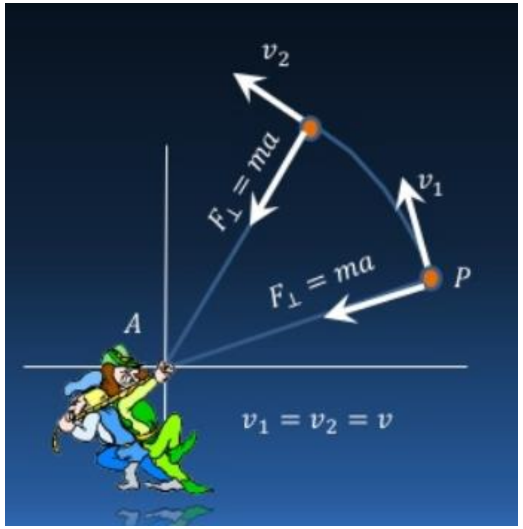
TWO BODY CENTRAL FORCE PROBLEM

- In this chapter, we will study
- Two-body problem,
- Reduction of two-body problem to equivalent one-body problem.
- Central Force
- Keppler's Laws and equation of the orbit
- First integrals

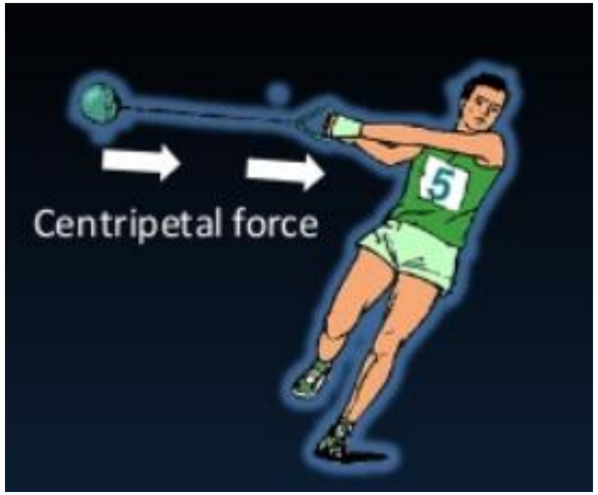


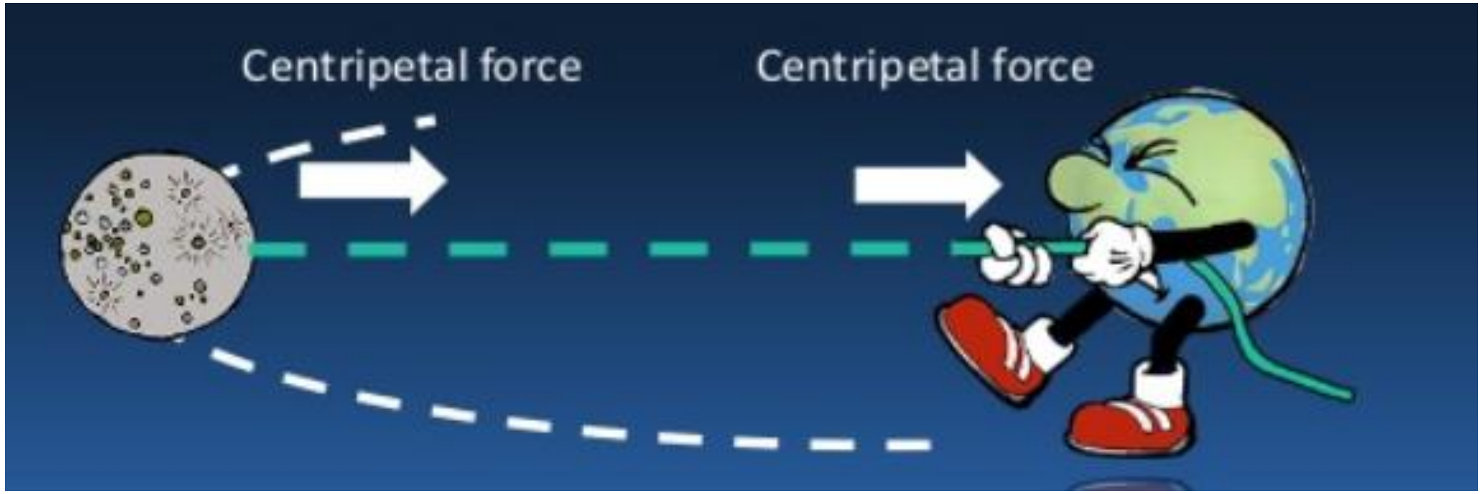
TWO BODY CENTRAL FORCE PROBLEM

4.1 Introduction



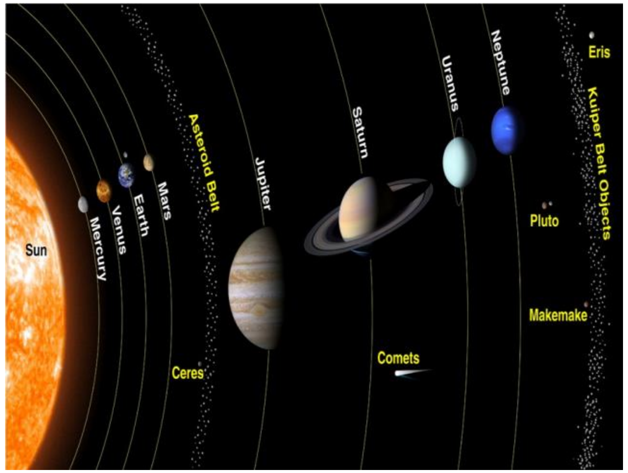
- One of the most important problems of classical mechanics is to understand the motion of a body moving under the influence of a central force field.
- Force which is always directed towards the centre or line joining two bodies
- The motion of the planets around the sun.





TWO BODY CENTRAL FORCE PROBLEM

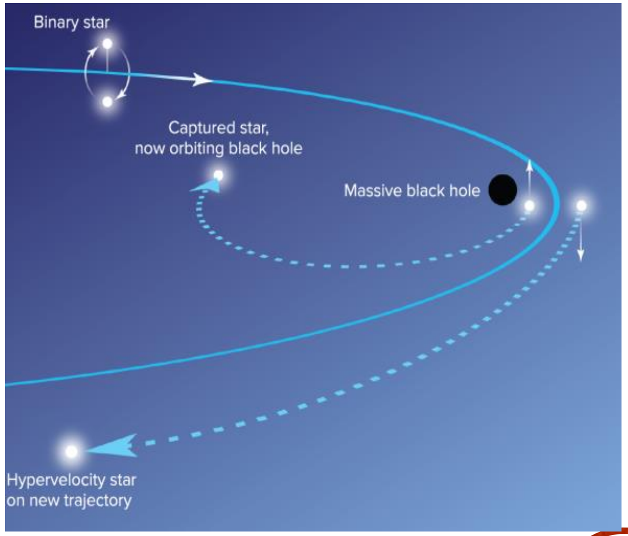
The motion in central force field can be classified as;



1) Bound motion

The distance between two bodies never exceeds a finite limit, e.g. motion of planets around the sun.

2) Unbound motion



The distance between two particles or bodies is infinite at initial and final stage.

The bodies move from infinite distance and approach to interact in close proximity

Finally move far from each other to an infinite distance.

For example, scattering of alpha particles by gold nuclei as studied by Rutherford.

TWO BODY CENTRAL FORCE PROBLEM

- It is always possible to reduce the motion of two bodies to that of an equivalent single-body problem.
- The exact solution and understanding of two bodies motion problem is possible.
- However, the presence of the third body complicates the situation and an exact solution to the problem become an impossibility.
- Therefore, one must adopt the approximate methods to solve the many bodies problem.
- We can always reduce many body systems to a two-body problem either by neglecting the effects of the others or by some other screening methods, where the effects of the other bodies don’t play prominent role.
- Such as the motion of a planets around the sun, where the effect due to the presence of other planets is neglected. However, we will restrict ourselves to the two bodies problem only.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

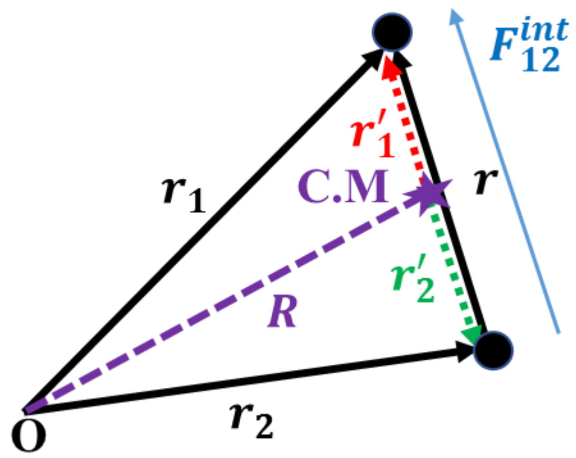
Consider the motion of two particles. Let $F^{(ext)}$ be the total external force acting on the system. Let F^{int} be the total internal force due to the interaction between two particles.

Total external force will be

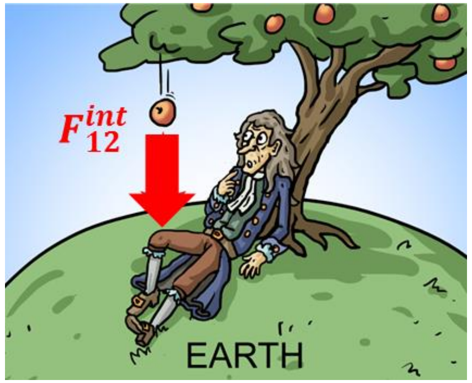
$$\boldsymbol{F}^{ext} = \boldsymbol{F}_1^{ext} + \boldsymbol{F}_2^{ext} \qquad \textbf{(4.1.1)}$$

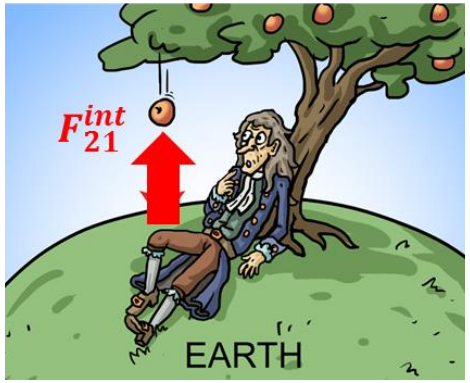
Further according to the Newton's 3rd law

$$\boldsymbol{F}_{12}^{int} = -\boldsymbol{F}_{21}^{int}` \qquad \textbf{(4.1.2)}$$



Action and reaction forces.





If the action and reaction forces are same Why only apple falls for earth?

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

The equations of motion can be written as

Force on Particle 1

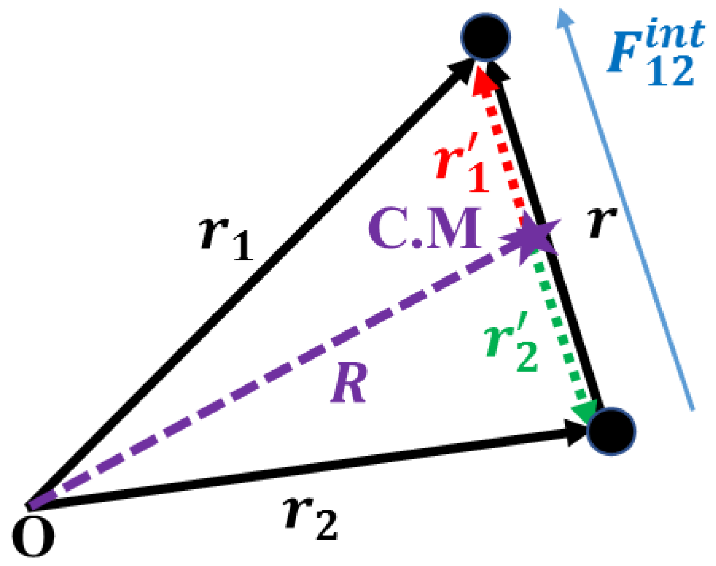
$$m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1^{ext} + \boldsymbol{F}_{12}^{int} \qquad \textbf{(4.1.3)}$$

Force on Particle 2

$$m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2^{ext} + \boldsymbol{F}_{21}^{int} \qquad \textbf{(4.1.4)}$$

Total force on system of Particles

$$\boldsymbol{F}^{ext} = M\ddot{\boldsymbol{R}} \qquad \textbf{(4.1.5)}$$



Total mass of the system

$$M = m_1 + m_2 \qquad \textbf{(4.1.6)}$$

Position vector of the centre of mass of the system is

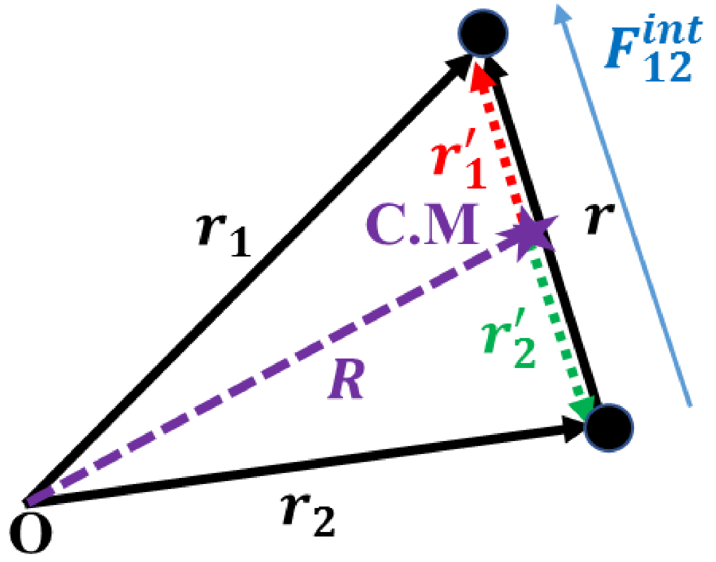
$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \qquad \textbf{(4.1.7)}$$

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Position vector of particle 1 relative to particle 2 be

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \textbf{(4.1.8)}$$

$$\boldsymbol{r}_1 = \boldsymbol{r} + \boldsymbol{r}_2 \qquad \textbf{(4.1.9)}$$



Putting Eq. (**4.1.9**) in Eq. (**4.1.7**) $\{\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2}\}$

$$\boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1\boldsymbol{r}}{m_1+m_2} \qquad \textbf{(4.1.10)}$$

Similarly, Eq. (**4.1.8**) can be written as

$$\boldsymbol{r}_2 = \boldsymbol{r}_1 - \boldsymbol{r} \qquad \textbf{(4.1.11)}$$

Putting in equation **(4.1.11)** in equation **(4.1.7)** $\{\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1+m_2}\}$

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2\boldsymbol{r}}{m_1+m_2} \qquad \textbf{(4.1.12)}$$

Multiplying Eq. (4.1.3) $\{m_1\ddot{r}_1 = F_1^{ext} + F_{12}^{int}\}$ by m_2 & Eq. (4.1.4) $\{m_2\ddot{r}_2 = F_2^{ext} + F_{21}^{int}\}$ by m_1 and subtracting,

$$m_1m_2(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = m_2\boldsymbol{F}_{12}^{int} - m_1\boldsymbol{F}_{21}^{int} + m_1m_2\left(\frac{\boldsymbol{F}_1^{ext}}{m_1} - \frac{\boldsymbol{F}_2^{ext}}{m_2}\right)$$

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Dividing the above equation by $(m_1 + m_2)$ and using $\boldsymbol{F}_{12}^{int} = -\boldsymbol{F}_{21}^{int}$

$$\frac{m_1 m_2}{(m_1+m_2)}(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \frac{(m_1+m_2)}{(m_1+m_2)}\boldsymbol{F}_{12}^{int} + \frac{m_1 m_2}{(m_1+m_2)}\left(\frac{\boldsymbol{F}_1^{ext}}{m_1} - \frac{\boldsymbol{F}_2^{ext}}{m_2}\right)$$

$$\Rightarrow \mu(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \boldsymbol{F}_{12}^{int} + \mu\left(\frac{\boldsymbol{F}_1^{ext}}{m_1} - \frac{\boldsymbol{F}_2^{ext}}{m_2}\right) \qquad \textbf{(4.1.13)}$$

Where μ is reduce mass of the system.

$$\mu = \frac{m_1 m_2}{(m_1+m_2)} \; or \; \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \qquad \textbf{(4.1.14)}$$

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Special case

If no external force is acting

$\boldsymbol{F}_1^{ext} = \boldsymbol{F}_2^{ext} = \boldsymbol{0}$ **(A)**

equation **(4.1.13)** will be reduced to

$\Rightarrow \mu(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \boldsymbol{F}_{12}^{int}$

$\Rightarrow \mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}^{int}$ **(4.1.15)a**

If the forces produce same acceleration

$\frac{\boldsymbol{F}_1^{ext}}{\boldsymbol{m}_1} = \frac{\boldsymbol{F}_2^{ext}}{\boldsymbol{m}_2}$ **(B)**

The condition **B** is realized if centre producing the external forces is at a considerable distance from the system and the force due to it on any mass is proportional to that of the mass.

Such as gravitational force. In Earth-moon mutual motion, force due to the sun is assumed such that it satisfy the condition mentioned in Eq. **B**.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Equation will be reduced to

$$\Rightarrow \mu(\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2) = \boldsymbol{F}_{12}^{int}$$

$$\Rightarrow \mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}^{int} \qquad \textbf{(4.1.15)b}$$

Eq. **(4.1.15)b** represent motion of a particle of mass equal μ and moving under the action of force $\boldsymbol{F}_{12}^{int}$.

The reduction is equivalent to replace the system of two bodies by a mass μ and considering the acceleration produced is due to the internal force.

Eq. **(4.1.15)a** ($\mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_{12}^{int}$) together with Eq. **(4.1.5)** ($\boldsymbol{F}^{ext} = M\ddot{\boldsymbol{R}}$) represents the motion of a two body system under the action of internal and external forces as long as the conditions mentioned in equations **A & B** are valid.

If the internal forces are attractive and these are the only forces acting on the system, the two bodies move around the centre of mass which acts as centre of force. i.e. directed towards the centre.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Condition on mass

If the mass of one of the particles is extremely large as compared to that of the other, say $m_1 >> m_2$, then the reduced mass is simply

$$\mu = \frac{m_1 m_2}{(m_1+m_2)} = \frac{m_1 m_2}{m_1(1+{}^{m_2}/_{m_1})}$$

$$\Rightarrow \mu = \frac{m_2}{(1+{}^{m_2}/_{m_1})} \qquad \text{as} \qquad {}^{m_2}/_{m_1} \approx 0$$

$$\Rightarrow \mu = m_2$$

In this case the centre of mass of the system coincides with the centre of mass of the heavier body.

This approximation is equivalent to neglecting the recoil of mass m_1. This is used in Bohr's theory of hydrogen atom and motion of satellites around the earth. It can be assumed for the motion of earth around the Sun.

REDUCTION OF TWO-BODY PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

Since mass $m_1 >> m_2$,
acceleration in mass m_1

$$\boldsymbol{a_1} = \frac{\boldsymbol{F_{12}^{int}}}{\boldsymbol{m_1}} \approx \boldsymbol{0} \qquad \text{or very small}$$

acceleration in mass m_2

$$\boldsymbol{a_2} = \frac{\boldsymbol{F_{21}^{int}}}{\boldsymbol{m_2}} > \boldsymbol{0}$$

That’s is why

“An apple appears to fall towards the earth and not the earth towards the apple”.

Lagrangian of the System

If $U(\boldsymbol{r},\dot{\boldsymbol{r}})$ is the function of "$\boldsymbol{r}$"and higher derivative of "$\dot{\boldsymbol{r}}$". Then Lagrangian of the system

$$L = T(\dot{\boldsymbol{R}},\dot{\boldsymbol{r}}) - U(\boldsymbol{r},\dot{\boldsymbol{r}}) \qquad \textbf{(4.1.16)}$$

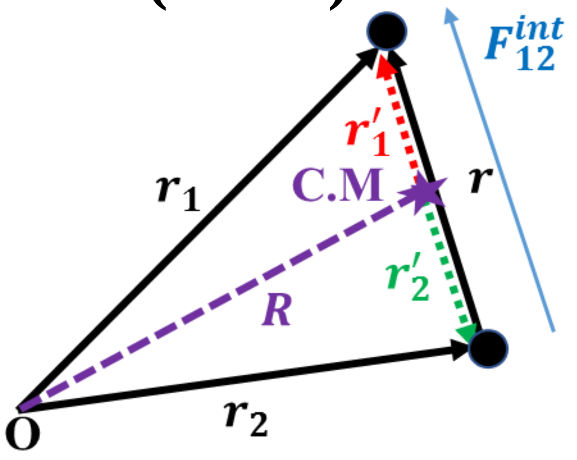
Where $\quad T(\dot{\boldsymbol{R}},\dot{\boldsymbol{r}}) = {}^{1}/_{2}\, M\dot{\boldsymbol{R}}^{\boldsymbol{2}} + T' = {}^{1}/_{2}\,(m_1+m_2)\dot{\boldsymbol{R}}^{\boldsymbol{2}} + T' \quad$ **(4.1.17)**

And $\quad T' = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1'^{\,\boldsymbol{2}} + \frac{1}{2}m_2\dot{\boldsymbol{r}}_{\boldsymbol{2}}'^{\,\boldsymbol{2}} \quad$ **(4.1.18)**

Where $\quad \boldsymbol{r}_{\boldsymbol{1}}' = \boldsymbol{r}_{\boldsymbol{1}} - \boldsymbol{R}$

$$\Rightarrow \boldsymbol{r}_1' = \boldsymbol{r}_{\boldsymbol{1}} - \frac{m_1\boldsymbol{r}_{\boldsymbol{1}}+m_2\boldsymbol{r}_{\boldsymbol{2}}}{(m_1+m_2)} = \frac{m_2(\boldsymbol{r}_{\boldsymbol{1}}-\boldsymbol{r}_{\boldsymbol{2}})}{(m_1+m_2)}$$

$$\Rightarrow \boldsymbol{r}_1' = \frac{m_2}{(m_1+m_2)}\boldsymbol{r} \qquad \textbf{(4.1.19)}$$



Similarly, $\quad \boldsymbol{r}_2' = \boldsymbol{r}_{\boldsymbol{2}} - \boldsymbol{R}$

$$\Rightarrow \boldsymbol{r}_{\boldsymbol{2}}' = \boldsymbol{r}_{\boldsymbol{2}} - \frac{m_1\boldsymbol{r}_{\boldsymbol{1}}+m_2\boldsymbol{r}_{\boldsymbol{2}}}{(m_1+m_2)} = -\frac{m_1(\boldsymbol{r}_{\boldsymbol{1}}-\boldsymbol{r}_{\boldsymbol{2}})}{(m_1+m_2)} = -\frac{m_1}{(m_1+m_2)}\boldsymbol{r} \qquad \textbf{(4.1.20)}$$

Lagrangian of the System

Therefore, the kinetic energy from Eq **(4.1.18)** can be written as

$$T' = \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \dot{\boldsymbol{r}} \right)^{\boldsymbol{2}} + \frac{1}{2} m_2 \left(- \frac{m_1}{m_1 + m_2} \dot{\boldsymbol{r}} \right)^{\boldsymbol{2}}$$

$$\Rightarrow T' = \frac{1}{2} (m_2 + m_1) \frac{m_1 m_2}{(m_1 + m_2)^2} \dot{\boldsymbol{r}}^{\boldsymbol{2}}$$

$$\Rightarrow T' = \frac{1}{2} \frac{m_1 m_2}{(m_2 + m_1)} \dot{\boldsymbol{r}}^{\boldsymbol{2}} \qquad \textbf{(4.1.21)}$$

The Lagrangian of the system can be written as;

$$L = T\left(\dot{\boldsymbol{R}}, \dot{\boldsymbol{r}} \right) - U(\boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$L = {}^1/_2 \, (m_1 + m_2) \dot{\boldsymbol{R}}^{\boldsymbol{2}} + \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{\boldsymbol{r}}^{\boldsymbol{2}} - U(\boldsymbol{r}, \dot{\boldsymbol{r}})$$

$$L = {}^1/_2 \, M \dot{\boldsymbol{R}}^{\boldsymbol{2}} + \frac{1}{2} \mu \dot{\boldsymbol{r}}^{\boldsymbol{2}} - U(\boldsymbol{r}, \dot{\boldsymbol{r}}) \qquad \textbf{(4.1.22)}$$

Where M is the total mass of the system and μ is the reduce mass of the system.

4.2 Properties of central Force

4.2.1a Under the central force, the angular momentum of the particle is conserved

a. In cartesian coordinates

The Torque on the system (if any) can be written as; $\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F}$ **(4.2.1)**

and the angular momentum of the body is $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P}$ **(4.2.2)**

We know that; $\frac{d\boldsymbol{l}}{dt} = \boldsymbol{N}$ **(4.2.3)**

Since the force acting on the body is central force and always directed towards the line joining the body with the centre therefore

$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F} = r\hat{r} \times F_r\hat{r} = rF_r(\hat{r} \times \hat{r}) = 0 \qquad \textbf{(4.2.4)}$$

$$\Rightarrow \frac{d\boldsymbol{l}}{dt} = \boldsymbol{N} = \boldsymbol{0} \Rightarrow \boldsymbol{l} = Constant \qquad \textbf{(4.2.5)}$$

Eq.**(4.2.4) & (4.2.5)** suggests that the total torque "$\boldsymbol{N}$" acting on the system will be zero and angular momentum "$\boldsymbol{l}$" of the body will be constant.

4.2 Properties of central Force

b. In Polar coordinates

$$\boldsymbol{F} = F_r\hat{r} + F_\theta\hat{\theta} \qquad \textbf{(4.2.6)}$$

And similarly, the torque acting on a particle in polar coordinates is

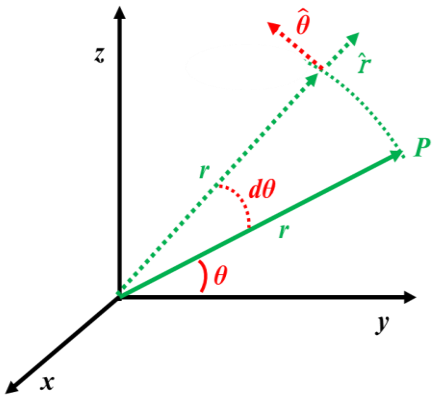
$$\boldsymbol{N} = \boldsymbol{r} \times \boldsymbol{F} = r\hat{r} \times \left[\left(m\ddot{r} - mr\dot{\theta}^2\right)\hat{r} + \left(mr\ddot{\theta} + 2m\dot{r}\dot{\theta}\right)\hat{\theta}\right]$$

$$\Rightarrow \boldsymbol{N} = r\left(m\ddot{r} - mr\dot{\theta}^2\right)(\hat{r} \times \hat{r}) + r\left(mr\ddot{\theta} + 2m\dot{r}\dot{\theta}\right)\left(\hat{r} \times \hat{\theta}\right)$$

$$\Rightarrow \boldsymbol{N} = 0 + r\left(mr\ddot{\theta} + 2m\dot{r}\dot{\theta}\right)\left(\hat{r} \times \hat{\theta}\right)$$

$$\Rightarrow \boldsymbol{N} = \left(mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}\right)\hat{n}$$ where $\hat{n}$ is $\perp$ to both $\hat{r}$ and $\hat{\theta}$

$$\Rightarrow \boldsymbol{N} = \frac{d}{dt}\left(mr^2\dot{\theta}\right)\hat{n} \qquad \textbf{(4.2.7)}$$



For Radial force, the angular part of the force is zero

$$\boldsymbol{N} = \frac{d\boldsymbol{l}}{dt} = 0 \Rightarrow \boldsymbol{l} = mr^2\dot{\theta} = Constant \qquad \textbf{(4.2.8)}$$

Note: Also, $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P} = \boldsymbol{r} \times m\boldsymbol{v} = \boldsymbol{r} \times mr\boldsymbol{\omega} = \boldsymbol{r} \times mr\dot{\boldsymbol{\theta}} \Rightarrow |\boldsymbol{L}| = mr^2\dot{\theta}$

4.2.2 The path of a particle moving under the central force must be a Plane

Consider the central force $\boldsymbol{F} = F_r \hat{r}$ **(4.2.9)**

Taking cross product with radius vector of above equation

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} F_r (\hat{r} \times \hat{r}) = 0$$

$$\Rightarrow \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{r} \times m \frac{d\boldsymbol{v}}{dt} = 0$$

$$\Rightarrow \boldsymbol{r} \times m \frac{d\boldsymbol{v}}{dt} = \mathrm{m} \frac{d}{dt} (\boldsymbol{r} \times \boldsymbol{v}) = 0$$

$$\Rightarrow \frac{d}{dt} (\boldsymbol{r} \times \boldsymbol{v}) = 0 \qquad \textbf{(4.2.10)}$$

Integrating above equation $\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{q} = constant$ **(4.2.11)**

Since the vector "$\boldsymbol{q}$" is perpendicular to both "$\boldsymbol{r}$" and "$\boldsymbol{v}$"

$$\boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{v}) = \boldsymbol{r} \cdot \boldsymbol{q} = 0$$

Therefore, the particle is in Plane.

4.2.3 The Areal velocity of the body under the central force is constant OR

The position vector of particle drawn from the origin sweeps equal area in equal interval of times. OR The rate of change of area is constant.

If the body move from position "A" to position "A'" and cover and angular displacement of "$d\theta$" and arc length "$rd\theta$".

The area of Triangle $\Delta AOA'$ in given figure is

$$d\boldsymbol{A} = \frac{1}{2}(\boldsymbol{r} \times rd\boldsymbol{\theta}) = \frac{1}{2}\left(r\hat{r} \times rd\theta\hat{\theta}\right)$$

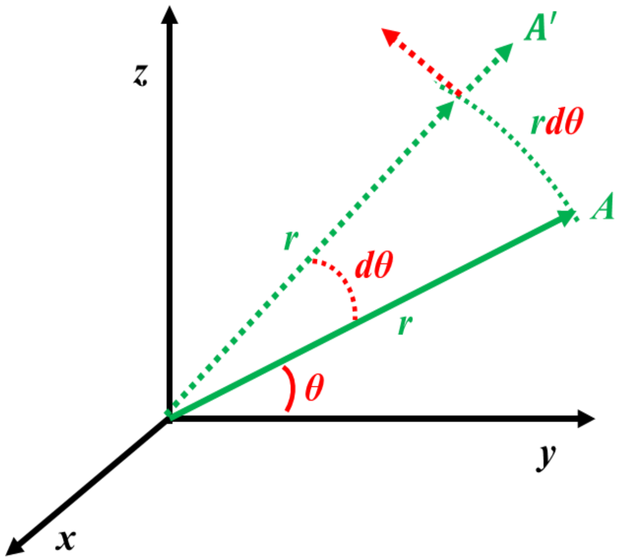
$$d\boldsymbol{A} = \frac{1}{2}r^2 d\theta\hat{n} \qquad \textbf{(4.2.12)}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}\hat{n}$$

Multiplying both sides with mass "μ" of the body

$$\text{m}\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}mr^2\frac{d\theta}{dt}\hat{n} = \frac{1}{2}mr^2\dot{\theta}\hat{n}$$

$$m\frac{d\boldsymbol{A}}{dt} = \frac{1}{2}\boldsymbol{l} \Rightarrow \frac{d\boldsymbol{A}}{dt} = \frac{\boldsymbol{l}}{2m} = \boldsymbol{constant} \qquad \textbf{(As required)}$$



Chapter 4

Lecture 2

Two body central Force Problem

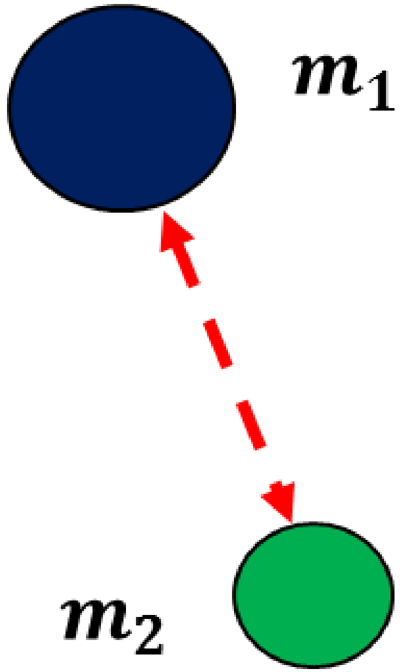
Akhlaq Hussain

4.3 Eq. of Motion for a body under the action of central force and First Integrals

Consider a conservative, where force can be drivable from potential "$V_{(r)}$".

The problem has spherical symmetry & angular momentum **($\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{P}$)** conserved.

Lagrangian of the system $$\boxed{L = T - V = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V_{(r)}} \qquad \textbf{(4.3.1)}$$



Using Lagrange's equation $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

And $\boxed{\dfrac{\partial L}{\partial \dot{\theta}} = P_\theta = \mu r^2 \dot{\theta},}$ and $\boxed{\dfrac{\partial L}{\partial \theta} = 0}$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\mu r^2 \dot{\theta}\right) = 0} \qquad \textbf{(4.3.2)}$$

$$\boxed{\Rightarrow \left(\mu r^2 \dot{\theta}\right) = P_\theta = \boldsymbol{l} = constant} \qquad \textbf{(4.3.3)}$$

Eq. (4.3.3) is **first integral of motion**

4.3 Eq. of Motion for a body under the action of central force and First Integrals

Lagrange's equation for radial part

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}(\mu \dot{r}) - \mu r \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\mu \ddot{r} - \mu r \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad \textbf{(4.3.4)}$$

Since $f_{(r)} = -\frac{dV}{dr}$ & $\dot{\theta} = \frac{\boldsymbol{l}}{\mu r^2}$ {from(**4.3.3**)}, Therefore, Eq. (**4.3.4**)

$$\text{Since} \Rightarrow \mu \ddot{r} - \frac{\boldsymbol{l}^2}{\mu r^3} = f_{(r)} \qquad \textbf{(4.3.6)}$$

4.3 Eq. of Motion for a body under the action of central force and First Integrals

$$\Rightarrow \mu\ddot{r} = \frac{\boldsymbol{l}^2}{\mu r^3} - \frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{\boldsymbol{l}^2}{2\mu r^2} + V\right)$$

Multiplying Both sides with "$\dot{r}$" $\quad \Rightarrow \mu\dot{r}\ddot{r} = -\dot{r}\frac{\partial}{\partial r}\left(\frac{\boldsymbol{l}^2}{2\mu r^2} + V\right)$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{dr}{dt}\frac{\partial}{\partial r}\left(\frac{\boldsymbol{l}^2}{2\mu r^2} + V\right) = -\frac{d}{dt}\left(\frac{\boldsymbol{l}^2}{2\mu r^2} + V\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + \frac{\boldsymbol{l}}{2\mu r^2} + V\right) = 0$$

$$\Rightarrow \frac{1}{2}\mu\dot{r}^2 + \frac{\boldsymbol{l}^2}{2\mu r^2} + V = Constant \qquad \textbf{(4.3.7)}$$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\boldsymbol{l}^2}{\mu r^2} + V_{(r)}} \qquad \textbf{(4.3.8)}$$

From eq. (**4.3.7**) and Eq. (**4.3.8**), total energy of a body under the action of central force is constant.

4.4 First Integrals

The Angular momentum, of the system is

$$\boldsymbol{l} = \mu r^2 \dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{\boldsymbol{l}}{\mu r^2} \Rightarrow \frac{d\theta}{dt} = \frac{\boldsymbol{l}}{\mu r^2}$$

$$\Rightarrow d\theta = \frac{\boldsymbol{l}}{\mu r^2} dt$$

Integrating above equation $\boxed{\Rightarrow \int_{\theta_o}^{\theta} d\theta = \int_0^t \frac{\boldsymbol{l}}{\mu r^2} dt}$ **(4.4.1)**

Now the total energy of a body moving under central force is given by

$$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\boldsymbol{l}^2}{\mu r^2} + V_{(r)} \qquad \textbf{(4.4.3)}$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{2\mu r^2} - V_{(r)}\right)} \qquad \textbf{(4.4.4)}$$

4.4 First Integral

$$\Rightarrow \frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{\mu r^2} - V_{(r)}\right)}$$

$$\boxed{\Rightarrow t = \int_{r_o}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - \frac{\boldsymbol{l}^2}{2\mu r^2} - V_{(r)}\right)}}} \qquad \textbf{(4.4.5)}$$

Eq. (**4.4.1**) & Eq. (**4.4.5**) are known as first integral for the motion in central force field. where $\boldsymbol{l}$, E, θ_o and r_o must be known initially.

Eq. (**4.4.1**) & Eq. (**4.4.5**) gives "r" and "θ" in terms of t. We are often interested to find "θ" in terms of "r" which will determine the shape of the orbit of the body.

4.4 First Integral

Since $\quad \frac{d\theta}{dt} = \frac{l}{\mu r^2} \qquad \Rightarrow \frac{d\theta}{dt}\frac{dr}{dr} = \frac{l}{\mu r^2}$

$$\Rightarrow d\theta = \frac{l}{\mu r^2 \dot{r}} dr \qquad \textbf{(4.4.6)}$$

From Eq. (**4.4.4**) we know that

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{\mu r^2} - V_{(r)}\right)} \qquad \textbf{(4.4.7)}$$

$$\Rightarrow d\theta = \frac{l}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{\mu r^2} - V_{(r)}\right)}} dr$$

$$\boxed{\Rightarrow \theta = \theta_o + \int_{r_o}^{r} \frac{l/_{r^2}}{\sqrt{2\mu\left(E - \frac{l^2}{2\mu r^2} - V_{(r)}\right)}} dr} \qquad \textbf{(4.4.8)}$$

Eq. 47 gives “θ” in terms of “r” which determine the shape of the orbit of the body under the action of central force field.

4.5 General Features of Motion Under Central Force

$$\begin{cases} \mu(\ddot{r} - r\dot{\theta}^2) = F_r \\ \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta \end{cases} \qquad \textbf{(4.5.1)}$$

The tangential component "F_θ" is zero because the force is radial

$$\mu(\ddot{r} - r\dot{\theta}^2) = F_r$$

$$\Rightarrow \mu\ddot{r} = F_r + \mu r\dot{\theta}^2$$

$$\boxed{\Rightarrow \mu\ddot{r} = F_r + \frac{\boldsymbol{l}^2}{\mu r^3}} \qquad \textbf{(4.5.2)}$$

$\frac{\boldsymbol{l}^2}{\mu r^3}$ is known as centrifugal force. It is a pseudo or false force since it does not arise from the interaction between the particles in the orbit. It appears due to accelerated motion of the body.

Since $\boldsymbol{l}^2 = \mu^2 r^4 \dot{\theta}^2$

$$\boxed{\Rightarrow \frac{\boldsymbol{l}^2}{\mu r^3} = \mu r\dot{\theta}^2 = \frac{\mu(r^2\dot{\theta}^2)}{r} = \frac{\mu v^2}{r} \text{ or } \frac{m v^2}{r}}$$

4.5 General Features of Motion Under Central Force

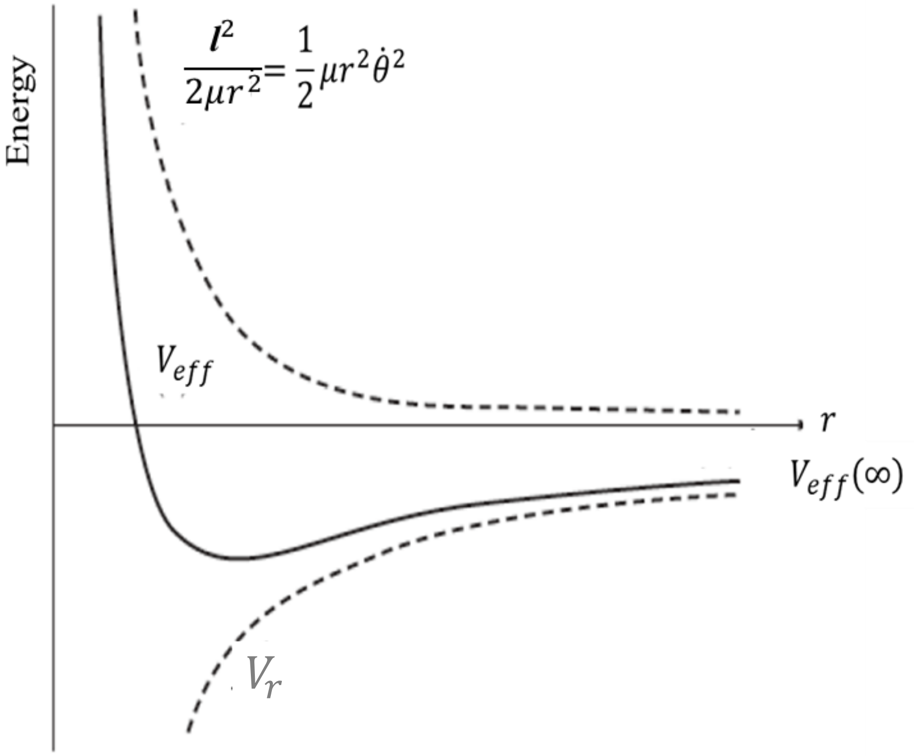
"$\mu\ddot{r}$" is the effective force responsible for the motion and can be derived from potential "V_{eff}"

$$\mu\ddot{r} = -\frac{dV_{eff}}{dr} \qquad \textbf{(4.5.3)}$$

Therefore Eq. (4.5.2) can be written as

$$-\frac{dV_{eff}}{dr} = F_r + \frac{l^2}{\mu r^3} \qquad \Rightarrow V_{eff} = -\int\left(-\frac{dV}{dr} + \frac{l^2}{\mu r^3}\right)dr$$

$$\Rightarrow V_{eff} = V + \frac{l^2}{2\mu r^2} \qquad \textbf{(4.5.4)}$$



For an inverse square law (gravitational or electrostatic force)

$$F_r = -\frac{k}{r^2} \quad \Rightarrow V = -\frac{k}{r}$$

Therefore, $$V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad \textbf{(4.5.5)}$$

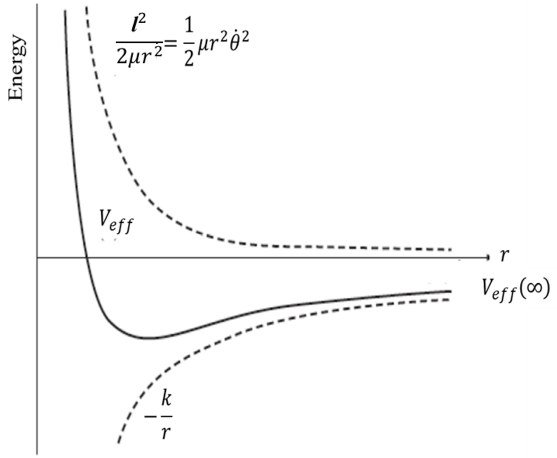
Note that the centrifugal potential reduces the effect of the inverse square law

4.5 General Features of Motion Under Central Force

Note: the total energy of the system is

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{eff}$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}\right)} \qquad \textbf{(4.5.6)}$$



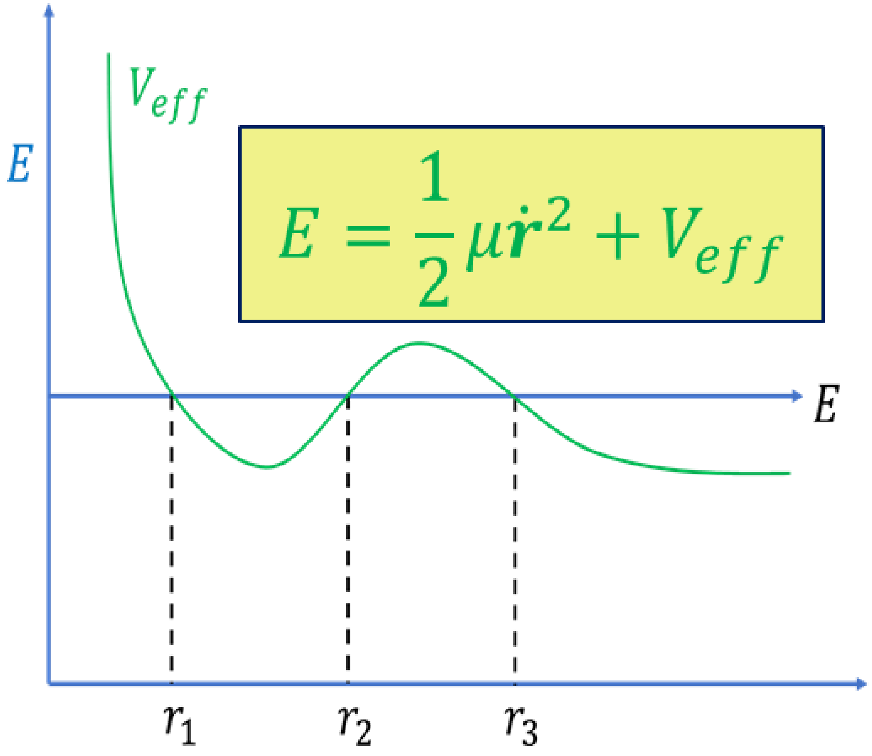
The centrifugal part gives a repulsive potential while the inverse square law part gives an attractive potential.

Centrifugal part decreases much faster with distance “r” as compared to the inverse attractive part.
The combine potential is given as the V_{eff} which decrease sharply from positive value to negative and then increase with r.
The V_{eff} approaches to zero value at infinite value of r.

4.6 Motion in arbitrary potential Field

Let an arbitrary potential V_{eff} which may or may not be same as the real problem and it might appear in different problems.

The Energy and potential curves intersect at "r_1", "r_2" and "r_3".



$$E = V_{eff} \qquad \textbf{(4.6.1)}$$

And $\frac{1}{2}\mu\dot{r}^2 = 0$ & $\dot{\boldsymbol{r}} = 0$

The curve can be divided into three regions.

<u>Region for $\boldsymbol{r < r_1}$</u>

$$E < V_{eff} \qquad \textbf{(4.6.2)}$$

& $T = \frac{1}{2}\mu\dot{r}^2 < 0$

& velocity has imaginary value. Hence motion in this region is not possible.

4.6 Motion in arbitrary potential Field

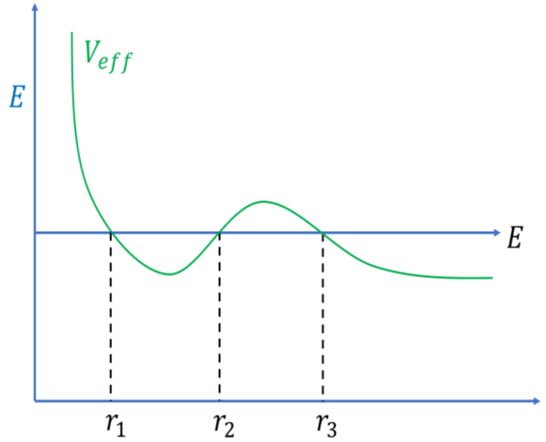
Region for $\mathbf{r_1 < r < r_2}$

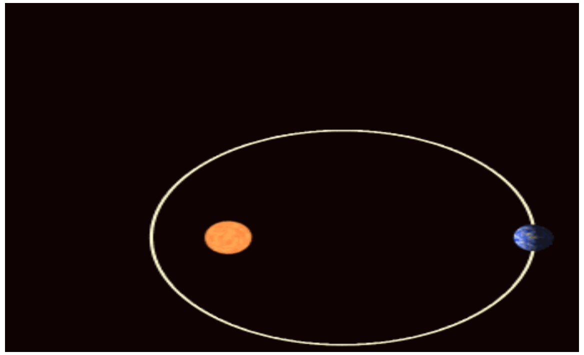
In this region $E > V_{eff}$

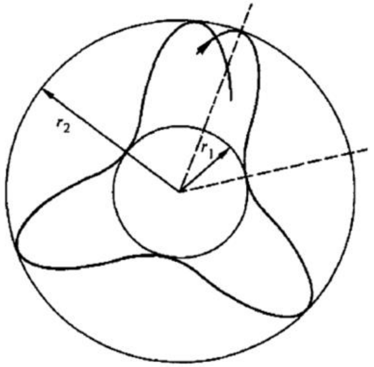
for $r < r_1$ and $r_2 < r$,

The kinetic energy T $= \frac{1}{2}\mu\dot{r}^2 < 0$

Which is not possible therefore the body will turn back on r_1 and r_2.







Region for $\boldsymbol{r_2 < r < r_3}$

In this region $E < V_{eff}$

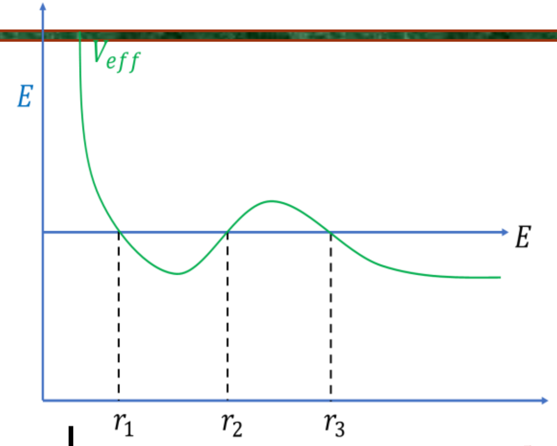
& $T = \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 < 0$ Therefore, the motion in this region is not possible.

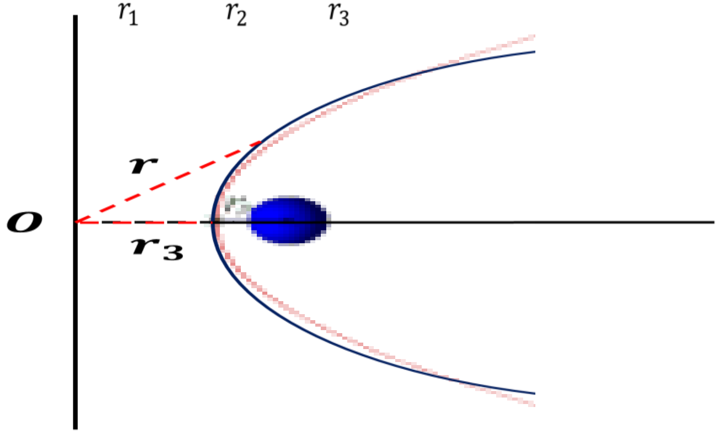
4.6 Motion in arbitrary potential Field

Region for $r > r_3$

Turning point is $r = r_3$.

The particle approaches to r_3 and rebounded.

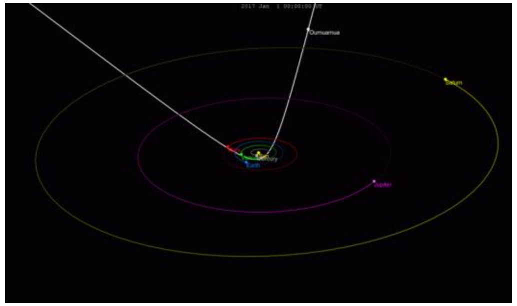




$$E = T + V_{eff} = 0 \Rightarrow T = -V_{eff}$$

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(-V_{eff}\right)} \qquad \textbf{(4.6.3)}$$

$\dot{r} =$ escape velocity; the initial velocity required to escape from the potential field V_{eff}.



The nature of motion of the particle discussed earlier with help of arbitrary potential will help to understand the nature of orbit.

4.7 Motion in Inverse Square Law Force Field

$$F_r = \frac{k}{r^2} \qquad \textbf{(4.7.1)}$$

$$\Rightarrow V_r = \frac{k}{r} \qquad \textbf{(4.7.2)}$$

Therefore, the effective potential V_{eff} is given by

$$V_{eff} = \frac{k}{r} + \frac{l^2}{2\mu r^2} \qquad \textbf{(4.7.3)}$$

The value of "k" depends on the nature of physical problem. For example,

i) gravitational force between two spherical bodies of mass m_1 and m_2

$$k = -Gm_1m_2 \qquad \textbf{(4.7.4)}$$

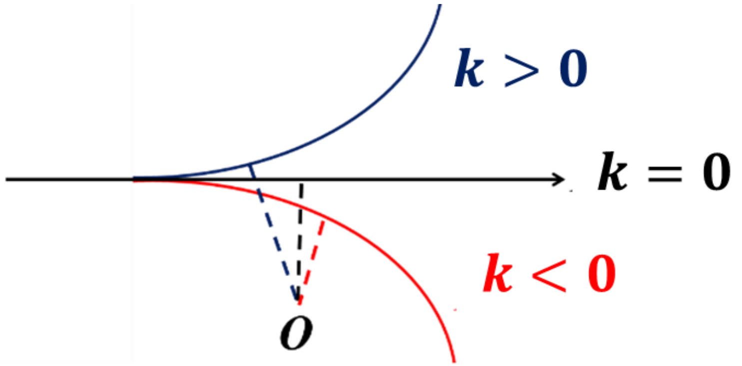
ii) Electrostatic force for two positive charges

$$k = \frac{q_1q_2}{4\pi\epsilon_o} \qquad \textbf{(4.7.5)}$$

The nature of the orbit depends on sign of "k". If $k > 0 \Rightarrow$ repulsive & for $k < 0 \Rightarrow$ attractive.

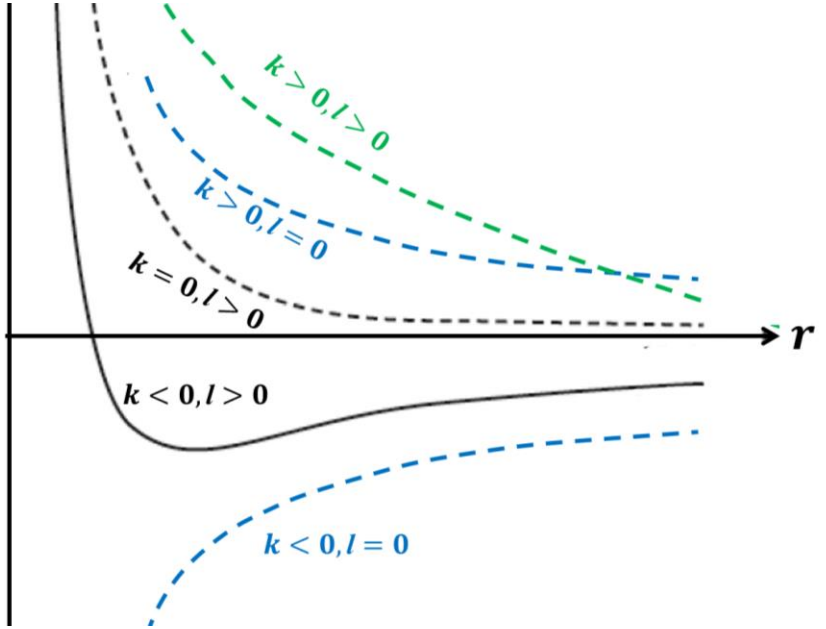
4.7 Motion in Inverse Square Law Force Field

A body with total energy $E > V_{eff}$ approaching to the centre of force from infinite distance. The particle will be deflected as given in figure.



If effective potential V_{eff} is plotted against "r" for different values of "k" and "l" following curves are obtained.

Case I	$k > 0, l > 0$
Case II	$k > 0, l = 0$
Case III	$k = 0, l > 0$
Case IV	$k < 0, l > 0$
Case V	$k < 0, l = 0$



These curves can be very helpful in understanding the nature of the orbit.

4.7 Motion in Inverse Square Law Force Field

(i) For E_1 at $r = r_1$

$$E_1 = V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

Turning point at $r = r_1$.

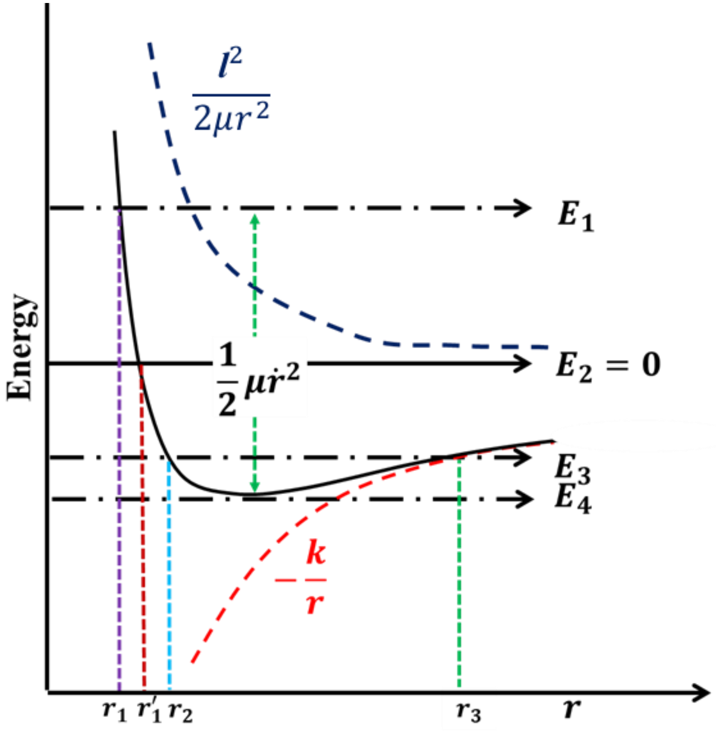
Motion represents scattering, where body is not bound to the centre and deflected away.

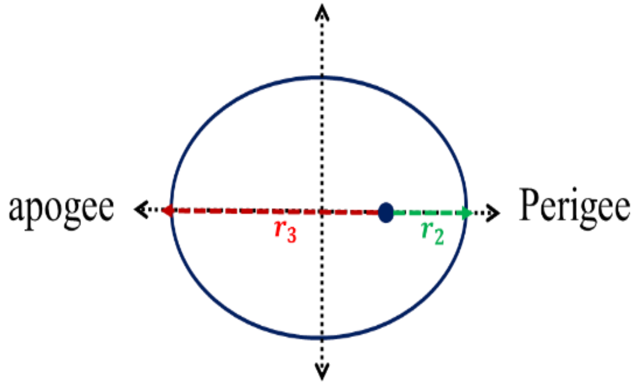
(ii) For $E_2 = 0$

Possible roots are $r = r_1'$ and $r = \infty$. The particle moves away & radial velocity fall continuously.

(iii) For $E_3 < 0$

Two roots $r = r_2$ and $r = r_3$ of equation are real and distinct.





4.7 Motion in Inverse Square Law Force Field

(iv) **For $E_4 = V_{eff}$,**

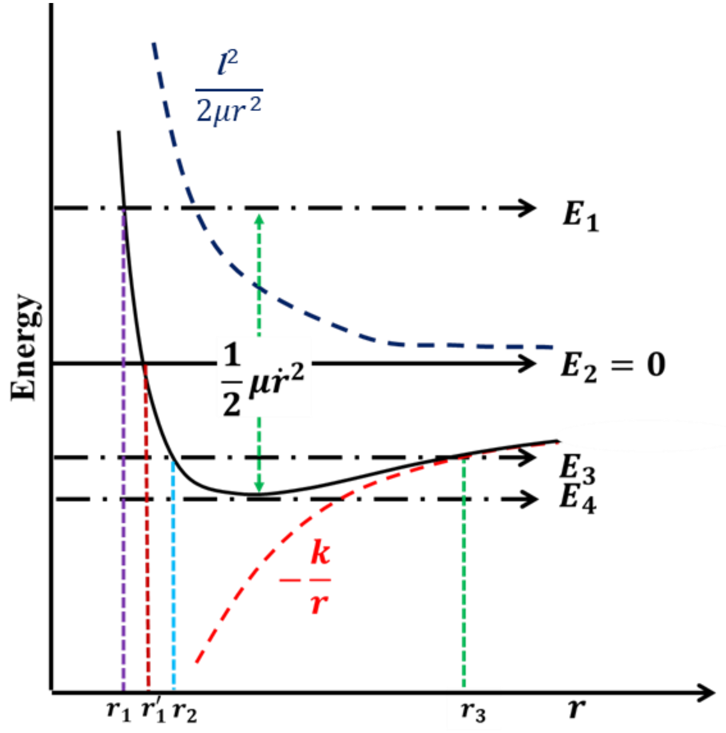
which is tangent of the potential energy curve.

Therefore, $$\frac{dV_{eff}}{dr} = 0$$

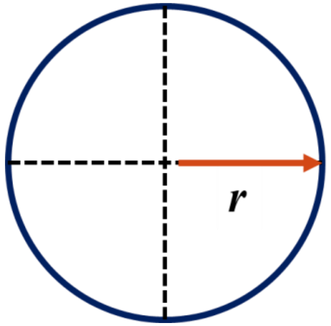
$$\Rightarrow \frac{dV}{dr} - \frac{l^2}{\mu r^3} = 0$$

$$\Rightarrow F_r = -\frac{dV}{dr} = -\frac{l^2}{\mu r^3} = -\mu r\dot{\theta}^2$$

$$\Rightarrow F_r = -\frac{\mu r^2\dot{\theta}^2}{r} = -\frac{\mu v^2}{r}$$



Thus, F_r is equal to the centrifugal force required to maintain circular motion of the body around the centre of the force. Thus, F_r is centripetal force that maintain the orbit.



Problem (Page 293, Classical Mechanics by Marion)

Find the force law for a central force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where "k" and "α" are constants. Also find value of $\theta_{(t)}$ and $r_{(t)}$. Also find Energy of the orbit.

Solution. Since we have verified that

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{\left(\frac{1}{u}\right)}}{l^2u^2}$$

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2} \qquad (1)$$

Now using

$$r = ke^{\alpha\theta} \quad \Rightarrow \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiating Twice with respect to θ

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k}e^{-\alpha\theta}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{\alpha^2}{k}e^{-\alpha\theta} = \alpha^2 u \qquad (2)$$

Putting value of u and $\frac{d^2u}{d\theta^2}$ in equation 1

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow (\alpha^2 + 1)u = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow f_{(r)} = -\frac{l^2}{\mu r^3}(\alpha^2 + 1) \qquad (3)$$

Eq. 3 represents the force responsible for motion.

Now the central potential responsible for the motion of the particle will be

$$V = -\int f_{(r)}\, dr = -\frac{l^2}{2\mu r^2}(\alpha^2 + 1) \qquad (4)$$

Total energy of the system is

$$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V \qquad (5)$$

Now

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \qquad = \frac{dr}{d\theta}\frac{l}{\mu r^2}$$

$$\dot{r} = k\alpha e^{\alpha\theta}\frac{l}{\mu r^2} = r\alpha\frac{l}{\mu r^2}$$

$$\dot{r} = \alpha\frac{l}{\mu r} \tag{6}$$

Now $\qquad E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{l\alpha}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2}{2\mu r^2}(\alpha^2 + 1)$$

$$\Rightarrow E = \frac{l^2}{2\mu r^2}(\alpha^2 + 1) - \frac{l^2}{2\mu r^2}(\alpha^2 + 1) = 0 \tag{7}$$

Eq. 7 gives the total energy of the system. Zero value of the system represent a bound system.

Now we will determine of $\theta_{(t)}$ and $r_{(t)}$

Since $\dot{\theta} = \frac{l}{\mu r} \Rightarrow \frac{d\theta}{dt} = \frac{l}{\mu r^2}$

$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \Rightarrow e^{2\alpha\theta} d\theta = \frac{l}{\mu k^2}\, dt$

Integrating both sides we get $\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C$

$$e^{2\alpha\theta} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow \theta_{(t)} = \frac{1}{2\alpha}\ln\left[2\alpha\left(\frac{lt}{\mu k^2} + C\right)\right] \quad (9)$$

Now $r = ke^{\alpha\theta}$

$\Rightarrow \frac{r}{k} = e^{\alpha\theta} \Rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$

$$\Rightarrow \frac{r^2}{k^2} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow r_{(t)} = \sqrt{2\alpha k^2\left(\frac{lt}{\mu k^2} + C\right)} \quad (10)$$

Chapter 4

Lecture 3

Two body central Force Problem

Akhlaq Hussain

4.8 Prove that for central force field the equation of motion can be written as;

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu f_{(u)}}{l^2u^2} \qquad \textbf{And} \qquad \frac{d^2u}{d\theta^2} + u = -\frac{f_{(u)}}{\mu h^2u^2}$$

where $h = r^2\dot{\theta}$ and $u = {}^1/_r$

Solution: Consider a particle of mass "μ" is at a distance "r" from the origin. The acceleration of the particle can have two components in the polar coordinates

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{4.8.1}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{4.8.2}$$

Since the central force is always directed along the radial vector "r". The radial force is responsible for the motion. Therefore;

$$f_{(r)} = \mu\left(\ddot{r} - r\dot{\theta}^2\right) \tag{4.8.3}$$

$$f_{(\theta)} = 0 \tag{4.8.4}$$

Let us consider a function "u" such that $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \dot{r} = -r^2\dot{\theta}\frac{du}{d\theta}$$

$$\boxed{\Rightarrow \dot{r} = -h\frac{du}{d\theta}} \qquad (4.8.5)$$

Differentiating above equation with respect to t

$$\frac{d\dot{r}}{dt} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d}{d\theta}\left(\frac{du}{dt}\right)$$

$$\Rightarrow \ddot{r} = -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\frac{d\theta}{dt}\right)$$

$$\Rightarrow \ddot{r} = -h\dot{\theta}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right) = -h\dot{\theta}\frac{d^2u}{d\theta^2} \qquad (4.8.6)$$

Since h $= r^2\dot{\theta} \Rightarrow \dot{\theta} = {}^{h}/_{r^2}$ $or\ \dot{\theta} = hu^2$, Putting in Eq. (4.8.6)

$$\boxed{\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}} \qquad (4.8.7)$$

Putting $r = \frac{1}{u}$, $\dot{\theta} = hu^2$ and Eq. (4.8.7) in Eq. (4.8.3)

$$f_{(r)} = \mu\left(\ddot{r} - r\dot{\theta}^2\right) \Rightarrow f_{(u)} = \mu\left(-h^2u^2\frac{d^2u}{d\theta^2}\right) - \mu\left(\frac{1}{u}\right)(hu^2)^2$$

$$\Rightarrow f_{(u)} = -\mu h^2u^2\frac{d^2u}{d\theta^2} - \mu h^2u^3$$

$$\Rightarrow f_{(u)} = -\mu h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\Rightarrow -\frac{f_{(u)}}{\mu h^2u^2} = \left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\boxed{\Rightarrow \left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{f_{(u)}}{\mu h^2u^2}} \qquad (4.8.8)$$

As required.

Since $l = \mu r^2\dot{\theta} = \mu h$ putting in Eq. (4.8.8)

$$\boxed{\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2}} \qquad (4.8.9)$$

As desired.

Problem (Page 293, Classical Mechanics by Marion)

Find the force law for a central force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where "k" and "α" are constants. Also find value of $\theta_{(t)}$ and $r_{(t)}$. Also find Energy of the orbit.

Solution. Since we have verified that

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2}$$

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2} \tag{1}$$

Now using

$$r = ke^{\alpha\theta} \quad \Rightarrow \mathrm{u} = \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiating Twice with respect to θ

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2}{k}e^{-\alpha\theta}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} = \frac{\alpha^2}{k}e^{-\alpha\theta} = \alpha^2 u \tag{2}$$

Putting value of u and $\frac{d^2u}{d\theta^2}$ in equation 1

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow (\alpha^2 + 1)u = -\frac{\mu r^2 f_{(r)}}{l^2}$$

$$\Rightarrow f_{(r)} = -\frac{l^2}{\mu r^3}(\alpha^2 + 1) \qquad (3)$$

Eq. 3 represents the force responsible for motion.

Now the central potential responsible for the motion of the particle will be

$$V = -\int f_{(r)}\, dr = -\frac{l^2}{2\mu r^2}(\alpha^2 + 1) \qquad (4)$$

Total energy of the system is

$$E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V \qquad (5)$$

Now

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt}$$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \qquad = \frac{dr}{d\theta}\frac{l}{\mu r^2}$$

$$\dot{r} = k\alpha e^{\alpha\theta}\frac{l}{\mu r^2} = r\alpha\frac{l}{\mu r^2}$$

$$\dot{r} = \alpha\frac{l}{\mu r} \qquad (6)$$

Now $\qquad E = T + V = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V$

$$\Rightarrow E = \frac{1}{2}\mu\left(\frac{l\alpha}{\mu r}\right)^2 + \frac{l^2}{2\mu r^2} - \frac{l^2}{2\mu r^2}(\alpha^2 + 1)$$

$$\Rightarrow E = \frac{l^2}{2\mu r^2}(\alpha^2 + 1) - \frac{l^2}{2\mu r^2}(\alpha^2 + 1) = 0 \qquad (7)$$

Eq. 7 gives the total energy of the system. Zero value of the system represent a bound system.

Now we will determine of $\theta_{(t)}$ and $r_{(t)}$

Since $\dot{\theta} = \frac{l}{\mu r^2} \Rightarrow \frac{d\theta}{dt} = \frac{l}{\mu r^2}$

$$\frac{d\theta}{dt} = \frac{l}{\mu k^2 e^{2\alpha\theta}} \Rightarrow e^{2\alpha\theta} d\theta = \frac{l}{\mu k^2}\, dt$$

Integrating both sides we get $\frac{e^{2\alpha\theta}}{2\alpha} = \frac{lt}{\mu k^2} + C$

$$e^{2\alpha\theta} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow \theta_{(t)} = \frac{1}{2\alpha} \ln\left[2\alpha\left(\frac{lt}{\mu k^2} + C\right)\right] \qquad (9)$$

Now $r = ke^{\alpha\theta}$

$$\Rightarrow \frac{r}{k} = e^{\alpha\theta} \Rightarrow \frac{r^2}{k^2} = e^{2\alpha\theta}$$

$$\Rightarrow \frac{r^2}{k^2} = 2\alpha\left(\frac{lt}{\mu k^2} + C\right) \Rightarrow r_{(t)} = \sqrt{2\alpha k^2\left(\frac{lt}{\mu k^2} + C\right)} \qquad (10)$$

4.9 Show That: a) $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$

b) Using results from part "a" also prove that the conservation of energy equation will be

$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{\mu h^2}$ **if** $u = \frac{1}{r}$

Solution: Let us consider a particle of mass "μ" and position vector "$\boldsymbol{r}$".

Since $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} \qquad = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$\Rightarrow \dot{r} = -r^2\dot{\theta}\frac{du}{d\theta} \Rightarrow \dot{r} = -h\frac{du}{d\theta}$$

Therefore, $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$$\Rightarrow v^2 = \left(-h\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(hu^2)^2 = h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2$$

$$\Rightarrow v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \qquad (4.9.1)$$

Since $E = T + V \Rightarrow T = E - V$

$$\Rightarrow \frac{1}{2}\mu v^2 = E - V$$

$$\Rightarrow \frac{1}{2}\mu h^2 \left(\left(\frac{du}{d\theta} \right)^2 + u^2 \right) = E - V$$

$$\Rightarrow \left(\frac{du}{d\theta} \right)^2 + u^2 = \frac{2(E-V)}{\mu h^2} \qquad (4.9.2)$$

Eq. (4.9.1) and Eq. (4.9.2) are as desired.

4.10 Equation of motion for a body under central force (inverse square law force)

Solve $\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{L^2u^2}$ and $\theta = \theta_o + \int \frac{l/r^2}{\sqrt{2\mu\left(E - V_{(r)} - \frac{l^2}{2\mu r^2}\right)}} dr$ and prove that the solution is the equation of conic. i.e. the motion under the inverse square law force represent motion on conic path. Also discuss the possibilities of bound and unbound system.

Let us consider a particle of mass "μ" is under inverse square law force. The equation of motion can be written as

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2} \qquad \textbf{(4.10.1)}$$

Since the inverse square attractive force

$$f_{(r)} = -\frac{k}{r^2} = -ku^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k u^2}{l^2u^2}$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}} \qquad \textbf{(4.10.2)}$$

4.10 Equation of motion for a body under central force

(inverse square law force)

Starting with equation Eq. (4.10.2)

$$\frac{d^2\boldsymbol{u}}{d\theta^2} + \boldsymbol{u} = \frac{\mu k}{l^2} \Rightarrow \boxed{\frac{d^2\boldsymbol{u}}{d\theta^2} + \boldsymbol{u} - \frac{\mu k}{l^2} = 0} \qquad (4.10.2)$$

Consider a function

$$\boxed{y = \boldsymbol{u} - \frac{\mu k}{l^2}} \qquad (4.10.3)$$

Differentiating above equation Twice

$$\boxed{\frac{d^2y}{d\theta^2} = \frac{d^2\boldsymbol{u}}{d\theta^2}} \qquad (4.10.4)$$

Now

$$\frac{d^2u}{d\theta^2} + \boldsymbol{u} - \frac{\mu k}{l^2} = \frac{d^2y}{d\theta^2} + y = 0$$

$$\boxed{\frac{d^2y}{d\theta^2} + y = 0} \qquad (4.10.5)$$

4.10 Equation of motion for a body under central force (inverse square law force)

It is a second order differential equation where "y" is a function of "θ"

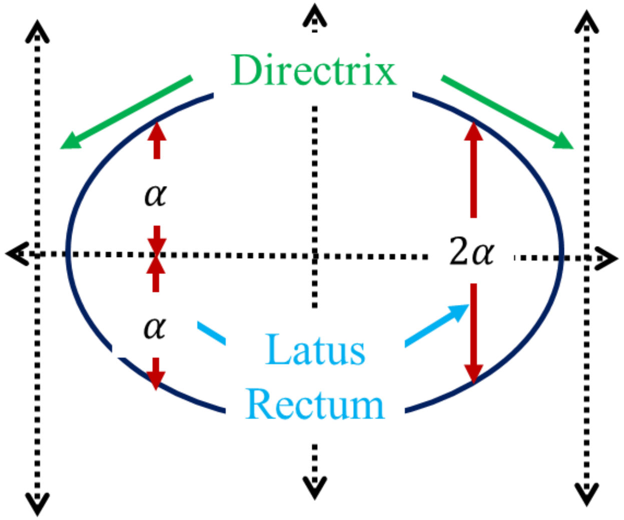
And $$y = A\cos(\theta - \theta_o) \qquad (4.10.6)$$

$$y = u - \frac{\mu k}{l^2} = A\cos(\theta - \theta_o)$$

$$\frac{1}{r} = \frac{\mu k}{l^2} + A\cos(\theta - \theta_o)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \frac{Al^2}{\mu k}\cos(\theta - \theta_o) \qquad (4.10.7)$$

Equation of conic. $$\boxed{\frac{\alpha}{r} = 1 + e\cos(\theta - \theta_o)} \qquad (4.10.7)a$$



where $\alpha = \frac{l^2}{\mu k}$ Semi latus rectum.

and $e = \frac{Al^2}{\mu k}$ is eccentricity which is defined as the measure of deviation from circular shape.

4.10 Equation of motion for a body under central force (inverse square law force)

Now consider the first integral for the motion under central force

$$\theta = \theta_o + \int \frac{{l}/{r^2}}{\sqrt{2\mu\left(E - V_{(r)} - \frac{l^2}{2\mu r^2}\right)}} dr \qquad (4.10.8)$$

Since $\boxed{du = -\frac{1}{r^2} dr}$ & $\boxed{V = -\frac{k}{r} = -ku}$ (4.10.9) & (4.10.10)

Putting Eq. (4.10.4) and Eq. (4.10.5) in Eq. (4.10.3)

$$\theta = \theta_o - \int \frac{du}{\sqrt{\left(\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2} u - u^2\right)}} \qquad (4.10.11)$$

Let $\frac{2\mu E}{l^2} = a, \frac{2\mu k}{l^2} = b$ and $-1 = c$

Then $$\theta - \theta_o = -\int \frac{du}{\sqrt{\left(\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2} u - u^2\right)}} = -\int \frac{du}{\sqrt{(a + bu + cu^2)}}$$

4.10 Equation of motion for a body under central force (inverse square law force)

$$\theta - \theta_o = -\left[\frac{1}{\sqrt{-c}} cos^{-1}\left\{-\left(\frac{b+2cu}{\sqrt{b^2-4ac}}\right)\right\}\right] = -\left[\frac{1}{\sqrt{1}} cos^{-1}\left\{-\left(\frac{-\frac{2\mu k}{l^2}+2(-1)u}{\sqrt{\left(\frac{2\mu k}{l^2}\right)^2-4\left(\frac{2\mu E}{l^2}\right)(-1)}}\right)\right\}\right]$$

$$\theta_o - \theta = \left[cos^{-1}\left\{\frac{-\frac{\mu k}{l^2}+u}{\sqrt{\left(\frac{\mu k}{l^2}\right)^2+\left(\frac{2\mu E}{l^2}\right)}}\right\}\right]$$

$$\frac{2\mu E}{l^2} = a, \frac{2\mu k}{l^2} = b \text{ and } -1 = c$$

$$u = \frac{\mu k}{l^2} + \frac{\mu k}{l^2}\sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)} cos(\theta_o - \theta)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = \left[1 + \sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)} cos(\theta_o - \theta)\right]$$

$$\boxed{\Rightarrow \frac{\alpha}{r} = [1 + ecos(\theta_o - \theta)] = [1 + ecos(\theta - \theta_o)]} \qquad (4.10.12)$$

We have shown that the solution of the first integral is an equation of conic

$\alpha = \frac{l^2}{\mu k}$ ⇨ semi latus rectum and $e = \sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)}$ is the eccentricity

4.10 Equation of motion for a body under central force (inverse square law force)

For Eq.(4.10.7)a & Eq.(4.10.12) if we assume $\theta_o = 0^o$, $\theta = 0^o$ & 180^o

$$\boxed{r_1 = \frac{\alpha}{1+e} = \frac{\alpha}{1+\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)}}} \quad \& \quad \boxed{r_2 = \frac{\alpha}{1-e} = \frac{\alpha}{1-\sqrt{1+\left(\frac{2l^2E}{\mu k^2}\right)}}} \qquad (4.10.13)\ \&\ (4.10.14)$$

For $e > 1$ of $E > 0$, r_2 is negative

And $e = 1$, $E = 0$, r_2 is infinity

Both cases ⇨ motion is unbound

Therefore $e < 1$ and $E < 0$ is necessary to keep a bounded motion.

The finite and positive values of r_1 and r_2 represents the turning points.

Comparing the equation of eccentricity

$$\boxed{\boldsymbol{A = \frac{\mu k}{l^2}\sqrt{1 + \left(\frac{2l^2E}{\mu k^2}\right)}}} \qquad (4.10.15)$$

4.10 Equation of motion for a body under central force (inverse square law force)

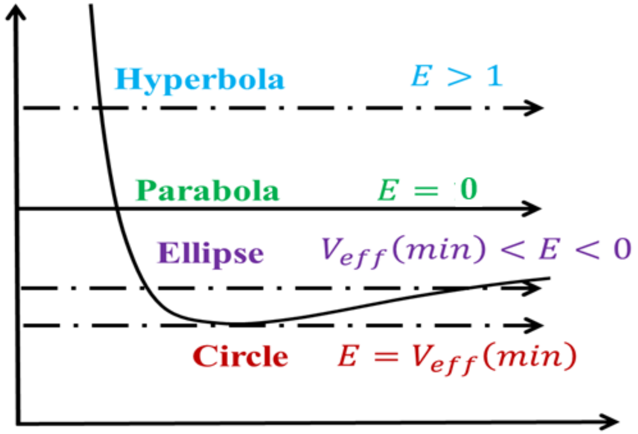
Nature of the Orbit

The nature of orbit is determined by eccentricity e which depend on energy

Value of E	Value of eccentricity	Nature of orbit
$E > 0$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$V_{eff}(min) < E < 0$	$0 < e < 1$	Ellipse
$E = V_{eff}(min)$	$e = 0$	Circle

we can always set $\theta_o = 0$ And $\frac{1}{c} = \alpha = \frac{L^2}{\mu k} \quad \Rightarrow \frac{1}{r} = C[1 + e\cos(\theta - \theta_o)]$

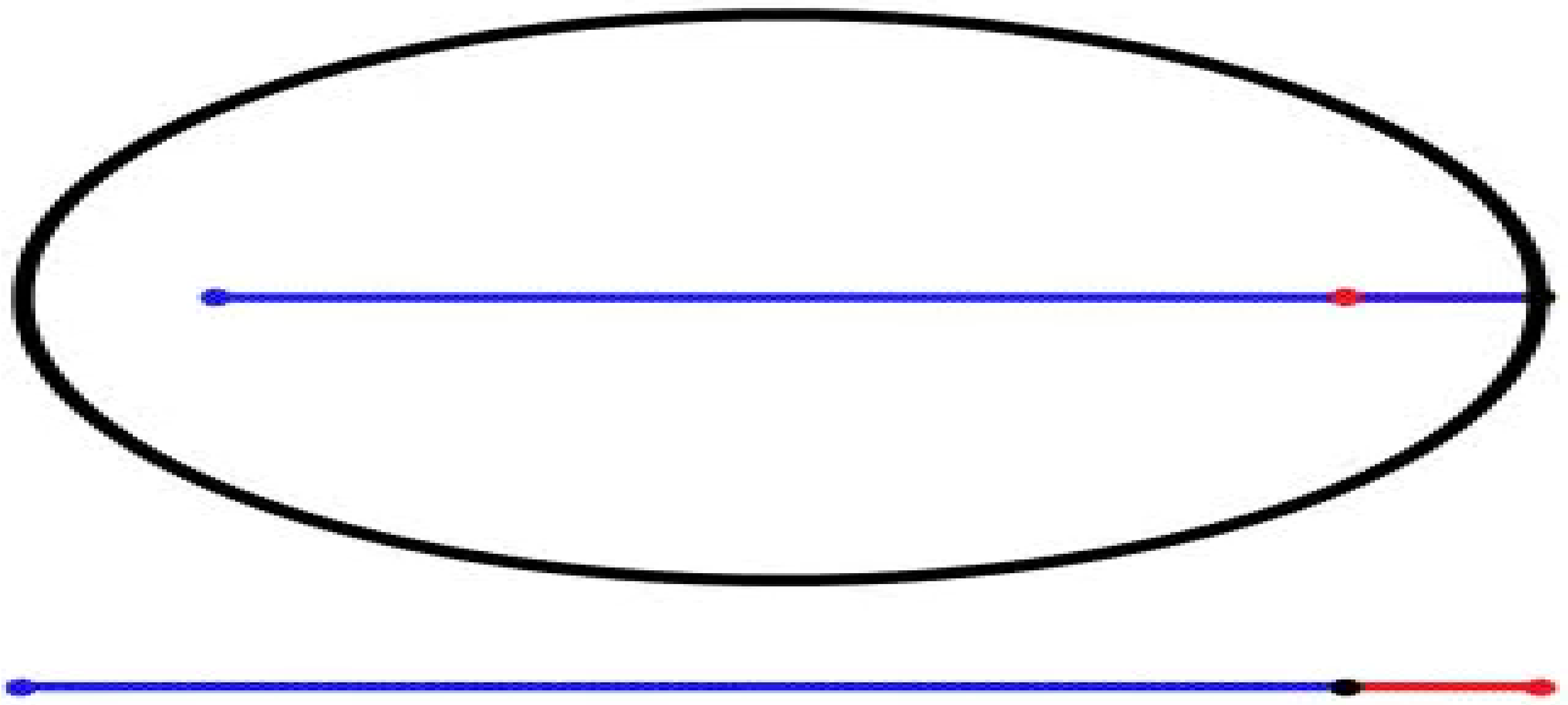
- Bound motion is possible only for Ellipse or circle.
- The motion of planets is either circular of elliptical.
- The variation of length of the day and seasonal changes suggest that the path of the planet is elliptical.



4.10 Equation of motion for a body under central force (inverse square law force)

Elliptic Orbit

The ellipse is a curve traced out by a particle moving in such a way that the sum of its distance from two fixed foci O and O' is always constant.



4.10 Equation of motion for a body under central force (inverse square law force)

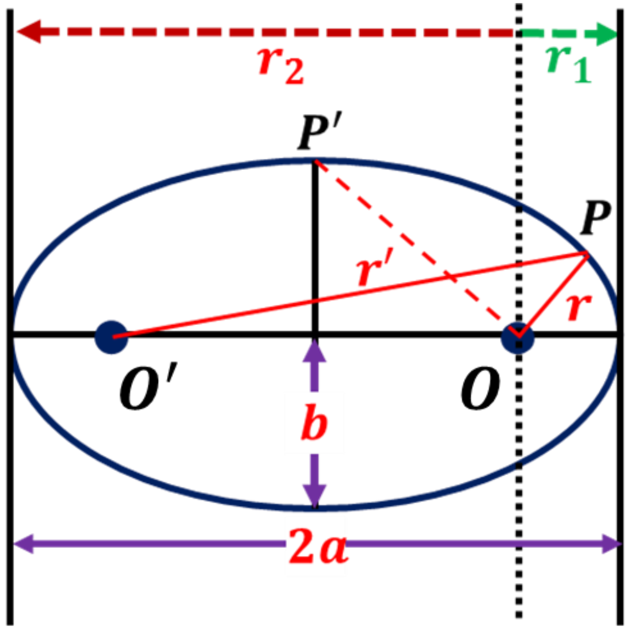
Elliptic Orbit

$r_2 + r_1 = \frac{2\alpha}{(1-e^2)} = 2a$ where a the semi-major axis is constant

Note the distance between two foci

$$\overline{OO'} = r_2 - r_1 = \frac{2\alpha}{(1-e^2)}e = 2ae$$

$$\Rightarrow \frac{\overline{OO'}}{2} = ae \qquad (4.10.16)$$



From the figure it is clear that $\overline{OP'} = \overline{O'P'}$ and

$\overline{OP'} + \overline{O'P'} = 2a$ & $\overline{OP'} = a$

Now from figure $b^2 = (\overline{OP'})^2 - \left(\frac{\overline{OO'}}{2}\right)^2 = a^2 - a^2e^2 = a^2(1-e^2)$

$$\Rightarrow b = a\sqrt{(1-e^2)}$$

4.10 Equation of motion for a body under central force (inverse square law force)

If $e \neq 0$ then,

Since $$e = \sqrt{1 + \left(\frac{2l^2 E}{\mu k^2}\right)}$$

Therefore, $$b = a\sqrt{(1 - e^2)} = a\sqrt{\left(1 - 1 - \left(\frac{2l^2 E}{\mu k^2}\right)\right)}$$

$$b = a\sqrt{\left(-\left(\frac{2l^2 E}{\mu k^2}\right)\right)}$$

The energy of the bounded system is less than zero therefore it will give a real value solution.

4.10 Equation of motion for a body under central force (inverse square law force)

If $\boldsymbol{e = 0}$ ellipse will become circle $\boldsymbol{b = a}$

(Note in the region when body passes through closest distance the curve is arc of a circle)

And $$1 + \left(\frac{2l^2E}{\mu k^2}\right) = 0$$

$$E = -\left(\frac{\mu k^2}{2l^2}\right) \Rightarrow l^2 = -\frac{\mu k^2}{2E}$$

Eq. (4.10.13) & (4.10.14) will be reduced to $r_1 = r_2 = a = \alpha$, therefore;

$$r_o = a = \alpha = \frac{l^2}{\mu k} = \frac{-\left(\frac{\mu k^2}{2E}\right)}{\mu k}$$

And $$r_o = -\frac{k}{2E}$$

And $$E = -\frac{k}{2a}$$

$$r_1 + r_2 = \frac{\alpha}{1+e} + \frac{\alpha}{1-e} = 2a$$

$$r_1 + r_2 = \frac{\alpha}{1+0} + \frac{\alpha}{1-0} = 2a$$

$$a = \alpha = \frac{l^2}{\mu k}$$

4.10 Equation of motion for a body under central force (inverse square law force)

Putting this value in equation of eccentricity we get

$$e = \sqrt{1 + \left(\frac{2l^2 E}{\mu k^2}\right)} = \sqrt{1 - \left(\frac{l^2}{\mu k a}\right)} \qquad \therefore E = -\frac{k}{2a}$$

Using this value, the semi-minor axis b can be written as.

$$b = a\sqrt{\left(1 - 1 + \left(\frac{l^2}{\mu k a}\right)\right)}$$

$$b = a\sqrt{\frac{l^2}{\mu k a}}$$

$$b = a^{1/2}\frac{l}{\sqrt{\mu k}}$$

Chapter 4

Lecture 4

Two body central Force Problem

Akhlaq Hussain

4.11 Keppler's Laws

Keppler First Law: "Every planet describes an ellipse with the sun at one of the foci"

Let us consider a particle of mass "μ" is under inverse square law force. Since the inverse square attractive force

$$f_{(r)} = -\frac{k}{r^2} = -ku^2 \quad (\text{if } r = \frac{1}{u}) \qquad (4.11.1)$$

For gravitational force $k = GmM_s$

$$\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{\mu f_{(u)}}{l^2u^2} \qquad (4.10.2)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k u^2}{l^2u^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2} \qquad (4.11.2)$$

We will now solve these equations Eq. (4.11.2) to understand the nature of the orbit.

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2} \Rightarrow \frac{d^2u}{d\theta^2} + u - \frac{\mu k}{l^2} = 0 \qquad (4.11.3)$$

4.11 Keppler's Laws

Consider a function $y = u - \frac{\mu k}{l^2}$ (4.11.4)

Differentiating above equation

$$\frac{dy}{d\theta} = \frac{du}{d\theta}$$

Differentiating above equation again

$$\frac{d^2y}{d\theta^2} = \frac{d^2u}{d\theta^2} \qquad (4.11.5)$$

Now

$$\frac{d^2y}{d\theta^2} + y = \frac{d^2u}{d\theta^2} + u - \frac{\mu k}{l^2} = 0$$

$$\frac{d^2y}{d\theta^2} + y = 0 \qquad (4.11.6)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu k}{l^2}$$

The general solution is;

$$u = B\cos\theta + C\sin\theta + \frac{\mu k}{l^2}$$

Where $B = A\cos\theta_o$ and $C = A\sin\theta_o$

$$u = A\cos(\theta - \theta_o) + \frac{\mu k}{l^2}$$

$$r = \frac{1}{\frac{\mu k}{l^2} + A\cos(\theta - \theta_o)} = \frac{l^2/_{\mu k}}{1 + \frac{Al^2}{\mu k}\cos(\theta - \theta_o)}$$

$$r = \frac{\alpha}{1 + e\cos(\theta - \theta_o)}$$

$$\frac{\alpha}{r} = 1 + e\cos\theta$$

4.11 Keppler's Laws

It is a second order differential equation where "y" is a function of "θ" And

$$y = A\cos(\theta - \theta_o) \tag{4.11.7}$$

where A and θ_o are constants.

$$y = u - \frac{\mu k}{l^2} = A\cos(\theta - \theta_o)$$

$$\boxed{u = \frac{\mu k}{l^2} + A\cos(\theta - \theta_o)}$$

$$u\left(\frac{l^2}{\mu k}\right) = 1 + \frac{Al^2}{\mu k}\cos(\theta - \theta_o) \tag{4.11.8}$$

Using Equation Eq. (4.9.2) $$\boxed{\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu(E-V)}{l^2}}$$

and using $V = -\frac{k}{r} = -ku$

4.11 Keppler's Laws

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2\mu(E+ku)}{l^2}$$

Using Eq. $\boxed{u = A\cos(\theta - \theta_o) + \frac{\mu k}{l^2}}$ and $\boxed{\frac{du}{d\theta} = -A\sin(\theta - \theta_o)}$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = [-A\sin(\theta - \theta_o)]^2 + \left[A\cos(\theta - \theta_o) + \frac{\mu k}{l^2}\right]^2 = \frac{2\mu}{l^2}(E + ku)$$

$$\Rightarrow A^2[\sin^2(\theta - \theta_o) + \cos^2(\theta - \theta_o)] + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu}{l^2}(E + ku)$$

$$\boxed{\Rightarrow A^2 + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu}{l^2}\left(E + kA\cos(\theta - \theta_o) + \frac{\mu k^2}{l^2}\right)}$$

$$\Rightarrow A^2 + \left(\frac{\mu k}{l^2}\right)^2 + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) = \frac{2\mu E}{l^2} + 2\frac{\mu k}{l^2}A\cos(\theta - \theta_o) + 2\left(\frac{\mu k}{l^2}\right)^2$$

$$\boxed{\Rightarrow A^2 = \frac{2\mu E}{l^2} + \left(\frac{\mu k}{l^2}\right)^2}$$

4.11 Keppler's Laws

$$\Rightarrow A = \frac{\mu k}{l^2}\sqrt{\frac{2El^2}{\mu k^2} + 1}$$

And $u\left(\frac{l^2}{\mu k}\right) = 1 + \frac{Al^2}{\mu k}cos(\theta - \theta_o)$ (4.11.8)

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \frac{\mu k}{l^2}\left(\sqrt{\frac{2El^2}{\mu k^2} + 1}\right)\frac{l^2}{\mu k}cos(\theta - \theta_o)$$

$$\Rightarrow \frac{\left(\frac{l^2}{\mu k}\right)}{r} = 1 + \left(\sqrt{\frac{2El^2}{\mu k^2} + 1}\right)cos(\theta - \theta_o)$$

$$\Rightarrow \frac{\alpha}{r} = 1 + ecos(\theta - \theta_o) \qquad (4.11.9)$$

Eq. (4.11.9) is equation of conic which describe the motion of planet around the sun.

4.11 Keppler's Laws

For Eq.(4.11.8) & Eq.(4.11.9) if we assume $\theta_o = 0$ & $\theta = 0^o$& 180^o

$$\boxed{r_1 = \frac{\alpha}{1+e}} \quad \& \quad \boxed{r_2 = \frac{\alpha}{1-e}} \qquad (4.11.10)\ \&\quad (4.11.11)$$

For $e > 1$ of $E > 0$, r_2 is negative

And $e = 1$, $E = 0$, r_2 is infinity

Both cases ⇨ motion is unbound

Therefore $e < 1$ and $E < 0$ is necessary to keep a bounded motion.

The finite and positive values of r_1 and r_2 represents the turning points.

4.11 Keppler's Laws

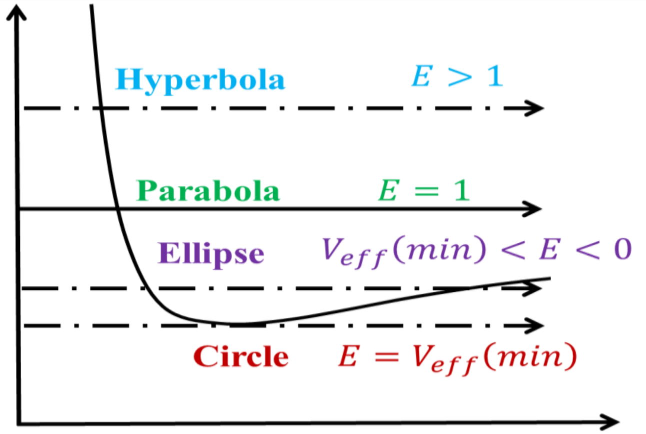
Nature of the Orbit

The nature of orbit is determined by eccentricity e which depend on energy

Value of E	Value of eccentricity	Nature of orbit
$E > 0$	$e > 1$	Hyperbola
$E = 0$	$e = 1$	Parabola
$V_{eff}(min) < E < 0$	$0 < e < 1$	Ellipse
$E = V_{eff}(min)$	$e = 0$	Circle

we can always set $\theta_o = 0$ And $\frac{1}{c} = \alpha = \frac{L^2}{\mu k} \quad \Rightarrow \frac{1}{r} = C[1 + e\cos(\theta - \theta_o)]$

- Bound motion is possible only for Ellipse or circle.
- The motion of planets is either circular of elliptical.
- The variation of length of the day and seasonal changes suggest that the path of the planet is elliptical.



4.11 Keppler's Laws

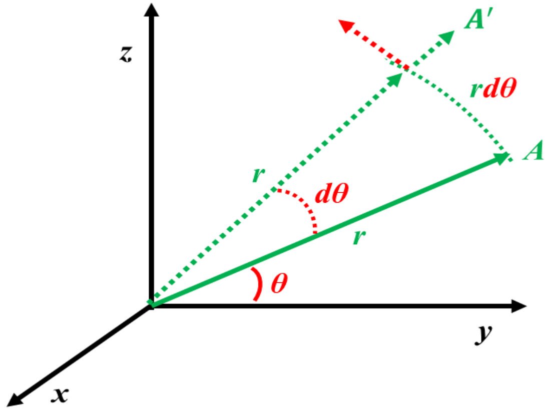
- Since the planet repeat its motion after a fixed period.
- During this period the variation in the length of day and night can only be explained if the orbit of the planet is elliptical.
- We conclude that the planet around the sun describe elliptical orbit with sun at one of its foci.
- Furthermore, the finite and positive values of r_1 and r_2 represents the turning points for the planet or the minimum and maximum radii of the planet during the motion which are called apogee and perigee for the earth orbit.

Keppler Second Law: “The position vector of particle drawn from the origin sweeps equal area in equal interval of times.” OR

“The Areal velocity of the body under the central force is constant.” OR

“The rate of change of area covered by the radial vector drawn from the centre to the planet under the central force is constant.”

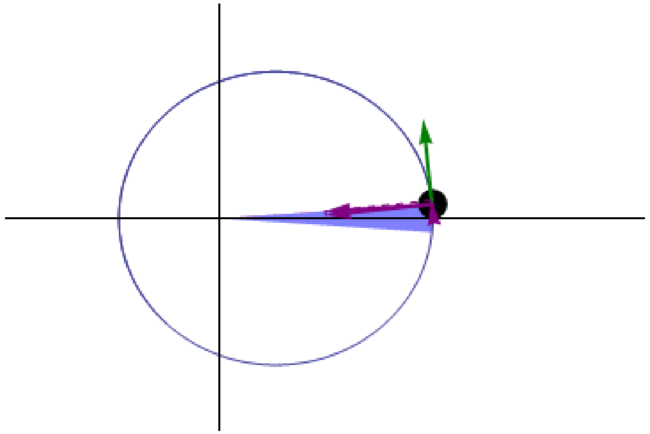
The area of Triangle $\Delta AOA'$ in given figure is



$$d\boldsymbol{A} = \frac{1}{2} r^2 d\theta \hat{n} \qquad \textbf{(4.11.12)}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \hat{n}$$

Multiplying both sides with mass “μ” of the body



$$\mu \frac{d\boldsymbol{A}}{dt} = \frac{1}{2} \mu r^2 \frac{d\theta}{dt} \hat{n} = \frac{1}{2} \mu r^2 \dot{\theta} \hat{n}$$

$$\mu \frac{d\boldsymbol{A}}{dt} = \frac{1}{2} \boldsymbol{l}$$

$$\frac{d\boldsymbol{A}}{dt} = \frac{1}{2\mu} \boldsymbol{l} = \boldsymbol{constant} \qquad \textbf{(4.11.13)}$$

4.11 Keppler's Laws

Kepler's Third Law: "The square of the time period of revolution of the planet is directly proportional to the cube of the semi-major axis of the orbit"

From the Kepler's second law, we know that Areal velocity of the body under the action of central force is constant

$$\dot{A} = \frac{l}{2\mu} = constant \qquad (4.11.13)$$

$$\frac{dA}{dt} = \frac{l}{2\mu} \Rightarrow \int \frac{dA}{dt} dt = \frac{l}{2\mu} \int dt$$

$$\Rightarrow \int_0^A dA = \frac{l}{2\mu} \int_0^\tau dt$$

Where τ is the time period of revaluation.

$$\boxed{\Rightarrow A = \frac{l}{2\mu} \tau} \qquad (4.11.14)$$

Since the area of the ellipse is $\boxed{\boldsymbol{A = \pi ab}}$

4.11 Keppler's Laws

And $b = a\sqrt{1-e^2}$

$$\Rightarrow A = \pi a^2\sqrt{1-e^2} \qquad (4.11.15)$$

And we also know that by using $r_o = a = -\frac{k}{2E}$

$\Rightarrow E = -\frac{k}{2a}$ putting this in $e = \sqrt{1+\frac{2El^2}{\mu k^2}}$

$$\Rightarrow e = \sqrt{1-\frac{l^2}{\mu k a}}$$

$$\Rightarrow e^2 = 1-\frac{l^2}{\mu k a} \Rightarrow \frac{l^2}{\mu k a} = 1-e^2$$

$$\Rightarrow \frac{l}{\sqrt{\mu k a}} = \sqrt{1-e^2} \qquad (4.11.16)$$

Therefore, $A = \pi a^2\sqrt{1-e^2} = \pi a^2\frac{l}{\sqrt{\mu k a}}$

4.11 Keppler's Laws

$$\Rightarrow \; A = \frac{\pi l}{\sqrt{\mu k}} a^{3/2} \qquad (4.11.17)$$

Comparing Equation for A

$$A = \frac{l}{2\mu}\tau = \frac{\pi l}{\sqrt{\mu k}} a^{3/2}$$

$$\Rightarrow \tau = 2\pi \sqrt{\frac{\mu}{k}} a^{3/2}$$

$$\Rightarrow \tau^2 = \frac{4\mu\pi^2}{k} a^3$$

$$\Rightarrow \tau^2 = (Constant) a^3$$

$\Rightarrow \tau^2 \propto a^3$ as desired. (4.11.18)

4.12 Virial Theorem

The **virial theorem** provides a general equation that relates the **average over time of the total Kinetic Energy (T)** of a system, bound by potential forces,

$$< T >= -\frac{1}{2} < \sum_{\boldsymbol{i=1}}^{\boldsymbol{N}} \boldsymbol{F_i} \cdot \boldsymbol{r_i} >$$

The word **virial** for the right-hand side of the equation derives from *vis*, the **Latin** word for "force" or "energy" and was given its technical definition by **Rudolf Clausius** in 1870.

significance : virial theorem is that it allows the average total kinetic energy to be calculated even for very complicated systems that defy an exact solution,

such as those considered in **Statistical mechanics**; this average total kinetic energy is related to the **Temperature** of the system by the **equipartition theorem**.

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Let us consider a system of points masses. Let the particle with mass "m_i", position vector "$\boldsymbol{r_i}$" and momentum " $\boldsymbol{P_i}$". We define a term "G" such that;

$$G = \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \boldsymbol{r_i} \qquad (4.12.1)$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \dot{\boldsymbol{P}}_i \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \dot{\boldsymbol{r}}_i$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} m_i \dot{r}_i^{\,2}$$

$$\frac{dG}{dt} = \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} + 2T \qquad (4.12.2)$$

$$I = \sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{1}{2}\left[\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i} + \sum_{i=1}^{N} \boldsymbol{m_i} \boldsymbol{r_i} \cdot \dot{\boldsymbol{r}}_i\right]$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{2}{2}\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \frac{2}{2}\sum_{i=1}^{N} \boldsymbol{m_i} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{r_i}$$

$$\frac{1}{2}\frac{dI}{dt} = \sum_{i=1}^{N} \boldsymbol{P_i} \cdot \boldsymbol{r_i} = \boldsymbol{G}$$

The time average over the time interval is obtained by integrating both sides of the equation.

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\, dt = \frac{1}{\tau}\left[G(\tau) - G(0)\right] \qquad (4.12.3)$$

4.12 Virial Theorem

If the motion is periodic, all coordinates repeat itself after a certain time "τ"

$$\Rightarrow \frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = 0 \qquad\qquad \text{because } G(\tau) = G(0)$$

If the motion is not periodic, even then for $\tau \gg$ the $\frac{1}{\tau}\,[G(\tau) - G(0)] \to 0$

In both cases left hand side is zero. Comparing Eq. (4.12.2) and (4.12.3)

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt =< \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} > +2 < T >= 0$$

$$\Rightarrow 2 < T >=-< \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < \sum_{i=1}^{N} \boldsymbol{F_i} \cdot \boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < \sum_{i=1}^{N} (-\boldsymbol{\nabla_r} V) \cdot \boldsymbol{r_i} >$$

$$\Rightarrow< T >= -\frac{1}{2} < -\frac{\boldsymbol{dV}}{\boldsymbol{dr}} \cdot \boldsymbol{r} > \qquad (4.12.4)$$

4.12 Virial Theorem

Since $V = \frac{k}{r}$ and $\frac{\boldsymbol{dV}}{\boldsymbol{dr}} = -\frac{k}{r^2}$ for central force

$\frac{\boldsymbol{dV}}{\boldsymbol{dr}} \cdot \boldsymbol{r} = -\frac{k}{r^2} \cdot r = -\frac{k}{r}$ putting this value in Eq. (4.12.4)

$$< T >= -\frac{1}{2} < -\frac{\boldsymbol{dV}}{\boldsymbol{dr}}.r >= -\frac{1}{2} < \frac{k}{\boldsymbol{r}} >$$

$$\Rightarrow < T >= -\frac{1}{2} < V > \qquad (4.12.5)$$

It is true for every system having potential

$V = kr^{n+1}$

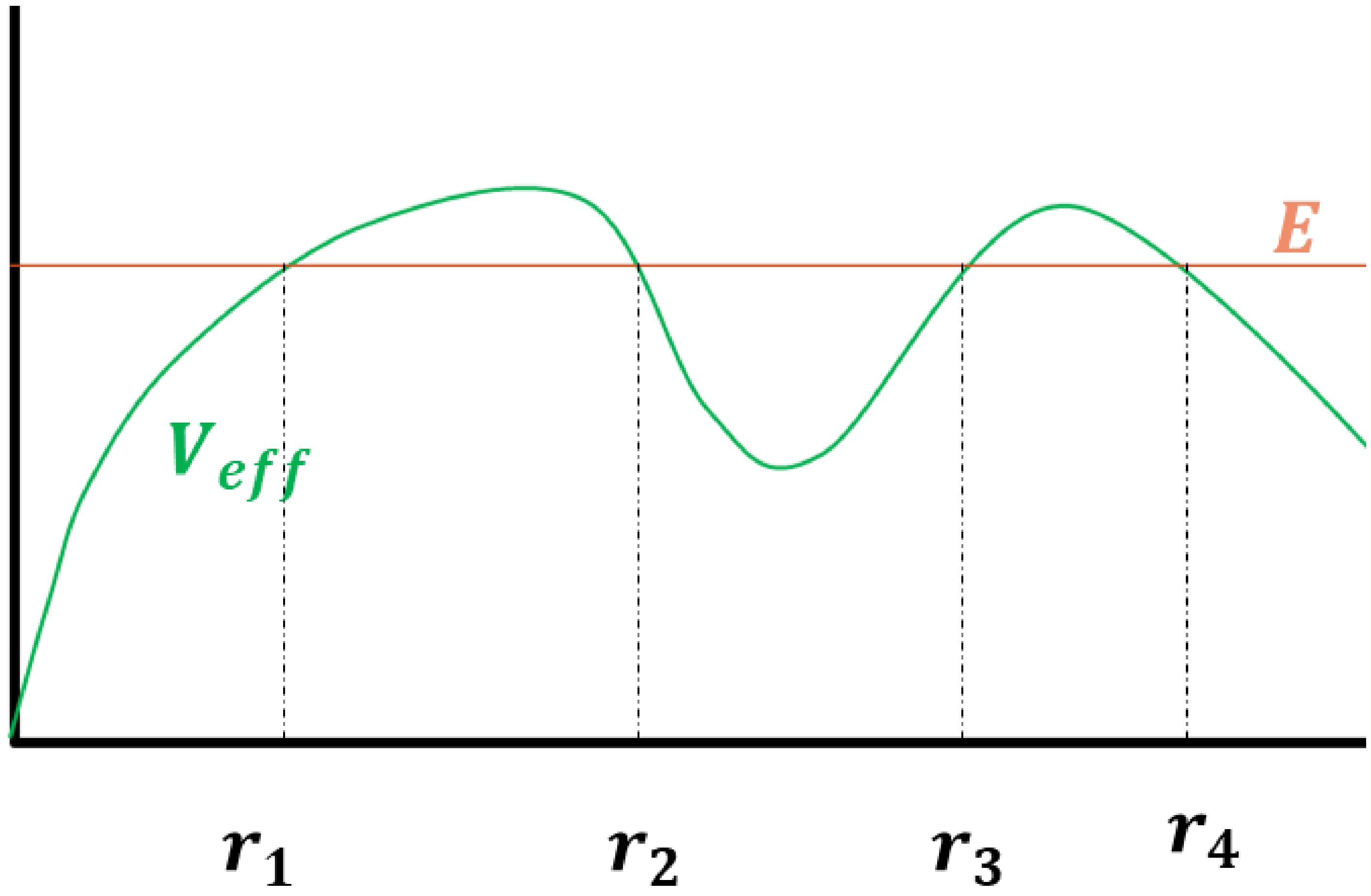
$$\Rightarrow < T >= \frac{n+1}{2} < V > \qquad (4.12.6)$$

$V = -\frac{k}{r}$ & $\frac{\boldsymbol{dV}}{\boldsymbol{dr}} = \frac{k}{r^2}$ for central attractive force

$\frac{\boldsymbol{dV}}{\boldsymbol{dr}} \cdot \boldsymbol{r} = \frac{k}{r^2} \cdot r = \frac{k}{r}$ putting this value in Eq. (4.12.4)

$$< T >= -\frac{1}{2} < -\frac{dV}{dr}.r >= -\frac{1}{2} < -\frac{k}{\boldsymbol{r}} >$$

$$\Rightarrow < T >= -\frac{1}{2} < V >$$


E
V_{eff}
r_1
r_2
r_3
r_4