Chapter 5
Lecture 1 & 2
Collision

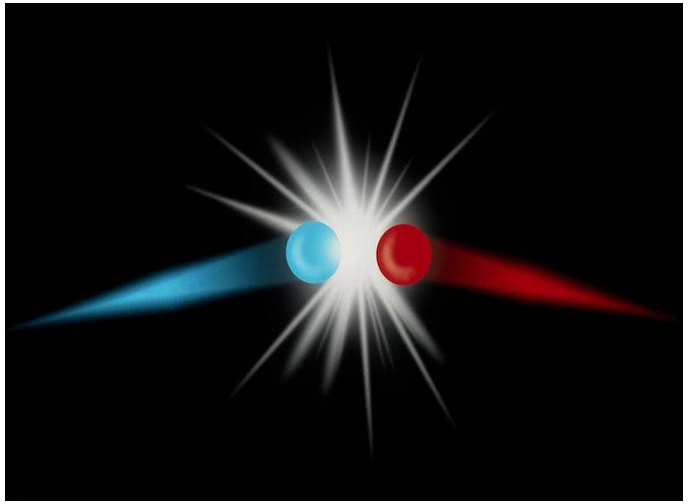
Akhlaq Hussain

5.1 Collision of Particles

Collision is when two bodies run into each other

Collision or **scattering** experiments are important to understand the nature of interaction between two particles.

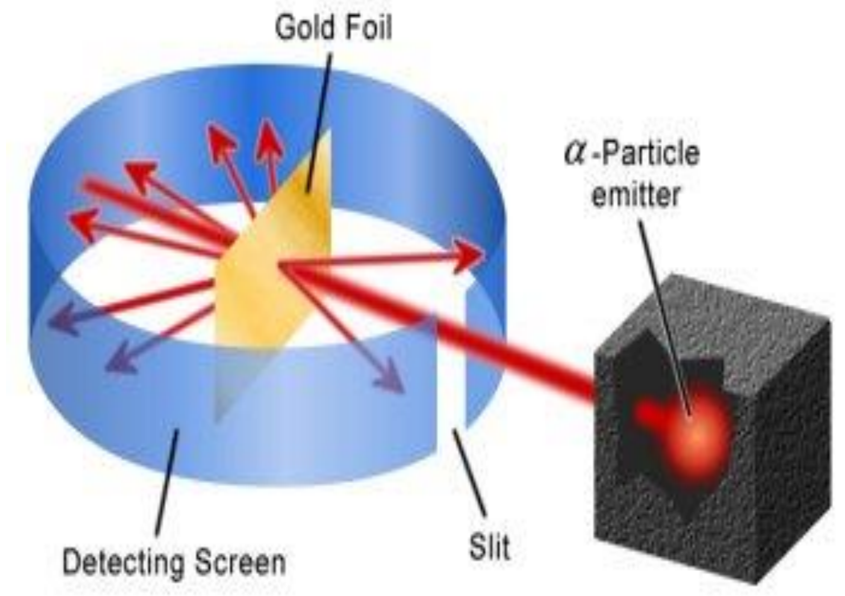
Nature of interacting Particles/bodies.



5.1 Collision of Particles

Rutherford's experiment of α-particles scattering by the atoms in a thin foil of gold revealed the existence of positively charged nucleus in the atom.

- The α-particles were scattered in all directions.
- Some passes undeflected by the foil.
- Some scattered through larger angles even back scattered.
- The only valid explanation of large angle deflection was if the total positive charge were concentrated at some small region.
- Rutherford called it nucleus.



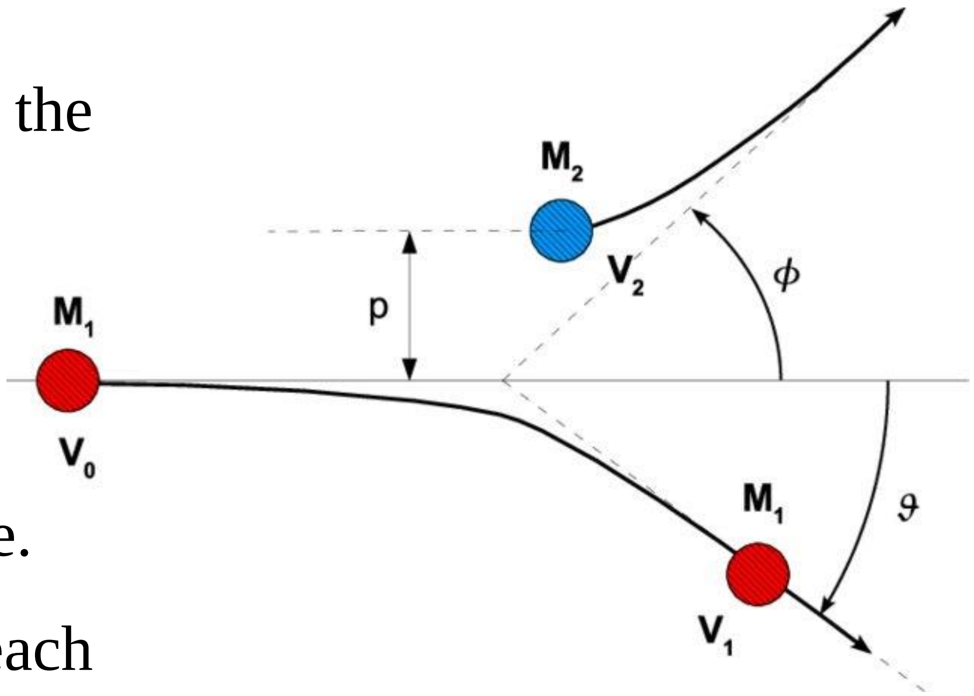
5.1 Collision of Particles

Collision or **scattering**

Initial Condition: the particles are for away from each other.

Target usually at rest and the other approaches the stationary target.

Particle interact in short interval of time.



The interaction forces appears and play their role.

Final condition; particles move away from each other and the incident particle is deflected through an angle called scattering angle.

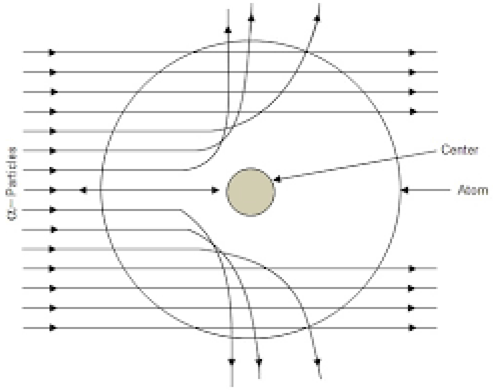
5.1 Collision of Particles

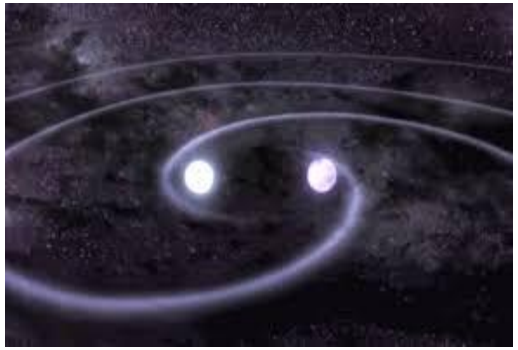
The force of interaction in different cases are due to different reasons.

- In the collision between two billiard balls, the force of interaction is due to elasticity. Which appear only when balls come into physical contact.

- In case of scattering of alpha particle by the nuclei, the force is due to electrostatic interaction.

- In deflection of stars, the force of interaction is due to of gravity,





5.1 Collision of Particles

Elastic and Inelastic Scattering

In Elastic Scattering; kinetic energy and momentum remain conserved and internal structure is not affected.

In Inelastic scattering; kinetic energy is not conserved. It is converted into other forms of energy such as heat or sd.

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Most of the interactions are inelastic particularly when large objects are interacting.

Similarly, in case of charge particles interaction, some amount of energy is released in form of electromagnetic radiation.

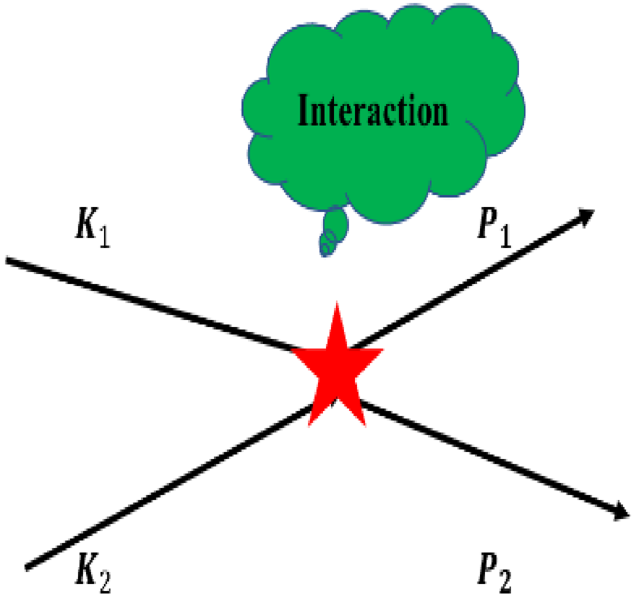
In some particles interaction other sub-atomic particles are also created.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Consider elastic scattering in **Laboratory** & **C.M** system.

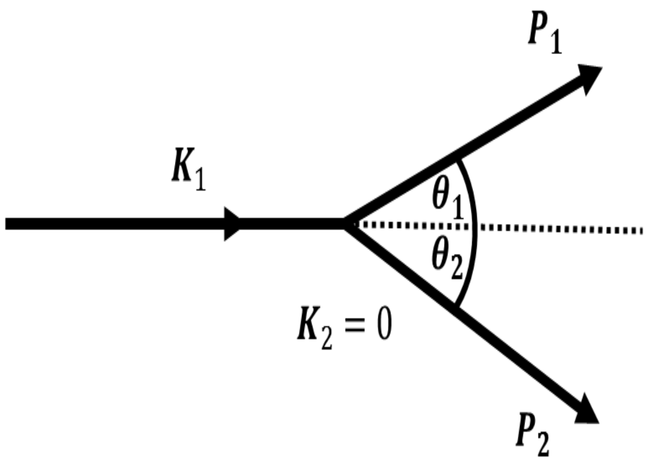
i) In general collision;

- particles move from certain distance towards each other.
- Both particles come closer and interact each other before scattering in final direction.



ii) **Laboratory frame**

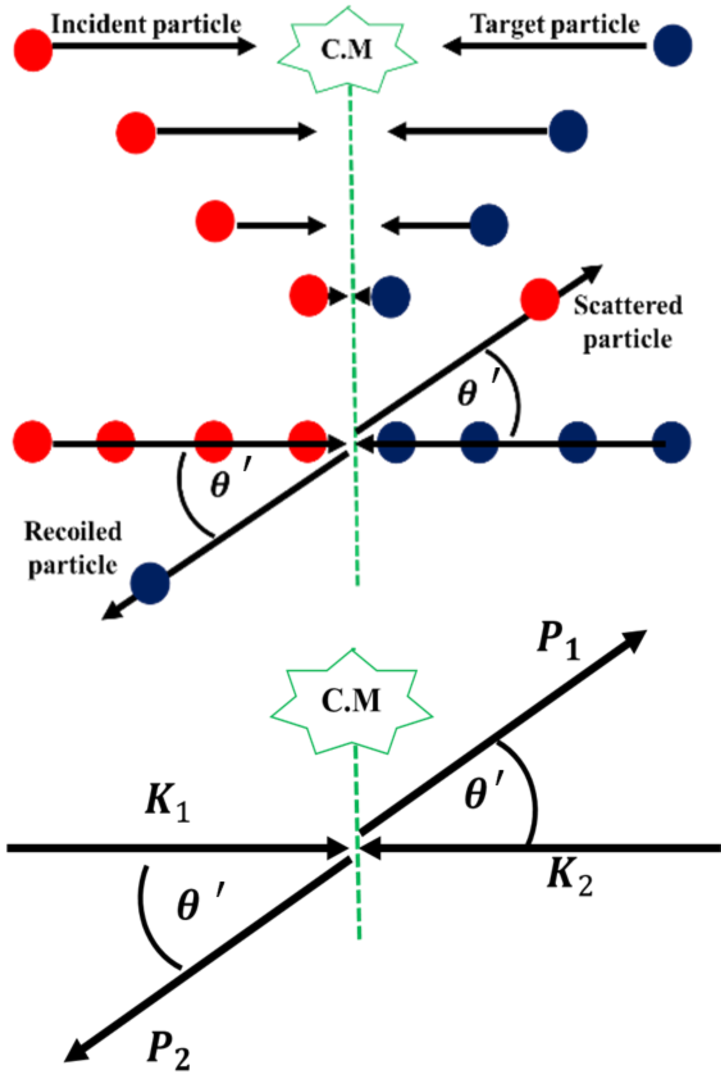
- usually target particle is taken at rest.
- Incident particle approach the target at rest.
- Reach closest distance & Scattered in different direction depending on nature of interaction..
- After collision Incident is called scattered & target is called recoiled particle



5.2 Elastic Scattering: Laboratory and Centre of Mass System

iii) **Centre of mass frame;**

- Centre of mass is assumed to be at rest.
- Both incident and target particle approach each other as observed from centre of mass.
- Regardless of the actual state of target particle which may be at rest or in motion.
- In centre of mass both particles approach each other and after interaction the are scattered in different directions.

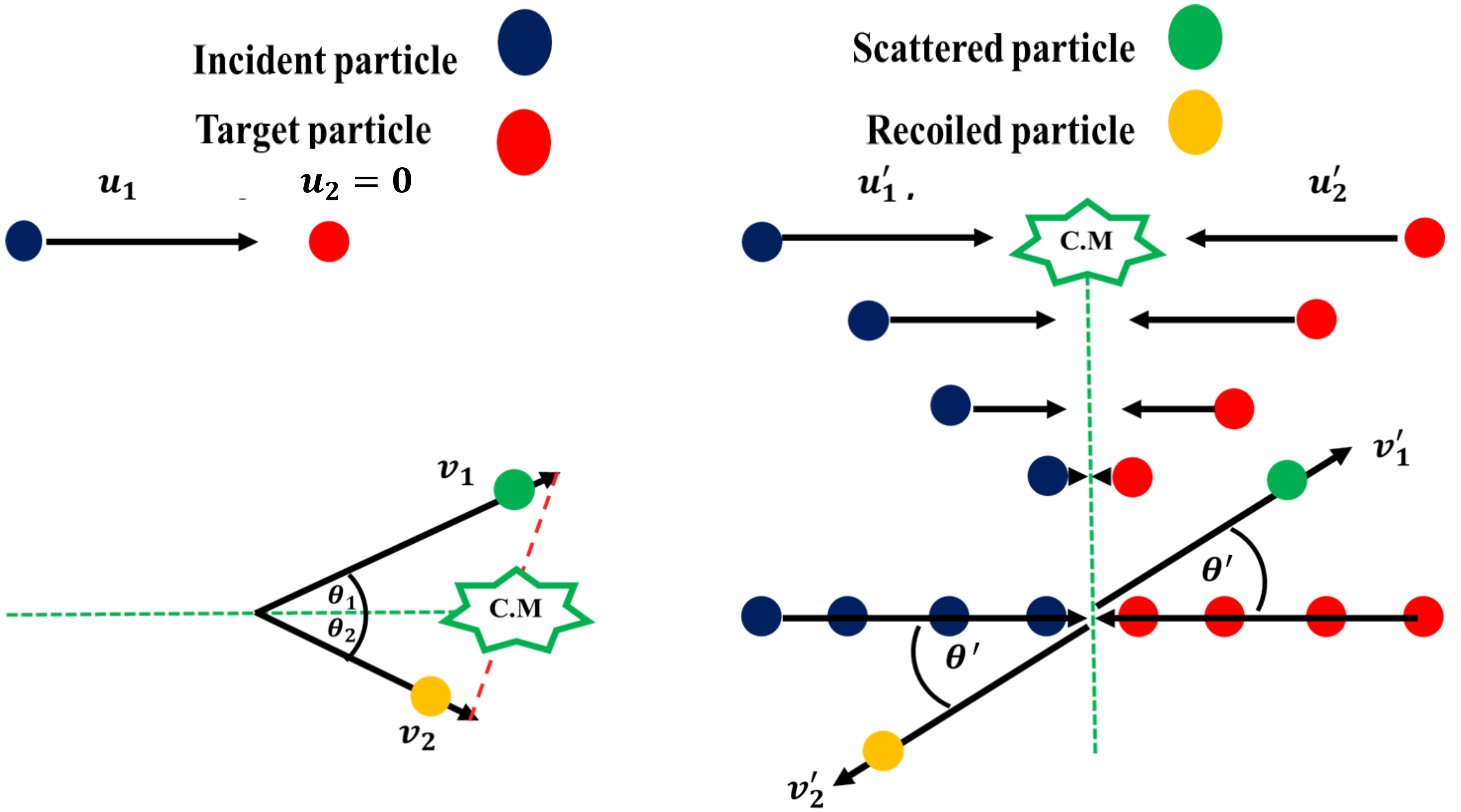


5.2 Elastic Scattering: Laboratory and Centre of Mass System

- **Choice of proper coordinated system is important in solving scattering problems.**
- **Each system has its own advantages and disadvantages.**
- **Actual measurements in experiments are made in the laboratory coordinate system.**
- **To avail the advantages & simplifications obtained in a C.M. coordinates system,**
- **The transformation relations between the quantities measured in the laboratory coordinate system and those measured in the C.M system.**

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations of C.M and Lab. System



5.2 Elastic Scattering: Laboratory and Centre of Mass System

In Laboratory Coordinate system

"**R**" is the position vector of Centre of Mass in the laboratory coordinates system.

$m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = M\boldsymbol{R}$

Differentiating above equation.

$\Rightarrow m_1\, \dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2 = M\dot{\boldsymbol{R}}$

$\Rightarrow m_1\boldsymbol{u}_1 + m_2\boldsymbol{u}_2 = M\boldsymbol{V}$

Where $\boldsymbol{u}_2 = 0$ and $m_1 + m_2 = M$

$\Rightarrow m_1\boldsymbol{u}_1 = (m_1 + m_2)\boldsymbol{V}$

$$\Rightarrow \boldsymbol{V} = \frac{m_1}{(m_1+m_2)}\boldsymbol{u}_1 \qquad \textbf{(5.2.1)}$$

C.M is moving towards m_2 with velocity $\frac{m_1}{(m_1+m_2)}\boldsymbol{u}_1$ in laboratory coordinate system.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

In C.M System

Since in C.M system, centre of mass is at rest and "m_2" is moving towards the centre of mass. Therefore, "m_2" must move with velocity "u_2'"

$$\boldsymbol{u}_2' = -\boldsymbol{V} = -\frac{m_1}{(m_1+m_2)}\boldsymbol{u}_1 \qquad \textbf{(5.2.2)}$$

And $\boldsymbol{u}_1' = \boldsymbol{u}_1 - V$

$$\Rightarrow \boldsymbol{u}_1' = \boldsymbol{u}_1 - \frac{m_1}{(m_1+m_2)}\boldsymbol{u}_1$$

$$\Rightarrow \boldsymbol{u}_1' = \frac{m_2}{(m_1+m_2)}\boldsymbol{u}_1 \qquad \textbf{(5.2.3)}$$

From Eq.s **(5.2.2)** & **(5.2.3)**

$$\boldsymbol{K}_2' = m_2\boldsymbol{u}_2' = -\frac{m_1 m_2}{(m_1+m_2)}\boldsymbol{u}_1 = -\frac{m_2}{(m_1+m_2)}\boldsymbol{K}_1$$

$$\boldsymbol{K}_1' = m_1\boldsymbol{u}_1' = \frac{m_2 m_1}{(m_1+m_2)}\boldsymbol{u}_1 = \frac{m_2}{(m_1+m_2)}\boldsymbol{K}_1$$

Momentum in C.M coordinates are equal and opposite

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations of C.M and Lab. System

Conservation of momentum in C.M system. Let "$\boldsymbol{K}'_1$" and "$\boldsymbol{K}'_2$" are initial momentum, where "$\boldsymbol{P}'_1$" and "$\boldsymbol{P}'_2$" final momentum in C.M,

$$\boldsymbol{K}'_1 + \boldsymbol{K}'_2 = \boldsymbol{P}'_1 + \boldsymbol{P}'_2 = \boldsymbol{0} \qquad \textbf{(5.2.12)}$$

$$\Rightarrow |\boldsymbol{K}'_1| = |\boldsymbol{K}'_2| \qquad \textbf{(5.2.13)}$$

and

$$\Rightarrow |\boldsymbol{P}'_1| = |\boldsymbol{P}'_2| \qquad \textbf{(5.2.14)}$$

Now conservation of energy

$$\frac{1}{2} m_1 \dot{\boldsymbol{u}}'_1 + \frac{1}{2} m_1 \dot{\boldsymbol{u}}'_2 = \frac{1}{2} m_1 \dot{\boldsymbol{v}}'_1 + \frac{1}{2} m_1 \dot{\boldsymbol{v}}'_2$$

$$\Rightarrow \frac{\boldsymbol{K}'^{\,2}_1}{2m_1} + \frac{\boldsymbol{K}'^{\,2}_2}{2m_2} = \frac{\boldsymbol{P}'^{\,2}_1}{2m_1} + \frac{\boldsymbol{P}'^{\,2}_2}{2m_2} \qquad \textbf{(5.2.15)}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations of C.M and Lab. System

Using equations, **Eq. (5.2.13)** & **Eq. (5.2.14)**

$$\frac{\boldsymbol{K}_1'^{\,2}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right) = \frac{\boldsymbol{P}_1'^{\,2}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right)$$

$$\Rightarrow \boldsymbol{K}_1'^{\,2} = \boldsymbol{P}_1'^{\,2}$$

$$\Rightarrow |\boldsymbol{K}_1'| = |\boldsymbol{P}_1'|$$

$$\frac{\boldsymbol{K}_2'^{\,2}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right) = \frac{\boldsymbol{P}_2'^{\,2}}{2}\left(\frac{m_1+m_2}{m_1 m_2}\right)$$

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$$\Rightarrow |\boldsymbol{K}_2'| = |\boldsymbol{P}_2'|$$

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$$|\boldsymbol{u}_1'| = |\boldsymbol{v}_1'| \quad \& \quad |\boldsymbol{u}_2'| = |\boldsymbol{v}_2'| \qquad \textbf{(5.2.16)}$$

Therefore, in C.M, the particle momentum and magnitude of velocity remain constant. Therefore, we don’t need the separate measurements before and after collision.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations of C.M and Lab. System

Now in Laboratory system

$$\boldsymbol{K_1} = \boldsymbol{P_1} + \boldsymbol{P_2} \quad \textbf{as} \quad \boldsymbol{K_2} = \mathbf{0} \qquad \textbf{(5.2.17)}$$

Now the Kinetic energy before and after the Collision

$$\Rightarrow \frac{1}{2} m_1 \boldsymbol{u_1^2} + \frac{1}{2} m_2 \boldsymbol{u_2^2} = \frac{1}{2} m_1 \boldsymbol{v_1^2} + \frac{1}{2} m_1 \boldsymbol{v_1^2}$$

$$\Rightarrow \frac{\boldsymbol{K_1^2}}{2m_1} = \frac{\boldsymbol{P_1^2}}{2m_1} + \frac{\boldsymbol{P_1^2}}{2m_2} \qquad \textbf{(5.2.18)}$$

In case of laboratory system, we need to measure the momentum before and after collision separately.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

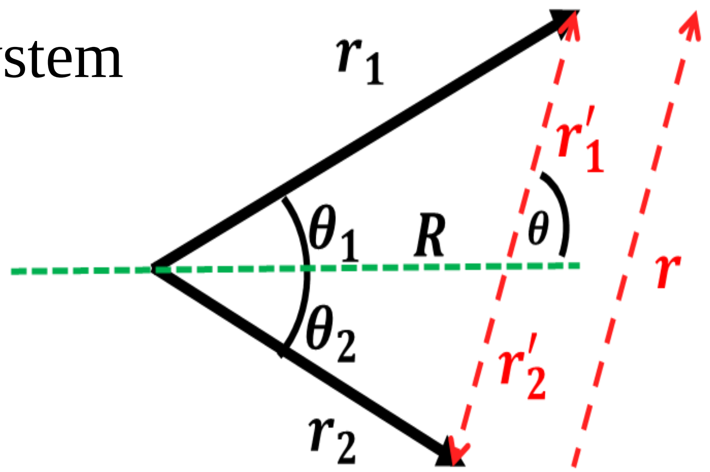
Transformation equations of C.M and Lab. System

Consider the following geometry of the problem

$\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vector of "m_1" and "m_1" in Lab. system. Where "$\boldsymbol{r}'_1$" and "$\boldsymbol{r}'_2$" are the position vectors in C.M system.

$$\boldsymbol{r} = \boldsymbol{r}'_1 - \boldsymbol{r}'_2 = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \textbf{(5.2.4)}$$

Since in C.M system $\boldsymbol{R}' = 0$ Position vector of C.M in C.M system



$$\Rightarrow m_1\, \boldsymbol{r}'_1 + m_2 \boldsymbol{r}'_2 = 0 \qquad \textbf{(5.2.5)}$$

Adding and subtracting $m_2 \boldsymbol{r}'_1$ in Eq. **(5.2.5)**

$$\Rightarrow (m_1 + m_2)\boldsymbol{r}'_1 - m_2(\boldsymbol{r}'_1 - \boldsymbol{r}'_2) = 0$$

$$\Rightarrow (m_1 + m_2)\boldsymbol{r}'_1 - m_2 \boldsymbol{r} = 0$$

$$\Rightarrow \boldsymbol{r}'_1 = \frac{m_2}{(m_1+m_2)}\boldsymbol{r} = \frac{\mu}{m_1}\,\boldsymbol{r} \qquad \textbf{(5.2.6)}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations of C.M and Lab. System

And adding and subtracting $m_1\boldsymbol{r}_2'$ we get $\boldsymbol{r}_2' = -\frac{m_2}{(m_1+m_2)}\boldsymbol{r} = -\frac{\mu}{m_2}\boldsymbol{r}$ **(5.2.7)**

Eq.(5.2.6) $\Rightarrow \dot{\boldsymbol{r}}_1' = \frac{\mu}{m_1}\dot{\boldsymbol{r}}$ **(5.2.8)**

And Eq. (**5.2.7**) $\Rightarrow \dot{\boldsymbol{r}}_2' = -\frac{\mu}{m_2}\dot{\boldsymbol{r}}$ **(5.2.9)**

From Eq. (**5.2.4**) $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_1' - \dot{\boldsymbol{r}}_2' = \dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2$

$$\Rightarrow \boldsymbol{u} = \dot{\boldsymbol{r}}_1' - \dot{\boldsymbol{r}}_2' = \dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{r}}_2 \quad \textbf{(5.2.10)}$$

"$\boldsymbol{u}$" is the relative velocity of "m_1" with respect to "m_2"

$$m_1\dot{\boldsymbol{r}}_1' = \mu\,\dot{\boldsymbol{r}} = -m_2\dot{\boldsymbol{r}}_2' \quad \textbf{(5.2.11)}$$

$$\boldsymbol{P}_1' = -\boldsymbol{P}_2' \qquad \& \qquad |\boldsymbol{P}_1'| = |\boldsymbol{P}_2'|$$

Characteristic property of C.M that total momentum before & after collision will be zero.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

Consider the figure of scattering of two particles we will now obtain the relation between velocities, momentum and scattering angles.

$$\boldsymbol{r_1} = \boldsymbol{R} + \boldsymbol{r'_1}$$

$$\boldsymbol{r_2} = \boldsymbol{R} + \boldsymbol{r'_2}$$

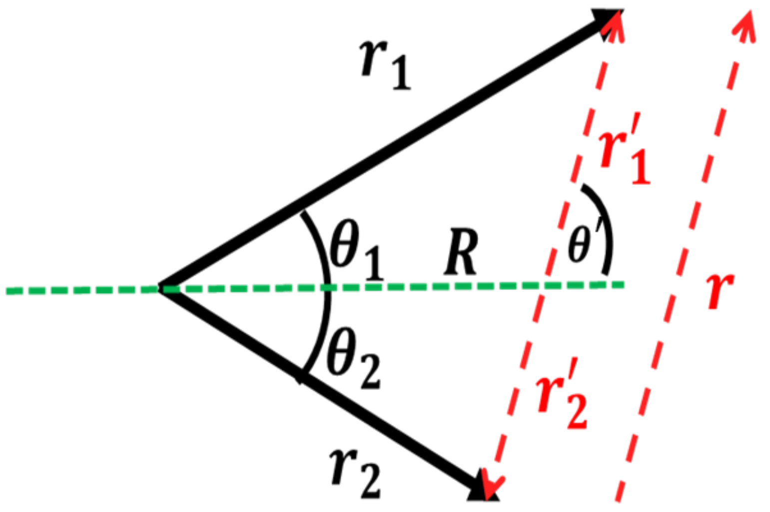
and $\dot{\boldsymbol{r}}_1 = \boldsymbol{v_1} = \boldsymbol{V} + \boldsymbol{v'_1} = \boldsymbol{V} + \frac{\mu}{m_1}\boldsymbol{u}$

$$\boldsymbol{P_1} = m_1\dot{\boldsymbol{r}}_1 = m_1\boldsymbol{v_1} = m_1\boldsymbol{V} + m_1\boldsymbol{v'_1} = m_1\,\boldsymbol{V} + \mu\boldsymbol{u}$$

$$\dot{\boldsymbol{r}}_2 = \boldsymbol{v_2} = \boldsymbol{V} + \boldsymbol{v'_2} = \boldsymbol{V} - \frac{\mu}{m_2}\boldsymbol{u}$$

$$\boldsymbol{P_2} = m_2\dot{\boldsymbol{r}}_2 = m_2\boldsymbol{v_2} = m_2\boldsymbol{V} + m_2\boldsymbol{v'_2} = m_2\,\boldsymbol{V} - \mu\boldsymbol{u}$$

$$M\boldsymbol{V} = \boldsymbol{P_1} + \boldsymbol{P_2} = \boldsymbol{K_1}$$



To understand the above equations, we solve the problem geometrically.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

Since $|\boldsymbol{P_1}'| = |\boldsymbol{P_2}'|$

Consider a circle having radius $\boldsymbol{P_1}' = \mu\boldsymbol{u}$ from Eq. (5.2.11)

From figure $\overline{AO} = m_1\boldsymbol{V}$ & $\overline{OC} = m_2\boldsymbol{V}$

$$\overline{AB} = \boldsymbol{P_1} = m_1\,\boldsymbol{V} + \mu\boldsymbol{u} = m_1\,\boldsymbol{V} + \boldsymbol{P_1}'$$

$$\overline{BC} = \boldsymbol{P_2} = m_2\,\boldsymbol{V} - \mu\boldsymbol{u} = m_2\,\boldsymbol{V} - \boldsymbol{P_1}'$$

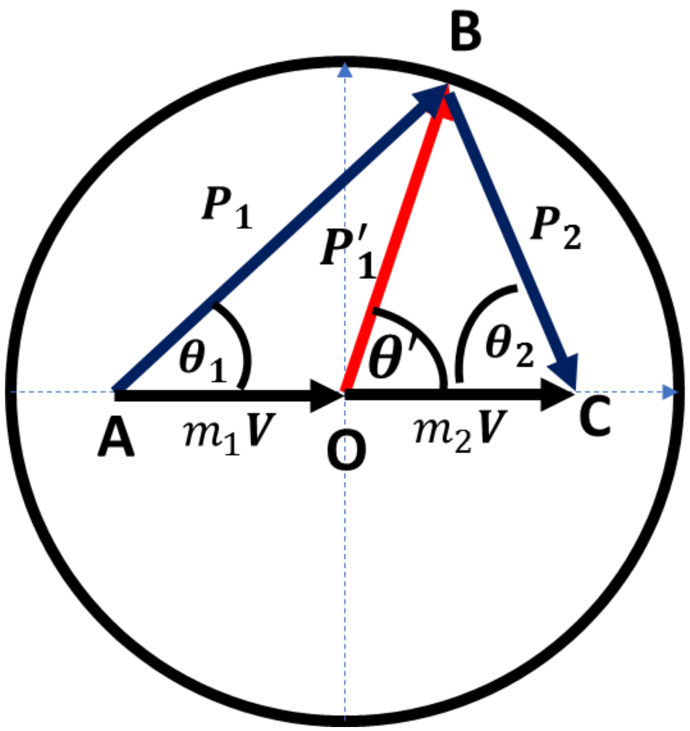
Since before the collision

$$m_1\boldsymbol{r_1} + m_2\boldsymbol{r_2} = M\boldsymbol{R} \Rightarrow m_1\boldsymbol{u_1} + m_2\boldsymbol{u_2} = (m_1 + m_2)\boldsymbol{V}$$

$$\Rightarrow \boldsymbol{K_1} + \boldsymbol{K_2} = (m_1 + m_2)\boldsymbol{V}$$

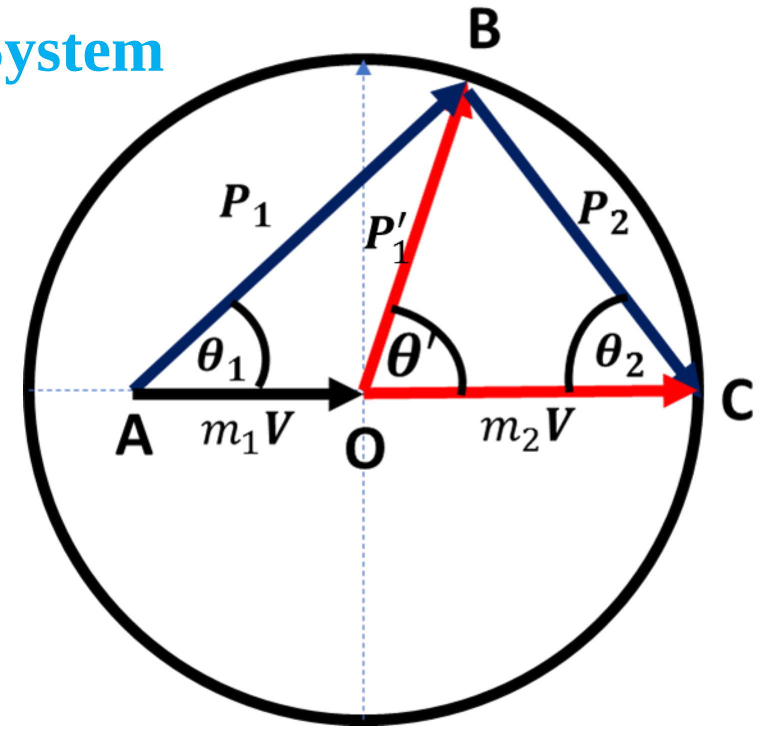
$$\Rightarrow \boldsymbol{K_1} = (m_1 + m_2)\boldsymbol{V}$$

$$\Rightarrow \mathbf{V} = \frac{\boldsymbol{K_1}}{(m_1+m_2)}$$



5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System



Now $\overline{OC} = m_2 \boldsymbol{V} = \frac{m_2 \boldsymbol{K_1}}{(m_1+m_2)}$

$$m_2 \boldsymbol{V} = \frac{\mu \boldsymbol{K_1}}{m_1} = \frac{\mu m_1 \boldsymbol{u_1}}{m_1}$$

$m_2 \boldsymbol{V} = \mu \boldsymbol{u_1} = \mu(\boldsymbol{u_1} - \boldsymbol{u_2})$ we know that $\boldsymbol{u_2} = \mathbf{0}$

$$m_2 V = \mu u = P_1'$$

Therefore, $\overline{OC} = \overline{OB}$

And Point C must be on circle. Therefore angle $< OBC$ & $< BCO$ are equal to $\boldsymbol{\theta_2}$

$$\boldsymbol{\theta_2} + \boldsymbol{\theta_2} + \boldsymbol{\theta'} = \boldsymbol{\pi}$$

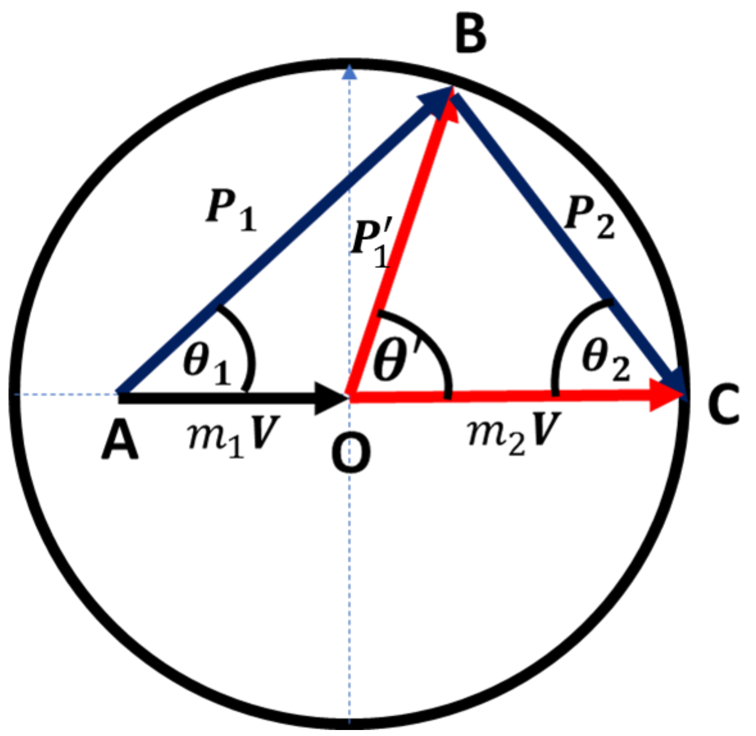
$$\Rightarrow \boldsymbol{\theta_2} = \frac{\boldsymbol{\pi} - \boldsymbol{\theta'}}{\mathbf{2}}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

Now position of A which will be decided as following

$$\frac{\overline{AO}}{\overline{OC}} = \frac{m_1 \boldsymbol{V}}{m_2 \boldsymbol{V}} = \frac{m_1}{m_2}$$



If $m_2 = m_1$ A must lay on the circle

If $m_2 > m_1$ A must lay inside the circle

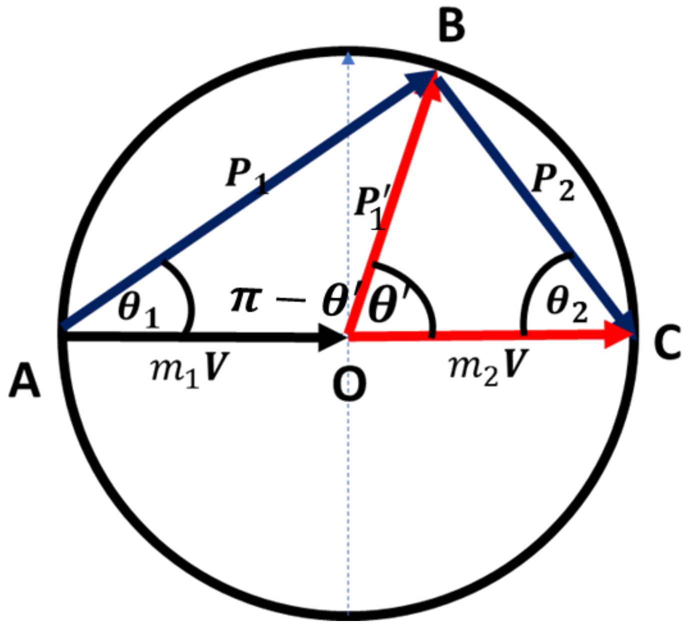
If $m_2 < m_1$ A must lay outside the circle

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

For $m_2 = m_1$ A must lay on the circle

$$\overline{AO} = \overline{OB}$$



Therefore, two angles of ΔAOB are equal and angle $< ABO$ is also equal to $\boldsymbol{\theta_1}$ and angle $< AOB$ is $(\boldsymbol{\pi - \theta'})$

$$\boldsymbol{2\theta_1 + (\pi - \theta') = \pi}$$

$$\boldsymbol{\theta_1 = \frac{\theta'}{2}}$$

And

$$\boldsymbol{\theta_1 + \theta_2 = \frac{\pi}{2}}$$

Thus, if the masses are equal, they will move at right angle to each other.

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

For $m_2 < m_1$ A must lay outside the circle

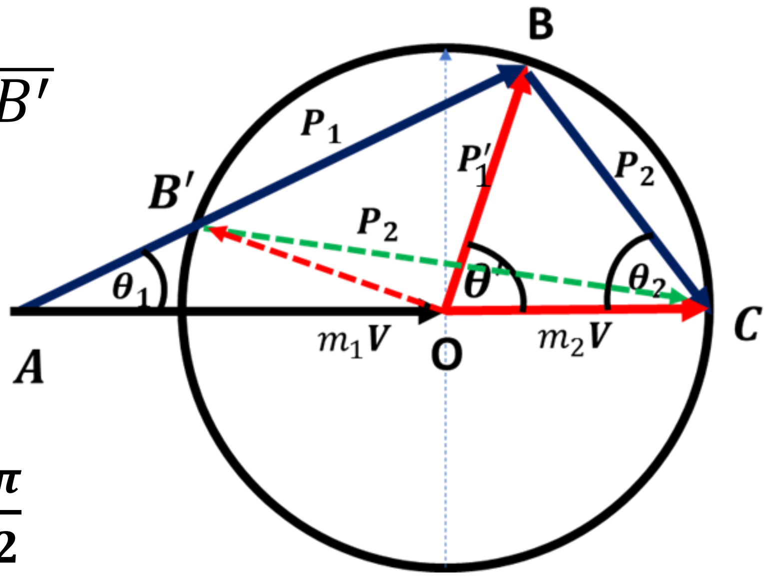
There are two values exist as represented by $\overline{AB}$ and $\overline{AB'}$ for first particle

And

$\overline{BC}$ and $\overline{B'C}$ for second particle

$\overline{AB}$ and $\overline{BC}$ represent forward scattering for which $\theta' < \frac{\pi}{2}$

$\overline{AB'}$ and $\overline{B'C}$ represents back scattering for which $\theta' > \frac{\pi}{2}$



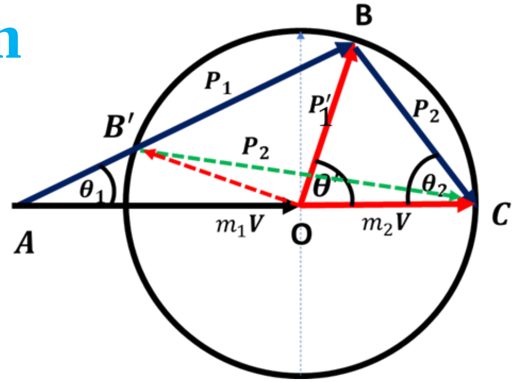
5.2 Elastic Scattering: Laboratory and Centre of Mass System

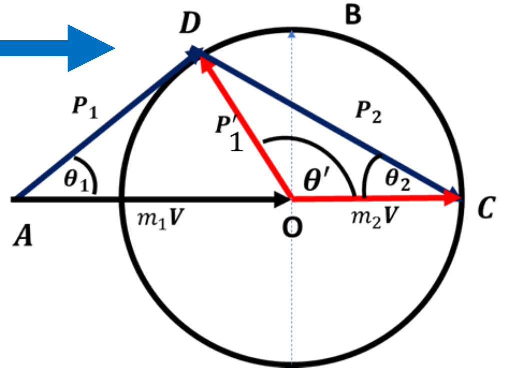
Transformation equations for angles in C.M and Lab. System

This is as for as C.M coordinates are concerned.

In Lab. System $\boldsymbol{\theta_1 < \frac{\pi}{2}}$ for both forward & back scattering.

This angle varies from zero (no scattering) to maximum when AB is tangent to the circle. $\overline{AB} = \overline{AD}$ as shown.

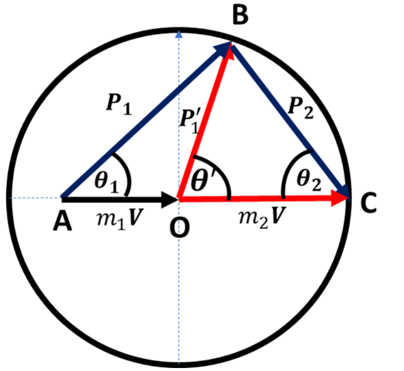




$$\sin\theta_{1(max)} = \frac{\overline{OD}}{\overline{OA}} = \frac{\overline{OC}}{\overline{OA}} = \frac{m_2}{m_1}$$

Which mean that larger the target is larger will be the scattering angle.

For $\boldsymbol{m_2 > m_1}$ A must lay inside the circle



Only one value of momentum exists as represented by $\overline{AB}$ in previous figure

5.2 Elastic Scattering: Laboratory and Centre of Mass System

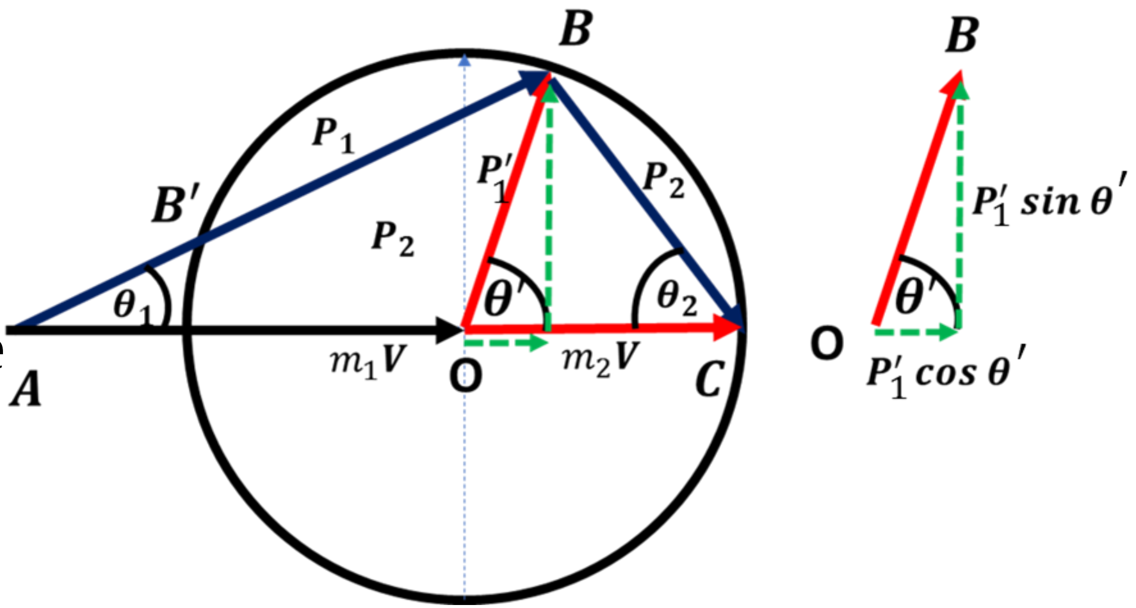
Transformation equations for angles in C.M and Lab. System

For General Solution of the problem

$$\tan\theta_1 = \frac{P_1' \sin\theta'}{m_1 V + P_1' \cos\theta'}$$

$$\tan\theta_1 = \frac{\sin\theta'}{\frac{m_1 V}{P_1'} + \cos\theta'}$$

$$\frac{m_1 V}{P_1'} = \frac{m_1 V}{m_2 V}$$ since $m_2 V = P_1'$ From figure



And $$\frac{m_1 V}{P_1'} = \frac{m_1}{m_2}$$

Therefore, $$\tan\theta_1 = \frac{\sin\theta'}{\frac{m_1}{m_2} + \cos\theta'}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

For $m_1 = m_2$

$$tan\,\theta_1 = \frac{sin\theta'}{1+cos\theta'} = \frac{2\,sin\frac{\theta'}{2}cos\frac{\theta'}{2}}{2\,cos^2\frac{\theta'}{2}}$$

$$\Rightarrow tan\,\theta_1 = \frac{sin\frac{\theta'}{2}}{cos\frac{\theta'}{2}}$$

$$\Rightarrow tan\,\theta_1 = tan\,\frac{\theta'}{2}$$

$$\Rightarrow \theta_1 = \frac{\theta'}{2}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

For $m_1 \neq m_2$

$$\tan\theta_1 = \frac{\sin\theta'}{\frac{m_1}{m_2} + \cos\theta'} \quad \textit{Squaring both sides}$$

$$\frac{\sin^2\theta_1}{\cos^2\theta_1} = \frac{\sin^2\theta'}{\left(\frac{m_1}{m_2} + \cos\theta'\right)^2}$$

$$\sin^2\theta_1\left(\frac{{m_1}^2}{{m_2}^2} + \cos^2\theta' + 2\frac{m_1}{m_2}\cos\theta'\right) = \sin^2\theta' \cos^2\theta_1$$

$$\sin^2\theta_1\left(\frac{{m_1}^2}{{m_2}^2} + \cos^2\theta' + 2\frac{m_1}{m_2}\cos\theta'\right) = (1 - \cos^2\theta')\cos^2\theta_1$$

$$(\sin^2\theta_1 + \cos^2\theta_1)\cos^2\theta' + \frac{{m_1}^2}{{m_2}^2}\sin^2\theta_1 + 2\frac{m_1}{m_2}\sin^2\theta_1\cos\theta' = \cos^2\theta_1$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

$$(sin^2\theta_1 + cos^2\theta_1)cos^2\theta' + \frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 + 2\frac{m_1}{m_2}sin^2\theta_1\, cos\theta' = cos^2\theta_1$$

$$\Rightarrow cos^2\theta' + 2\frac{m_1}{m_2}sin^2\theta_1\, cos\theta' + \left(\frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 - cos^2\theta_1\right) = 0$$

This equation is quadratic in $cos\theta'$ *the solution of the equation will be as follow*

$$cos\theta' = -\frac{m_1}{m_2}sin^2\theta_1 \pm \sqrt{\frac{m_1{}^2}{m_2{}^2}sin^4\theta_1 - \frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 + cos^2\theta_1}$$

$$cos\theta' = -\frac{m_1}{m_2}sin^2\theta_1 \pm \sqrt{\frac{m_1{}^2}{m_2{}^2}sin^2\theta_1\,(1 - cos^2\theta_1) - \frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 + cos^2\theta_1}$$

$$cos\theta' = -\frac{m_1}{m_2}sin^2\theta_1 \pm \sqrt{\frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 - \frac{m_1{}^2}{m_2{}^2}sin^2\theta_1\, cos^2\theta_1 - \frac{m_1{}^2}{m_2{}^2}sin^2\theta_1 + cos^2\theta_1}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

$$cos\theta' = -\frac{m_1}{m_2}sin^2\theta_1 \pm \sqrt{-\frac{{m_1}^2}{{m_2}^2}sin^2\theta_1\,cos^2\theta_1 + cos^2\theta_1}$$

$$cos\theta' = -\frac{m_1}{m_2}sin^2\theta_1 \pm cos\theta_1\sqrt{1-\frac{{m_1}^2}{{m_2}^2}sin^2\theta_1}$$

For $\theta_{1(max)}$, $\sin\theta_{1(max)} = \frac{m_2}{m_1}$ *putting in above equation*

$$cos\theta' = -\frac{m_2}{m_1}\left(\frac{m_2}{m_1}\right)^2 \pm cos\theta_{1(max)}\sqrt{1-1}$$

$$cos\theta' = -\frac{m_2}{m_1} = -\sin\theta_{1(max)}$$

$$cos\theta' = \cos\left(\theta_{1(max)} + \frac{\pi}{2}\right)$$

$$\Rightarrow \theta_{1(max)} = \theta' - \frac{\pi}{2}$$

5.2 Elastic Scattering: Laboratory and Centre of Mass System

Transformation equations for angles in C.M and Lab. System

Now for θ_2

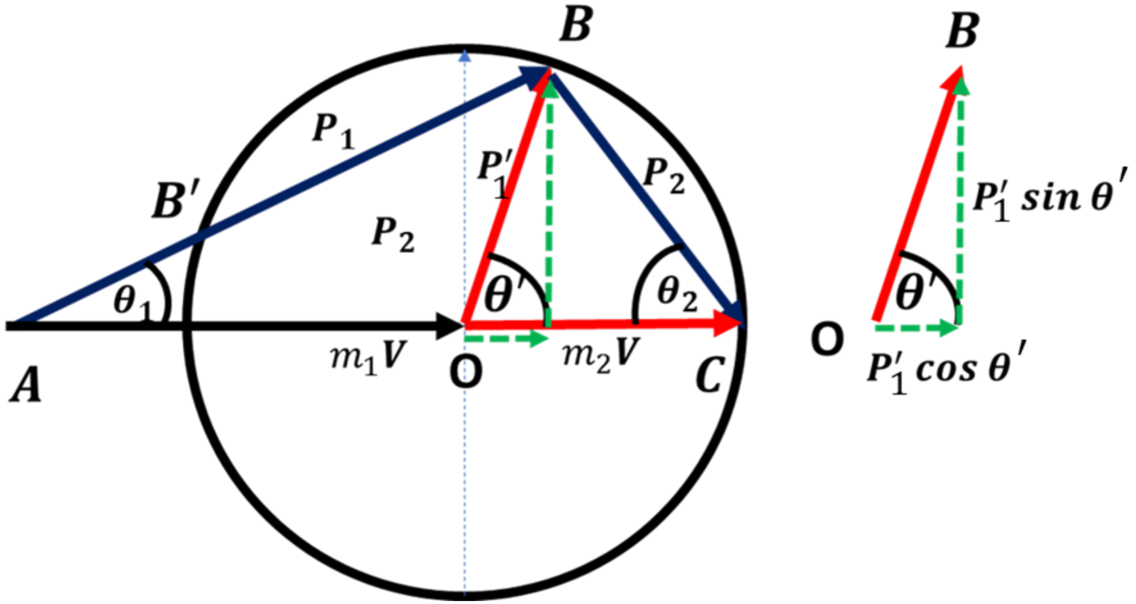
$$\tan\theta_2 = \frac{P_1'\, sin\theta'}{m_2 \boldsymbol{V} - P_1' cos\theta'}$$

$$\Rightarrow \tan\theta_2 = \frac{sin\theta'}{\frac{m_2 \boldsymbol{V}}{P_1'} - cos\theta'}$$

$$\Rightarrow \tan\theta_2 = \frac{sin\theta'}{1 - cos\theta'}$$

$$\Rightarrow \tan\theta_2 = \cot\frac{\theta'}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta'}{2}\right)$$

$$\Rightarrow \theta_2 = \frac{\boldsymbol{\pi - \theta'}}{\boldsymbol{2}}$$



5.2 Elastic Scattering: Laboratory and Centre of Mass System

The transformation equation of angle in two frames of reference are defined

$$cos\theta' = -\frac{m_1}{m_2} sin^2\theta_1 \pm cos\theta_1 \sqrt{1 - \frac{{m_1}^2}{{m_2}^2} sin^2\theta_1}$$

We have predicted the maximum scattering angle $\theta_{1(max)} = \theta' - \frac{\pi}{2}$ and

the possible angle for the particles of equal mass.

Where $\theta_1 = \frac{\theta'}{2}$ and $\theta_2 = \frac{\pi - \theta'}{2}$

The angle θ' varies from 0 to π

Angle θ_1 from 0 to $\frac{\pi}{2}$

And θ_2 from $\frac{\pi}{2}$ to 0

Such that the sum of θ_1 and θ_2 will not exceed $\frac{\pi}{2}$.

Chapter 5
Lecture 3
Cross Section

Akhlaq Hussain

5.3 Cross-Section

Collision is when two bodies run into each other

Collision or **scattering** experiments are important to understand the nature of interaction between two particles.

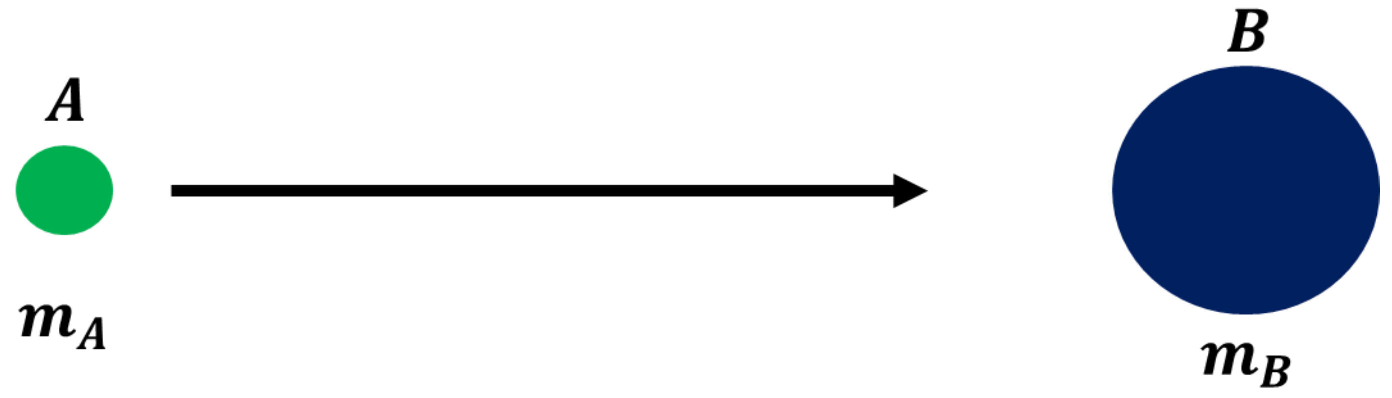
Nature of interacting Particles/bodies.

The most important in a collision or scattering is the cross-section of collision/Scattering.

To understand the cross section, we will start from very simple examples.

5.3 Cross-Section

Let us consider a point mass A is incident on a body B. The probability that ball/mass.



A will make collision with B depends on area of B .

If the area of B is large.

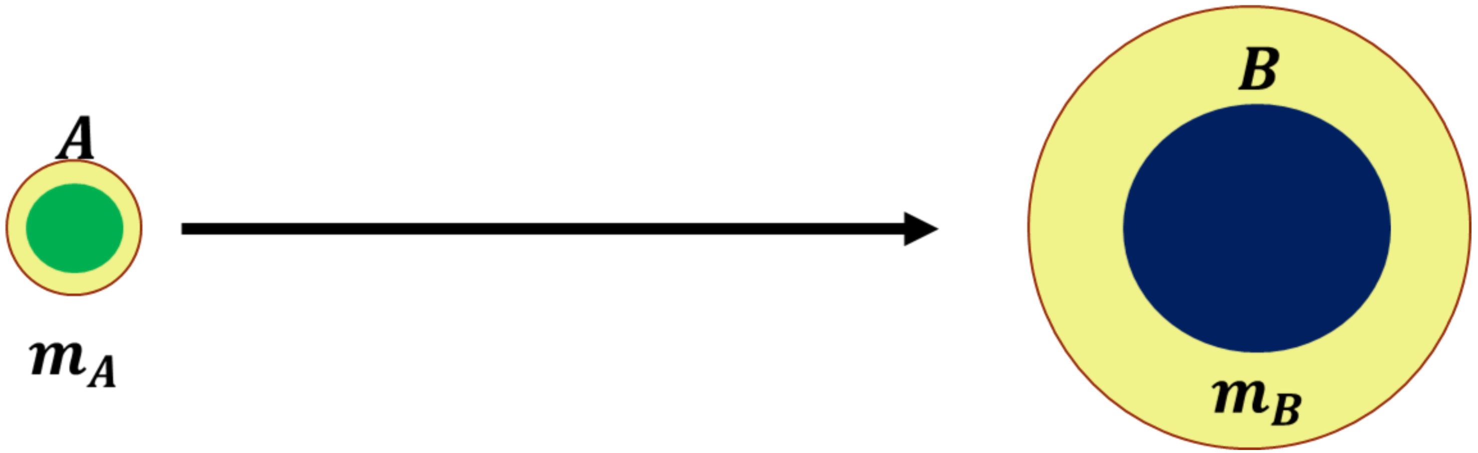
The chance that A will hit B is more or vice versa.

Therefore, the probability that A will collide with body B is more in case the cross-section area of B is large.

5.3 Cross-Section

If we replace balls by two charges.

Then the probability of collision will be more as charge particles can be scattered without physical collision.



Therefore,

The cross-section of B will be the area of an imaginary disc associated with B.

A more convenient way to express the probability of scattering is cross-section

5.3 Cross-Section

In collision experiment of microparticles (subatomic)

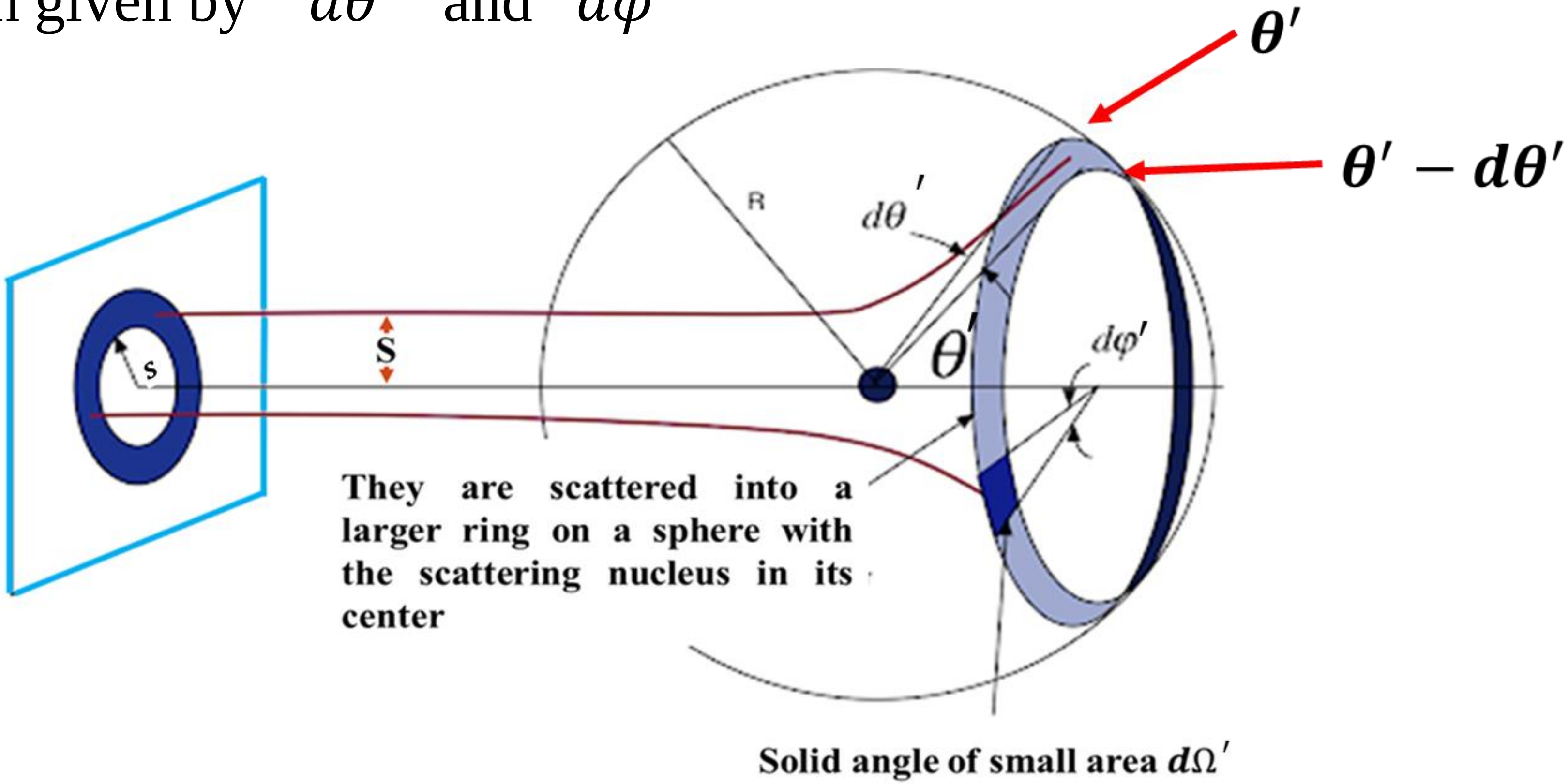
- Study individual particles through the scattering process is not possible.
- Particles are identical and cannot be distinguished.
- The concept of probability is introduced in terms of the cross-section.

Cross-section can be defined as the measure of probability that collision will take place when a single particle is bombarded on a target having one target particle per unit volume.

5.3 Differential Cross-Section in Canter of Mass Coordinates

Let "N" be the number of particles incident per unit area per second on target particles which are at rest.

Let "dN" be the number of particles scattered per second in solid angle "$d\Omega'$" along the direction given by "$d\theta'$" and "$d\varphi'$"



5.3 Differential Cross-Section in Canter of Mass Coordinates

The No. of scattered particles in solid angle "$d\Omega'$" will be proportional to the intensity of the incident particles and the magnitude of solid angle

$$dN \propto N d\Omega'$$

$$dN = \sigma(\theta') N d\Omega'$$

The constant is known as differential cross section

$$\sigma(\theta') = \frac{1}{N}\frac{dN}{d\Omega'}$$

Where solid angle is $d\Omega' = \sin\theta' \, d\theta' d\varphi'$

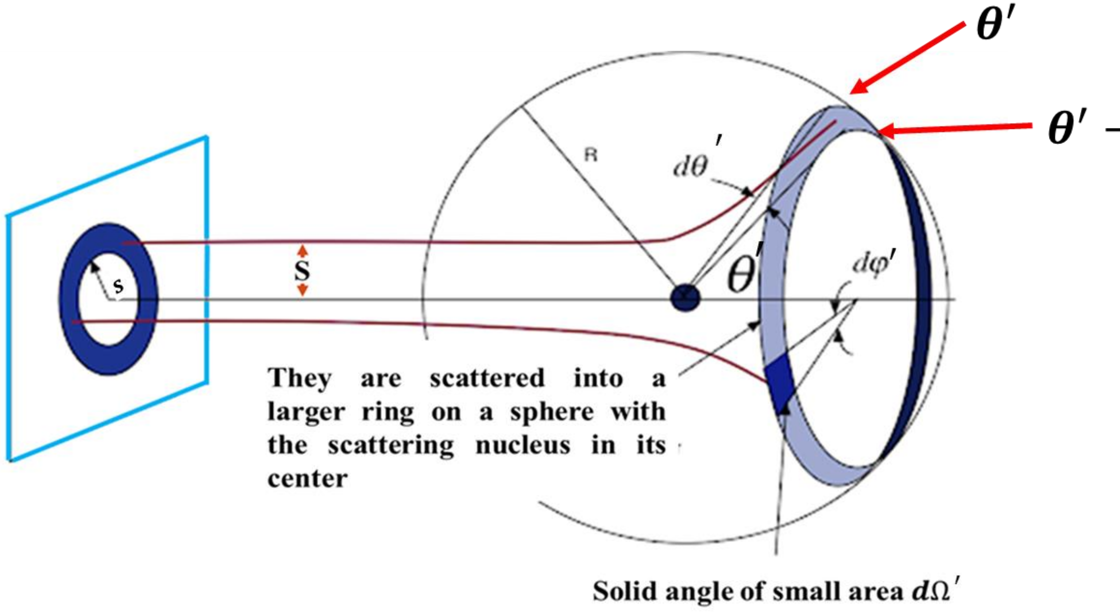
For Azimuthal symmetry

$$\int_0^{2\pi} d\varphi' = 2\pi$$

$$\Rightarrow d\Omega' = 2\pi \sin\theta' \, d\theta'$$

5.3 Differential Cross-Section in Canter of Mass Coordinates

- The particles with impact parameter "s" are scattered through angle "θ'"
- Those having impact parameter "$s + ds$" will be scattered through smaller angle "$\theta' - d\theta'$"
- Where all the particles in ring having radius between "s" and "$s + ds$" and area $2\pi s ds$ will be scattered in solid angle.
- For N is the total particles in the beam of area $A = \pi r^2$ (say: r is the radius of the circular beam), particles on surface of the ring between "s" and "$s + ds$" and area $2\pi s ds$ will be



$$dN = 2\pi s ds N$$

5.3 Differential Cross-Section in Canter of Mass Coordinates

Therefore,

$$dN = 2\pi s ds N = dN = \sigma(\theta')N d\Omega'$$

$$\Rightarrow 2\pi s ds N = \sigma(\theta')N(2\pi \sin\theta' \, d\theta')$$

$$\Rightarrow 2\pi s ds N = 2\pi N\sigma(\theta') \, \sin\theta' \, d\theta'$$

$$\Rightarrow s ds = \sigma(\theta') \, \sin\theta' \, d\theta'$$

$$\Rightarrow \sigma(\Omega') = \frac{d\sigma(\Omega')}{d\Omega'} = \frac{s}{\sin\theta'}\left|\frac{ds}{d\theta'}\right|$$

5.3 Differential Cross-Section in Canter of Mass Coordinates

Now the total Cross section will be

$$\boldsymbol{\sigma_T = \int \sigma(\theta')\, d\Omega'}$$

$$\boldsymbol{\sigma_T = 2\pi \int \sigma(\theta')\ \sin\theta'\, d\theta'}$$

This represent the total probability of particles scattered in all directions per unit intensity of incident bean per second.

5.4 Relation between cross section in C.M. and Lab. Coordinate system

Since the particles scattered in a solid angle $dN = 2\pi N\sigma(\theta_1)\sin\theta_1\, d\theta_1$ in laboratory system is equal to that scattered in corresponding solid angle "$d\Omega'$" in C.M system and equal to $dN = 2\pi N\sigma(\theta')\ \sin\theta'\, d\theta'$. therefore

$$2\pi N\sigma(\theta')\ \sin\theta'\, d\theta' = 2\pi N\sigma(\theta_1)\sin\theta_1\, d\theta_1$$

$$\sigma(\theta') = \sigma(\theta_1)\frac{\sin\theta_1}{\sin\theta'}\frac{d\theta_1}{d\theta'}$$

Since the scattering angle in Lab. And C.M system are related by relation

$$\tan\theta_1 = \frac{\sin\theta'}{\frac{m_1}{m_2} + \cos\theta'}$$

If $m_1 = m_2$

$$\theta_1 = \frac{\theta'}{2}$$

5.4 Relation between cross section in C.M. and Lab. Coordinate system

$$\sigma(\theta') = \sigma(\theta_1)\frac{\sin\theta_1}{\sin\theta'}\frac{d\theta_1}{d\theta'} = \sigma(\theta_1)\frac{\sin\frac{\theta'}{2}}{\sin\theta'}\frac{\frac{d\theta'}{2}}{d\theta'}$$

$$\sigma(\theta') = \frac{1}{2}\sigma(\theta_1)\frac{\sin\frac{\theta'}{2}}{2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}}\frac{d\theta'}{d\theta'}$$

$$\boxed{\sigma(\theta') = \frac{1}{4\cos\frac{\theta'}{2}}\sigma(\theta_1) = \frac{1}{4\cos\theta_1}\sigma(\theta_1)}$$

Now if $m_2 \gg m_1 \Rightarrow \ \theta_1 = \theta'$

$$\sigma(\theta') = \sigma(\theta_1)\frac{\sin\theta_1}{\sin\theta'}\frac{d\theta_1}{d\theta'} = \sigma(\theta_1)\frac{\sin\theta'}{\sin\theta'}\frac{d\theta'}{d\theta'}$$

$$\boxed{\sigma(\theta') = \sigma(\theta_1)}$$

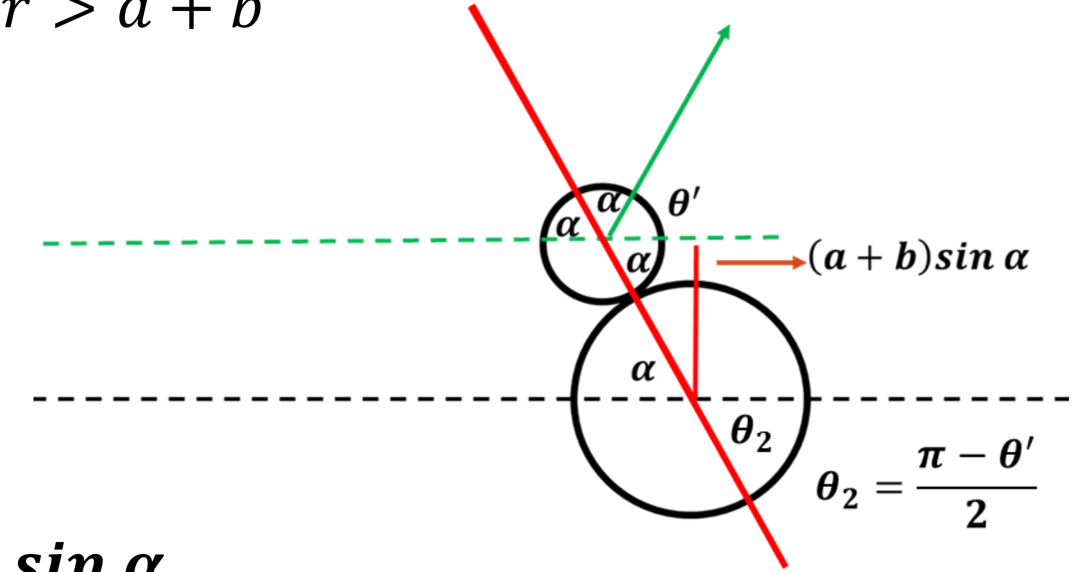
5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

Consider an elastic scattering of sphere having mass m_1 and radius b by a target sphere of mass m_2 and radius a in the C.M and Lab. System.

The law of force

$$V = \begin{cases} \infty & r < a + b \\ 0 & r > a + b \end{cases}$$

The incident sphere will get scattered after rebounding from the surface of the target sphere.



The impact parameter in this case is;

$$\boldsymbol{s = (a+b)\,sin\,\alpha}$$

$$\boldsymbol{s = (a+b)\,sin\left(\frac{\pi - \theta'}{2}\right)}$$

5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

$$s = (a+b)\cos\frac{\theta'}{2}$$

Differentiating above equation

$$\frac{ds}{d\theta'} = -\frac{1}{2}(a+b)\sin\frac{\theta'}{2}$$

$$\sigma(\theta') = \frac{s}{\sin\theta'}\left|\frac{ds}{d\theta'}\right|$$

$$\sigma(\theta') = \frac{(a+b)\cos\frac{\theta'}{2}}{\sin\theta'}\left|-\frac{1}{2}(a+b)\sin\frac{\theta'}{2}\right|$$

$$\sigma(\theta') = \frac{(a+b)^2\cos\frac{\theta'}{2}\sin\frac{\theta'}{2}}{2\ \sin\theta'}$$

5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

$$\sigma(\theta') = \frac{(a+b)^2}{4} \frac{2sin\frac{\theta'}{2}cos\frac{\theta'}{2}}{\sin\theta'}$$

$$\sigma(\theta') = \frac{(a+b)^2}{4} \frac{\sin\theta'}{\sin\theta'}$$

$$\boxed{\sigma(\theta') = \frac{(a+b)^2}{4}}$$

Now the total scattering Cross section in C.M system

$$\sigma_{C.M} = \int \sigma(\theta')\, d\Omega'$$

$$\sigma_{C.M} = \int \frac{(a+b)^2}{4} d\Omega'$$

5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

$$\sigma_{C.M} = \frac{(a+b)^2}{4}\int_0^{\pi} 2\pi \ \sin\theta' \, d\theta'$$

$$\boxed{\sigma_{C.M} = \pi(a+b)^2}$$

Now in the lab system

If $m_1 = m_2$

$$\sigma(\theta') = \sigma(\theta_1)\frac{1}{4\cos\theta_1}$$

$$\sigma(\theta_1) = \sigma(\theta')\, 4\cos\theta_1$$

$$\sigma(\theta_1) = \frac{(a+b)^2}{4} 4\cos\theta_1$$

5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

$$\sigma(\theta_1) = (a+b)^2 \cos\theta_1$$

Integrating above equation

$$\sigma(\theta_1)_T = \int \sigma(\theta_1)\, d\Omega = 2\pi(a+b)^2 \int_o^{\frac{\pi}{2}} \sin\theta_1 \cos\theta_1\, d\theta_1$$

$$\frac{1}{2}$$

$$\sigma(\theta_1)_T = 2\pi(a+b)^2 \int_o^{\frac{\pi}{2}} \sin\theta_1 \cos\theta_1\, d\theta_1$$

$$\sigma(\theta_1)_T = \pi(a+b)^2$$

5.4 Effective Cross Section for scattering of particle with radius b by a sphere of radius a

If $m_1 \ll m_2$ $$\boldsymbol{\sigma(\theta') = \sigma(\theta_1) = \frac{(a+b)^2}{4}}$$

$$\boldsymbol{\sigma(\theta_1)_T = \int \sigma(\theta_1)\, d\Omega = \frac{(a+b)^2}{4} \int d\Omega}$$

$$\boldsymbol{\sigma(\theta_1)_T = 2\pi \frac{(a+b)^2}{4} \int_o^{\frac{\pi}{2}} \sin\theta_1\, d\theta_1}$$

$$\boldsymbol{\sigma(\theta_1)_T = \frac{\pi}{2}(a+b)^2 \int_o^{\frac{\pi}{2}} \sin\theta_1\, d\theta_1 = \frac{\pi}{2}(a+b)^2}$$

If $m_1 \ll m_2$ mean $a \gg b$ $$\boldsymbol{\sigma(\theta_1)_T = \frac{\pi}{2} a^2}$$