Chapter 6
Lecture 1

Canonical Transformations

Akhlaq Hussain

6.1 Canonical Transformations

Hamiltonian formulation $H(q_i, p_i) = \sum_{i=1}^{N} p_i \dot{q}_i - L$ (Hamiltonian)

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

(*Hamiton's Equations*)

one can get the same differential equations to be solved as are provided by the Lagrangian procedure.

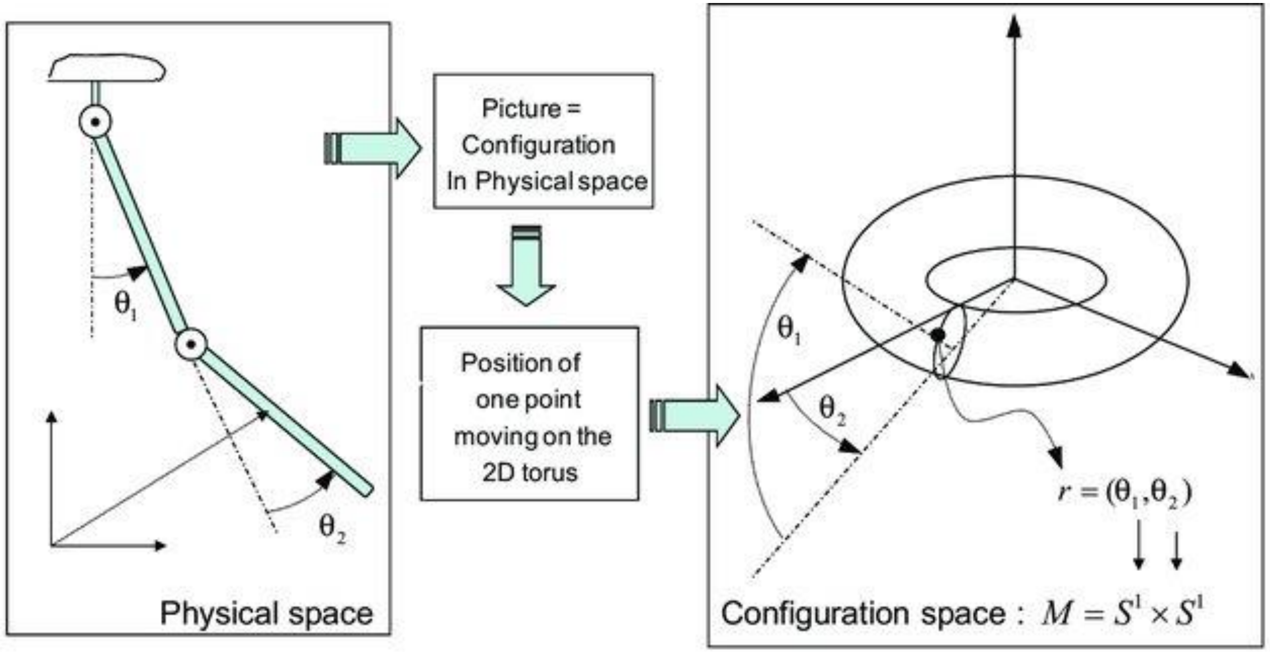
$$L(\dot{q}_i, q_i) = T - V \qquad \text{(Lagrangian)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{(Lagrange's equation)}$$

Therefore, the Hamiltonian formulation does not decrease the difficulty of solving problems. The advantages of Hamiltonian formulation is not its use as a calculation tool, but rather in deeper insight it offers into the formal structure of the mechanics.

6.1 Canonical Transformations

- In Lagrangian mechanics $\{L(\dot{q}_i, q_i)\}$ system is described by "q_i" and velocities" $\dot{q}_i$" in configurational space,
- The parameters that define the configuration of a system are called generalized coordinates and the vector space defined by these coordinates is called configuration space.
- The position of a single particle moving in ordinary Euclidean Space (3D) is defined by the vector $q = q(x, y, z)$ and therefore its configuration space is $Q = \mathbb{R}^3$
- For n disconnected, non-interacting particles, the configuration space is $\mathbb{R}^{3n}$.

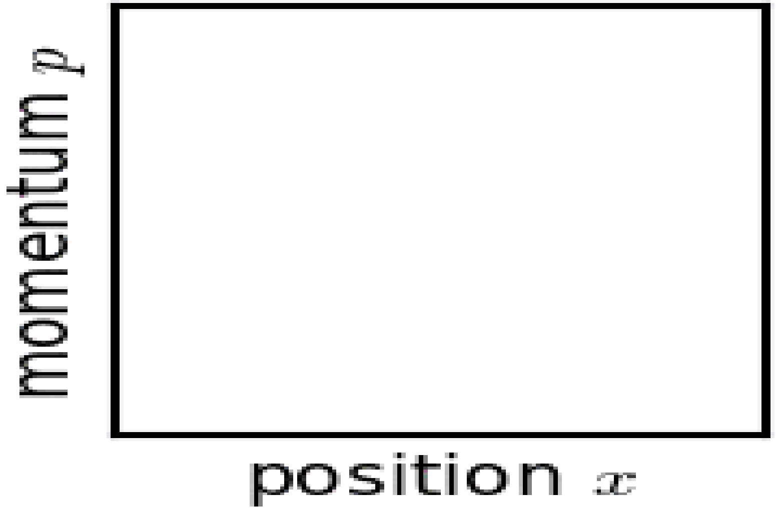


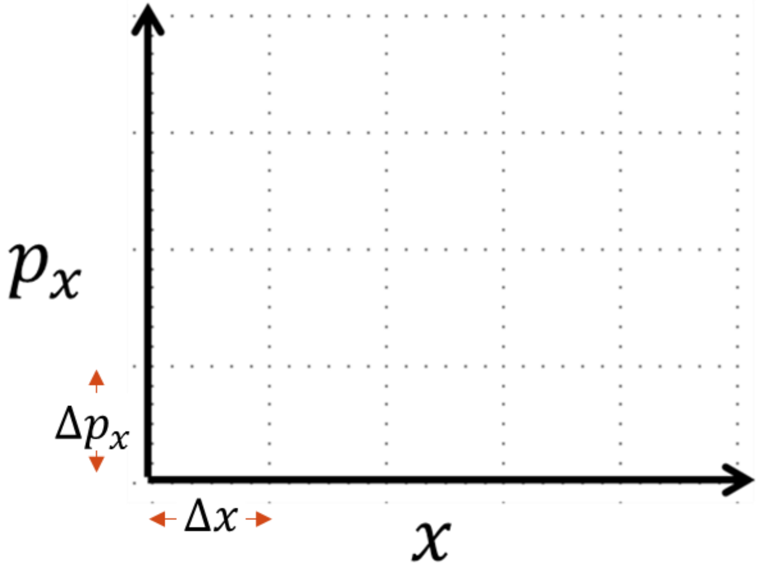
6.1 Canonical Transformations

- In Hamiltonian $\{H(q_i, p_i)\}$ we describe the state of the system in Phase space by generalized coordinates and momenta.

- In dynamical system theory, a Phase space is a space in which all possible states of a system are represented with each possible state corresponding to one unique point in the phase space.

- There exist different momenta for particles with same position and vice versa.





6.1 Canonical Transformations

- To understand the importance of Hamiltonian let us consider a problem for which solution of Hamilton's equations are trivial (simple) and Hamiltonian is constant of motion.
- For this case all the coordinates "q_i" of the problem will be cyclic and all conjugate "p_i" momenta will be constant.

Since $$p_i = \alpha_i = Constant$$

And $$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

$$q_i = \omega_i t + \beta_i$$

β_i is constant and can be find by the initial conditions.

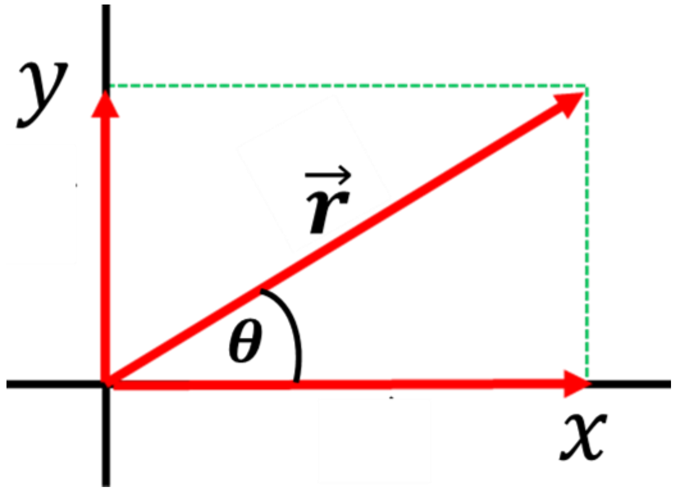
But in real problem it is not necessary that all the coordinates are cyclic.

6.1 Canonical Transformations

- Practically, it rarely happens that all the coordinates are cyclic.
- However, a system can be described by more than one set of generalized coordinates.
- The motion of particle in plane is described by generalized coordinates.

In cartesian coordinates

$$q_1 = x, \quad \& \quad q_1 = y$$



In polar coordinates

$$q_1 = r, \quad \& \quad q_2 = \theta$$

Both choices are equally valid, but one of the set may be more convenient for the problem under the consideration. Not that for the central force neither x, nor y is cyclic while the second set does contain a cyclic coordinate θ

6.1 Canonical Transformations

- The number of cyclic coordinates thus depend on choice of generalized coordinates,
- For each problem there may be one choice for which more than one or even all the coordinates are cyclic.
- Since the generalized coordinates suggested by the problem will not be cyclic normally,
- They can be replaced by set of cyclic coordinates.
- We must first derive a specific procedure for transforming from one set of variables to some other set that may be more suitable.

6.1 Canonical Transformations

- Let us consider transformation equations

$$Q_i = Q_i(q_i, p_i, t), \;\&\; \; P_i = P_i(q_i, p_i, t)$$

- Such that the general dynamical theory is invariant under these transformations.

- Let us consider a function $K(Q_i, P_i, t)$ such that

$$\boxed{\dot{P}_i = -\frac{\partial K}{\partial Q_i}} \quad \& \quad \boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i}}$$

Q_i & P_i are called canonical coordinates and transformation $q_i \longrightarrow Q_i$ & $p_i \longrightarrow P_i$

$Q_i = Q_i(q_i, p_i, t), \;\&\; \; P_i = P_i(q_i, p_i, t)$ are known as canonical transformations.

6.1 Canonical Transformations

Here "$K(Q_i, P_i, t)$" play role of Hamiltonian and Q_i & P_i must satisfy Hamilton's principle.

$$\delta \int_{t_1}^{t_2} \left[\sum P_i \dot{Q}_i - K(Q_i, P_i, t)\right] dt = 0 \qquad (1)$$

$$\delta \int_{t_1}^{t_2} \left[\sum p_i \dot{q}_i - H(q_i, p_i, t)\right] dt = 0 \qquad (2)$$

Equation (1) and Equation (2) may not be qual; therefore, we can find a function "F" such that

$$\int_{t_1}^{t_2} \frac{dF}{dt} dt = F(t_2) - F(t_1)$$

and $\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0$ where $\delta F(t_2) = \delta F(t_1)$

and

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = \sum P_i \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF}{dt} \qquad (3)$$

Function "F" is called generating function.

6.1 Canonical Transformations

- There are four different possibilities for "F"

1) $F_1(q_i, Q_i, t)$ Provided that q_i, Q_i are treated as independent

2) $F_2(q_i, P_i, t)$ Provided that q_i, P_i are treated as independent

3) $F_3(p_i, Q_i, t)$ Provided that p_i, Q_i are treated as independent

4) $F_4(p_i, P_i, t)$ Provided that p_i, P_i are treated as independent

6.1 Canonical Transformations

Case I: Eq. 3 can be written as

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = \sum P_i \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt} \qquad (4)$$

Since $\frac{dF_1(q_i,Q_i,t)}{dt} = \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$ Therefore Eq. (4)…

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = \sum P_i \dot{Q}_i - K(Q_i, P_i, t) + \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{dQ_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

Comparing coefficient of "$\dot{q}_i$" & "$\dot{Q}_i$" on both sides

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad (5)a$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \qquad (5)b$$

And

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_1}{\partial t} \qquad (5)c$$

6.1 Canonical Transformations

- From Eq. (5a) we can determine "p_i" in terms of q_i, Q_i and t and the inverse transformation Q_i in terms of q_i, p_i and t

Eq. (5)a ⇒ $Q_i = Q_i(q_i, p_i, t)$

Eq. (5)b ⇒ $P_i = P_i(q_i, p_i, t)$

& Eq. (5)c provide connection between new and old Hamiltonian

6.1 Canonical Transformations

Case II: For $F_2(q_i, P_i, t)$ generating function

Since $P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore, $$F_1(q_i, Q_i, t) + \sum P_i\, Q_i = F_2(q_i, P_i, t)$$

Since $$\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}\left[F_2(q_i, P_i, t) - \sum P_i\ Q_i\right]$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_2}{dt} - \sum P_i\, \dot{Q}_i - \sum \dot{P}_i\, Q_i$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = -K(Q_i, P_i, t) + \frac{dF_2}{dt} - \sum \dot{P}_i\, Q_i$$

Since $$\frac{d}{dt} F_2(q_i, P_i, t) = \sum \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t}$$

6.1 Canonical Transformations

Putting in previous equation

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = -K(Q_i, P_i, t) + \sum \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} - \sum \dot{P}_i Q_i$$

Comparing coefficient of "$\dot{q}_i$" & "$\dot{P}_i$" on both sides

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad (6)\text{a}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \qquad (6)\text{b}$$

And

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_2}{\partial t} \qquad (6)\text{c}$$

6.1 Canonical Transformations

Case III: For $F_3(p_i, Q_i, t)$ generating function

Since $$p_i = \frac{\partial F_1}{\partial q_i}$$

Therefore, we can write $$F_1(q_i, Q_i, t) - \sum p_i\, q_i = F_3(p_i, Q_i, t)$$

Since $$\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}\left[F_3(p_i, Q_i, t) + \sum p_i\, q_i\right]$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_3}{dt} + \sum p_i\, \dot{q}_i + \sum \dot{p}_i\, q_i$$

$$\Rightarrow -H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_3}{dt} + \sum \dot{p}_i\, q_i$$

Since $\frac{d}{dt} F_3(p_i, Q_i, t) = \sum \frac{\partial F_3}{\partial p_i} \dot{p}_i + \sum \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t}$, putting in above equation.

6.1 Canonical Transformations

Putting in previous equation

$$-H(q_i,p_i,t) = \sum P_i\,\dot{Q}_i - K(Q_i,P_i,t) + \sum \frac{\partial F_3}{\partial p_i}\dot{p}_i + \sum \frac{\partial F_3}{\partial Q_i}\dot{Q}_i + \frac{\partial F_3}{\partial t} + \sum \dot{p}_i\,q_i$$

Comparing coefficient of "$\dot{p}_i$" & "$\dot{Q}_i$" on both sides

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad (7)a$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \qquad (7)b$$

And

$$K(Q_i,P_i,t) = H(q_i,p_i,t) + \frac{\partial F_3}{\partial t} \qquad (7)c$$

6.1 Canonical Transformations

Case IV: For $F_4(p_i, P_i, t)$ generating function

Since $p_i = \frac{\partial F_1}{\partial q_i}$ & $P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore, we can write $F_1(q_i, Q_i, t) - \sum p_i\, q_i + \sum P_i\, Q_i = F_4(p_i, P_i, t)$

Since $\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}\left[F_4(p_i, P_i, t) + \sum p_i\, q_i - \sum P_i\, Q_i\right]$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_4}{dt} + \sum p_i\, \dot{q}_i + \sum \dot{p}_i\, q_i - \sum P_i\, \dot{Q}_i - \sum \dot{P}_i\, Q_i$$

$$-H(q_i, p_i, t) = -K(Q_i, P_i, t) + \frac{dF_4}{dt} + \sum \dot{p}_i\, q_i - \sum \dot{P}_i\, Q_i$$

Since $\frac{d}{dt} F_4(p_i, P_i, t) = \sum \frac{\partial F_4}{\partial p_i}\dot{p}_i + \sum \frac{\partial F_4}{\partial P_i}\dot{P}_i + \frac{\partial F_4}{\partial t}$, putting in above equation.

6.1 Canonical Transformations

Putting in previous equation

$$-H(q_i, p_i, t) = -K(Q_i, P_i, t) + \sum \frac{\partial F_4}{\partial p_i}\dot{p}_i + \sum \frac{\partial F_4}{\partial P_i}\dot{P}_i + \frac{\partial F_4}{\partial t} + \sum \dot{p}_i\, q_i - \sum \dot{P}_i\, Q_i$$

Comparing coefficient of "$\dot{p}_i$" & "$\dot{P}_i$" on both sides

$$q_i = -\frac{\partial F_4}{\partial p_i} \quad \text{(8)a}$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \quad \text{(8)b}$$

And

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_4}{\partial t} \quad \text{(8)c}$$

6.1 Canonical Transformations

Properties of the Four basic canonical transformations

Generating function	Derivatives of generating function	Trivial special cases	Transformation
$F_1(q_i, Q_i, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$p_i = Q_i,$ $P_i = -q_i$
$F_2(q_i, P_i, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$p_i = P_i$ $Q_i = q_i$
$F_3(p_i, Q_i, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$q_i = -Q_i$ $P_i = -p_i$
$F_4(p_i, P_i, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$q_i = -P_i$ $Q_i = p_i$

Chapter 6
Lecture 2

Canonical Transformations

Akhlaq Hussain

6.1 Canonical Transformations

Properties of the Four basic canonical transformations

Generating function	**Derivatives of generating function**	**Trivial special cases**	**Transformation**
$F_1(q_i, Q_i, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$p_i = Q_i,$ $P_i = -q_i$
$F_2(q_i, P_i, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$p_i = P_i$ $Q_i = q_i$
$F_3(p_i, Q_i, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$q_i = -Q_i$ $P_i = -p_i$
$F_4(p_i, P_i, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$q_i = -P_i$ $Q_i = p_i$

6.2 Conditions for the transformation to be canonical

Conditions for the transformation to be canonical

For $F_1(q_i, Q_i, t) \Rightarrow \quad dF_1 = \sum p_i dq_i - \sum P_i dQ_i$

For $F_2(q_i, P_i, t) \Rightarrow \quad dF_2 = \sum p_i dq_i + \sum Q_i dP_i$

For $F_3(p_i, Q_i, t) \Rightarrow \quad dF_3 = -\sum q_i dp_i - \sum P_i dQ_i$

For $F_4(p_i, P_i, t) \Rightarrow \quad dF_4 = -\sum q_i dp_i + \sum Q_i dP_i$

6.2 Conditions for the transformation to be canonical

The transformation from (q_i, p_i) to (Q_i, P_i) will be canonical if

$$\sum \boldsymbol{p_i dq_i} - \sum \boldsymbol{P_i dQ_i}$$

is an exact differential

Solution: Consider the generating function $F_1(q_i, Q_i)$

$$dF_1 = \sum \frac{\partial F_1}{\partial q_i} dq_i + \sum \frac{\partial F_1}{\partial Q_i} dQ_i$$

Since $\quad p_i = \frac{\partial F_1}{\partial q_i} \quad$ and $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore,

$$dF_1 = \sum p_i dq_i - \sum P_i dQ_i$$

which is an exact differential equation.

6.2 Conditions for the transformation to be canonical

Similarly, considering generating function $F_4(p_i, P_i, t)$

$$dF_4(p_i, P_i, t) = \sum \frac{\partial F_4}{\partial p_i} dp_i + \sum \frac{\partial F_4}{\partial P_i} dP_i$$

Since $q_i = -\frac{\partial F_4}{\partial p_i}$ and $Q_i = \frac{\partial F_4}{\partial P_i}$

Therefore, $dF_4(p_i, P_i, t) = -\sum q_i dp_i + \sum Q_i dP_i$ which is an exact differential

Now subtracting dF_4 from dF_1

$$dF_1 - dF_4 = \sum p_i dq_i - \sum P_i dQ_i + \sum q_i dp_i - \sum Q_i dP_i$$

$$\Rightarrow dF_1 - dF_4 = \left(\sum q_i dp_i + \sum p_i dq_i\right) - \left(\sum Q_i dP_i + \sum P_i dQ_i\right)$$

6.2 Conditions for the transformation to be canonical

$$\Rightarrow \; dF_1 - dF_4 = d(q_i p_i) - d(Q_i P_i)$$

$$\Rightarrow d(F_1 - F_4) = d(q_i p_i - Q_i P_i)$$

Which is exact differential.

Therefore, the transformation is canonical.

And $\Rightarrow F_1 = F_4 + q_i p_i - Q_i P_i$

Examples [Conditions for the transformation to be canonical]

Show that transformation is canonical

$$P = \frac{1}{2}(p^2 + q^2) \text{ and } Q = tan^{-1}\frac{q}{p}$$

Solution: The transformation is canonical if $[pdq - PdQ]$ is an exact differential

$$pdq - PdQ = pdq - \frac{1}{2}(p^2 + q^2)\left[\frac{1}{1+\frac{q^2}{p^2}} d\left(\frac{q}{p}\right)\right]$$

$$pdq - PdQ = pdq - \frac{1}{2}(p^2 + q^2)\left[\frac{1}{1+\frac{q^2}{p^2}} . \left(\frac{pdq - qdp}{p^2}\right)\right]$$

$$\Rightarrow pdq - PdQ = pdq - \frac{1}{2}(p^2 + q^2)\left[\frac{p^2}{p^2+q^2} . \frac{pdq - qdp}{p^2}\right]$$

$$\Rightarrow pdq - PdQ = pdq - \frac{1}{2}[pdq - qdp]$$

Examples [Conditions for the transformation to be canonical]

$$\Rightarrow pdq - PdQ = \frac{1}{2}pdq + \frac{1}{2}qdp$$

$$\Rightarrow pdq - PdQ = \frac{1}{2}(pdq + qdp)$$

$$\Rightarrow pdq - PdQ = \frac{1}{2}d(pq)$$

Hence the transformation is canonical.

Second Approach

Symplectic approach to canonical transformation

Let

$$Q_i = Q_i(q_j, p_j)$$
$$P_i = P_i(q_j, p_j)$$

The inverse transformation are

$$q_j = q_j(Q_i, P_i)$$
$$p_j = p_j(Q_i, P_i)$$

As the transformation does not involve time, therefore the Hamiltonian does not change in this case.

$$K(Q_i, P_i) = H(q_j, p_j)$$

To verify $\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial P_i}$

We Know that $H = H(q_j, p_j)$

$$\frac{\partial H}{\partial P_i} = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} \qquad (1)$$

Second Approach

$$\frac{\partial H}{\partial P_i} = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i} \qquad (1)$$

Now
$$\dot{Q}_i = \sum_j \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial Q_i}{\partial p_j}\dot{p}_j$$

$$\Rightarrow \dot{Q}_i = \sum_j \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_j \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \qquad (2)$$

Comparing eq (1) and eq (2) we concludes that

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}$$

The transformation is Canonical only if

$$\left(\frac{\partial Q_i}{\partial p_j}\right)_{q_j, p_j} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q_i, P_i} \quad \text{and} \quad \left(\frac{\partial Q_i}{\partial q_j}\right)_{q_j, p_j} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q_i, P_i}$$

Similarly, we verify
$$\dot{P}_i = -\frac{\partial H}{\partial Q_i}$$

Second Approach

$$\frac{\partial H}{\partial Q_i} = \sum_j \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q_i} \qquad (3)$$

and

$$\dot{P}_i = \sum_j \frac{\partial P_i}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial P_i}{\partial p_j}\dot{p}_j$$

$$\Rightarrow \dot{P}_i = \sum_j \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_j \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j} \qquad (4)$$

Comparing eq (3) and eq (4) we concludes that

$$\dot{P}_i = -\frac{\partial H}{\partial Q_i}$$

The transformation is Canonical only if

$$\left(\frac{\partial P_i}{\partial p_j}\right)_{q_j,\, p_j} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q_i,\, P_i} \quad \text{and} \quad \left(\frac{\partial P_i}{\partial q_j}\right)_{q_j,\, p_j} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q_i,\, P_i}$$

Examples

Show that transformation is canonical

$$Q = \log\left(\frac{1}{q}\sin p\right) \text{ and } P = q\cot p$$

Solution: since

$$Q = \log\left(\frac{1}{q}\sin p\right)$$

$$\Rightarrow \dot{Q} = \frac{q}{\sin p}\frac{d}{dt}\left(\frac{1}{q}\sin p\right)$$

$$\Rightarrow \dot{Q} = \frac{q}{\sin p}\left[\frac{q\dot{p}\cos p - \dot{q}\sin p}{q^2}\right]$$

$$\Rightarrow \dot{Q} = \frac{\dot{p}\cos p}{\sin p} - \frac{\dot{q}}{q}$$

$$\Rightarrow \dot{Q} = \dot{p}\cot p - \dot{q}\frac{1}{q}$$

$$\Rightarrow \dot{Q} = -\frac{\partial H}{\partial q}\cot p - \frac{\partial H}{\partial p}\frac{1}{q}$$

Examples

$$\Rightarrow \dot{Q} = \left(-\cot p \frac{\partial P}{\partial q}\right)\frac{\partial H}{\partial P} - \frac{1}{q}\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$$

$$\Rightarrow \dot{Q} = (-\cot^2 p)\frac{\partial H}{\partial P} - \frac{1}{q}\frac{\partial H}{\partial P}(-q\, cosec^2\, p)$$

$$\Rightarrow \dot{Q} = (cosec^2\, p - \cot^2 p)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \frac{\partial H}{\partial P} \qquad (1)$$

Now

$$P = q \cot p$$

$$\Rightarrow \dot{P} = \dot{q}\cot p - q\dot{p}\, cosec^2 p$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial p}\cot p + q\, cosec^2 p \frac{\partial H}{\partial q}$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\cot p + q\, cosec^2 p \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}$$

Examples

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}\left(\frac{q}{\sin p}\frac{\cos p}{q}\right)\cot p + q\, cosec^2 p\,\frac{\partial H}{\partial Q}\left(-\frac{q}{\sin p}\frac{\sin p}{q^2}\right)$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}(\cot p)\cot p + cosec^2 p\,\frac{\partial H}{\partial Q}(-1)$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q} cot^2\, p - cosec^2 p\,\frac{\partial H}{\partial Q}$$

$$\Rightarrow \dot{P} = \frac{\partial H}{\partial Q}(cot^2\, p - cosec^2 p)$$

$$\Rightarrow \dot{P} = -\frac{\partial H}{\partial Q}(cosec^2 p - cot^2\, p)$$

$$\Rightarrow \dot{P} = -\frac{\partial H}{\partial Q} \qquad (2)$$

From equation (1) and (2) we conclude that the transformation is canonical.

Examples (From: Goldstein Page 378)

Solve simple hormonic oscillator in one dimension whose Hamiltonian

$$H = \frac{p^2}{2m} + \frac{mw^2}{2} q^2$$

And generating function $F_1 = \frac{m}{2} w q^2 \cot Q$

Where m and w are constants.

Solution: Since the generating function $F_1 = F_1(q, Q)$

Therefore

$$p = \frac{\partial F_1}{\partial q} = mwq\, cot\, Q \qquad (1)$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{mwq^2}{2} (-cosec^2 Q)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{mwq^2}{2}\, cosec^2 Q \qquad (2)$$

$$\Rightarrow q = \sqrt{\frac{2P}{mw}} \sin Q$$ Putting in equation 1

Examples (From: Goldstein Page 378)

$$p = \frac{\partial F_1}{\partial q} = \sqrt{2mwP}\cos Q \qquad (3)$$

Since $$H = \frac{p^2}{2m} + \frac{mw^2}{2} q^2$$

Putting eq (2) and eq (3) in above equation

$$\Rightarrow H = \frac{2mwP cos^2 Q}{2m} + \frac{mw^2}{2}\frac{2P}{mw} sin^2 Q$$

$$\Rightarrow H = Pw(cos^2 Q + sin^2 Q)$$

$$\Rightarrow H = Pw \text{ or } P = \frac{H}{w}$$

Since $$\dot{Q} = \frac{\partial H}{\partial P} = \mathrm{w}$$

Integrating above equation $$Q = \mathrm{wt} + \alpha$$

Examples (From: Goldstein Page 378)

Now Putting in equation (2) $$q = \sqrt{\frac{2P}{mw}} \sin(wt + \alpha)$$

since $$P = \frac{H}{w}$$

Therefore, above equation is $$q = \sqrt{\frac{2H}{mw^2}} \sin(wt + \alpha)$$

& $$p = \sqrt{2mwP} \cos Q = \sqrt{2mH} \cos(wt + \alpha)$$

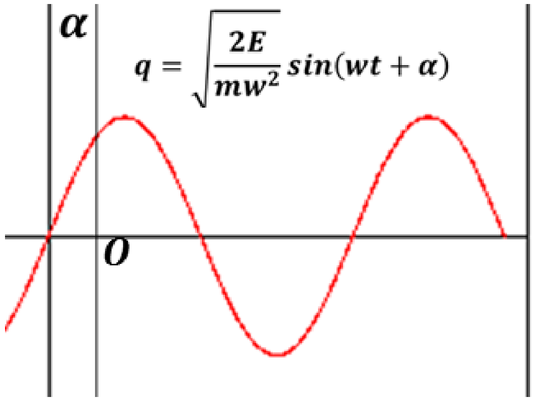
Since H is not exploit function of time therefore $H = E = constant$

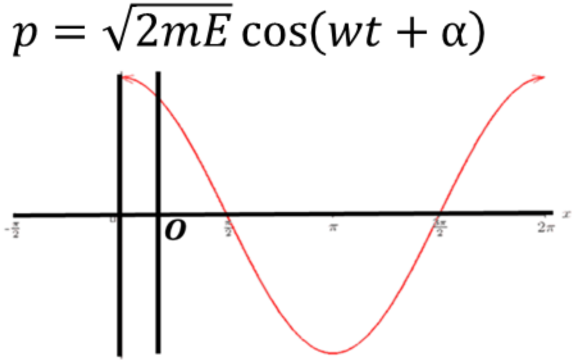
Therefore, above equation is $$q = \sqrt{\frac{2E}{mw^2}} \sin(wt + \alpha)$$

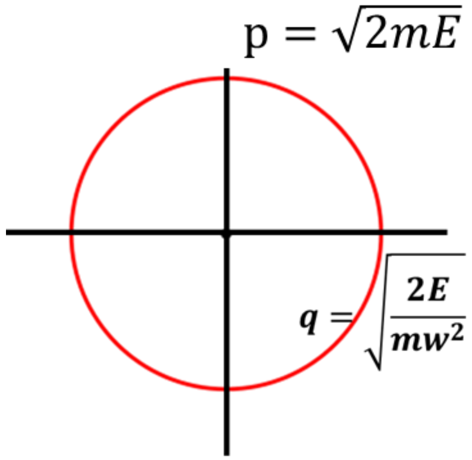
This is the one-dimensional solution of Simple Hormonic Oscillator

Examples (From: Goldstein Page 378)

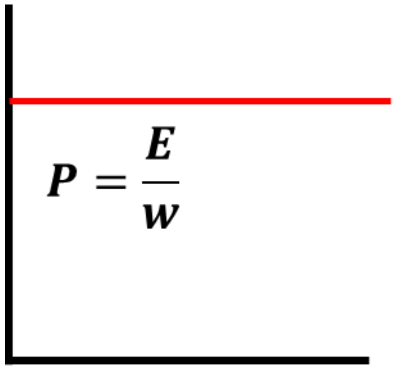
Since $q = \sqrt{\frac{2E}{mw^2}}\sin(wt + \alpha)$ & $p = \sqrt{2mE}\cos(wt + \alpha)$

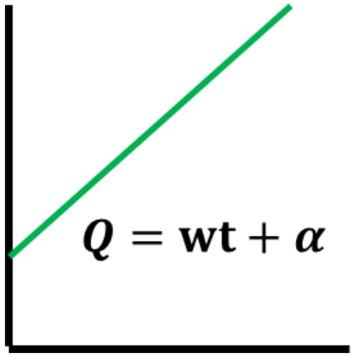


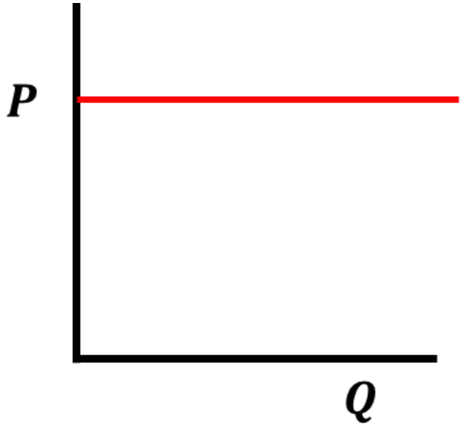




$P = \frac{H}{w} = \frac{E}{w}$ & $Q = \mathrm{wt} + \alpha$







Examples

For what value of α and β, equations

$$Q = q^{\alpha} \cos \beta p \qquad \text{and} \qquad P = q^{\alpha} \sin \beta p$$

Represents a canonical transformation. Find the generating Function F_3

Solution: The transformations will be canonical if it satisfies the following conditions.

$$\dot{Q} = \frac{\partial H}{\partial P} \qquad \& \qquad \dot{P} = -\frac{\partial H}{\partial Q}$$

Now if we take derivative of $Q = q^{\alpha} \cos \beta p$

$$\dot{Q} = \alpha q^{\alpha-1} \dot{q} \cos \beta p - \beta q^{\alpha} \dot{p} \sin \beta p$$

And

$$\dot{q} = \frac{\partial H}{\partial p} \qquad \& \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

$$\Rightarrow \dot{Q} = \alpha q^{\alpha-1} \frac{\partial H}{\partial p} \cos \beta p + \beta q^{\alpha} \frac{\partial H}{\partial q} \sin \beta p$$

Examples

$$\Rightarrow \dot{Q} = \alpha q^{\alpha-1}\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\cos\beta p + \beta q^{\alpha}\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} sin\,\beta p$$

$$\Rightarrow \dot{Q} = \left(\alpha q^{\alpha-1}\frac{\partial P}{\partial p}\cos\beta p + \beta q^{\alpha}\frac{\partial P}{\partial q} sin\,\beta p\right)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = [\alpha q^{\alpha-1}(\beta q^{\alpha}\cos\beta p)\cos\beta p + \beta q^{\alpha}(\alpha q^{\alpha-1} sin\,\beta p)\, sin\,\beta p]\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = [\alpha\beta q^{2\alpha-1}\cos^2\beta p + \alpha\beta q^{2\alpha-1} sin^2\beta p]\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \alpha\beta q^{2\alpha-1}[\cos^2\beta p + sin^2\beta p]\frac{\partial H}{\partial P} = \alpha\beta q^{2\alpha-1}\frac{\partial H}{\partial P}$$

The transformation is canonical if

$$\alpha\beta q^{2\alpha-1} = 1$$

$$\Rightarrow q^{2\alpha-1} = 1$$

Examples

$$\Rightarrow 2\alpha - 1 = 0$$

$$\Rightarrow \alpha = {}^{1}/_{2} \qquad \& \qquad \Rightarrow \beta = 2$$

Therefore, $Q = q^{1/2}\cos 2p$ and $P = q^{1/2}\sin 2p$

Now the generating function $\frac{\partial F_3}{\partial p} = -q$

$$\Rightarrow F_3 = -qp$$

Since

$$Q^2 + P^2 = q(\cos^2 2p + sin^2 2p) = q \qquad \& \qquad \frac{P}{Q} = \tan 2p \Rightarrow p = \frac{1}{2}tan^{-1}\frac{P}{Q}$$

Therefore, the generating function

$$\boxed{F_3 = -qp = -\frac{1}{2}(Q^2 + P^2)tan^{-1}\frac{P}{Q}}$$

Examples (Book: Classical mechanics by Takwal)

Show that the transformation is canonical

$$Q = \log\left(1 + q^{1/2}\cos p\right) \quad \text{and} \quad P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p$$

Also Show that the generating function $F_3 = -(e^Q - 1)^2 \tan p$

Solution: The transformations will be canonical if it satisfies the following conditions.

$$\dot{Q} = \frac{\partial H}{\partial P} \quad \& \quad \dot{P} = -\frac{\partial H}{\partial Q}$$

Now if we take derivative of $Q = \log\left(1 + q^{1/2}\cos p\right)$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\frac{d}{dt}\left(1 + q^{1/2}\cos p\right)$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\dot{q}\cos p - q^{1/2}\dot{p}\sin p\right)$$

Examples (Book: Classical mechanics by Takwal)

And $\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$ & $\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$

Therefore, $\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\cos p + q^{1/2}\frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\sin p\right)$

$$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial P}{\partial p}\cos p + q^{1/2}\frac{\partial P}{\partial q}\sin p\right)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\frac{\partial}{\partial p}\left(2q^{1/2}\sin p + q\sin 2p\right)\cos p + q^{1/2}\frac{\partial}{\partial q}\left(2q^{1/2}\sin p + q\sin 2p\right)\sin p\right)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\frac{1}{2}q^{-1/2}\left(2q^{1/2}\cos p + 2q\cos 2p\right)\cos p + q^{1/2}\left(2\frac{1}{2}q^{-1/2}\sin p + \sin 2p\right)\sin p\right)\frac{\partial H}{\partial P}$$

$$\Rightarrow \dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(\cos^2 p + q^{1/2}\cos 2p\cos p + \sin^2 p + q^{1/2}\sin 2p\sin p\right)\frac{\partial H}{\partial P}$$

Examples (Book: Classical mechanics by Takwal)

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left((cos^2 p + \sin^2 p) + q^{1/2}(cos\, 2p \cos p + \sin 2p \sin p)\right)\frac{\partial H}{\partial P}$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(1 + q^{1/2}\, cos(2p - p)\right)\frac{\partial H}{\partial P}$$

$$\dot{Q} = \frac{1}{\left(1+q^{1/2}\cos p\right)}\left(1 + q^{1/2}\, cos\, p\right)\frac{\partial H}{\partial P} = \frac{\partial H}{\partial P}$$

Similarly, $P = 2\left(1 + q^{1/2}\cos p\right)q^{1/2}\sin p$

$$\dot{P} = 2\left(1 + q^{1/2}\cos p\right)\left(\frac{1}{2}q^{-1/2}\dot{q}\, sin\, p + q^{1/2}\dot{p}\, cos\, p\right) + 2q^{1/2}\, sin\, p\left(\frac{1}{2}q^{-1/2}\dot{q}\cos p - q^{1/2}\dot{p}\, sin\, p\right)$$

$$\dot{P} = q^{-1/2}\dot{q}\, sin\, p + \dot{q}\cos p\, sin\, p + 2q^{1/2}\dot{p}\, cos\, p + 2q\cos p\cos p\,\dot{p} + \dot{q}\, sin\, p\cos p - 2q\sin p\sin p\,\dot{p}$$

$$\Rightarrow \dot{P} = \left(q^{-1/2}\, sin\, p + \sin 2p\right)\dot{q} + 2\left(q^{1/2}\, cos\, p + q\cos 2p\right)\dot{p}$$

And $\dot{q} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{dp}$ & $\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial Q}\frac{\partial Q}{dq}$

Examples (Book: Classical mechanics by Takwal)

Putting and solving we get $\dot{P} = -\frac{\partial H}{\partial Q}$ (Home work)

Hence the transformation is canonical.

Now $\frac{\partial F_3}{\partial Q} = \frac{\partial}{\partial Q}\left[-(e^Q - 1)^2 \tan p\right]$

$$\frac{\partial F_3}{\partial Q} = \left[-2(e^Q - 1)e^Q \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(e^{\log\left(1+q^{1/2}\cos p\right)} - 1\right)e^{\log\left(1+q^{1/2}\cos p\right)} \tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(1 + q^{1/2}\cos p - 1\right)\left(1 + q^{1/2}\cos p\right)\tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = \left[-2\left(q^{1/2}\cos p\right)\left(1 + q^{1/2}\cos p\right)\tan p\right]$$

$$\frac{\partial F_3}{\partial Q} = -\left[2\left(1 + q^{1/2}\cos p\right) q^{1/2}\sin p\right] = -P$$

Chapter 6
Lecture 3

Legendre Transformations Lagrange & Poisson Bracket

Dr. Akhlaq Hussain

6.3 Legendre Transformations

Lagrangian of the system $L = L(q_{i,} \dot{q}_i, t)$

Lagrange's equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Where momentum $p_i = \frac{\partial L(q_{i,} \dot{q}_i, t)}{\partial \dot{q}_i}$

In Lagrangian to Hamiltonian transition variables changes from $(q_{i,} \dot{q}_i, t)$ to $(q_{i,} p_i, t)$,

where p_i is related to q_i and $\dot{q}_i$ by above equation.

This transformation process is known as Legendre transformation.

Consider a function of only two variable $f(x, y)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$df = udx + vdy$$

6.3 Legendre Transformations

If we wish to change the basis of description from x, y to a new set of variable y,u

Let $g(y, u)$ be a function of u and y defined as $g = f - ux$ (I)

Therefore $$dg = df - xdu - udx$$

since $$df = udx + vdy$$

So $$dg = udx + vdy - xdu - udx$$

$$dg = vdy - xdu$$

And the quantities v and x are function of y and u respectively.

And $$dg = \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial u}du$$

$$\frac{\partial g}{\partial y} = v \qquad \& \qquad \frac{\partial g}{\partial u} = -x$$

6.3 Legendre Transformations

Equation I represent Legendre transformation. The Legendre transformation is frequently used in thermodynamics as

$$dU = TdS - PdV \qquad \text{for } U(S,V)$$

And $$H = U + PV \qquad H(S,P)$$

$$dH = TdS + VdP$$

And $$F = U - TS \qquad F(T,V)$$

$$G = H - TS \qquad G(T,P)$$

U = Internal energy	T = Temperature	S = Entropy	P = Pressure
V = Volume	H = Enthalpy	F = Helmholtz Free Energy	G=Gibbs free energy

6.3 Legendre Transformations

The transformation of $(q, \dot{q}, t)$ to (q, p, t) is however different.

Since $$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

And $$\frac{\partial L}{\partial \dot{q}} = p \qquad \Rightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \dot{p}$$

Therefore, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \frac{d}{dt} p - \frac{\partial L}{\partial q} = 0$$

$$\Rightarrow \frac{\partial L}{\partial q} = \dot{p}$$

Therefore , $$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt$$

$$dL = \dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial t} dt$$

And $$H = \dot{q} p - L$$

6.4 Lagrange's Bracket

Lagrange brackets were introduced by Joseph Louis Lagrange in 1808–1810.

- Mathematical formulation of Classical Mechanics.
- The Lagrange bracket is not use in modern mechanics
- The Lagrange's bracket are defined as if u and v are functions depending on q_i and p_i then

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right) = \sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)}$$

And invariant under canonical transformation.

6.4 Lagrange's Bracket

Some properties

i. $\{u, v\}_{q,p} = -\{v, u\}_{q,p}$ Lagrange's bracket are anti commutative

ii. $\{q_i, q_j\}_{q,p} = 0 = \{p_i, p_j\}_{q,p}$ For identical function it is zero

iii. $\{q_i, p_j\}_{q,p} = \delta_{ij}$ $\quad \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ Equal to Kronecker delta function.

iv. $\{u, v\}_{q,p} = \{u, v\}_{Q,P}$ Invariance of Lagrange Bracket

6.4 Lagrange's Bracket

Proof:

$$\{u,v\}_{q,p} = -\{v,u\}_{q,p}$$

$$\{u,v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$$

Taking negative common $$\{u,v\}_{q,p} = -\sum_i \left(-\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} + \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right)$$

$$\{u,v\}_{q,p} = -\sum_i \left(\frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u} - \frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v}\right)$$

$$\{u,v\}_{q,p} = -\{v,u\}_{q,p}$$ ***proved***

6.4 Lagrange's Bracket

Proof: $\{q_i, q_j\}_{p,q} = 0 = \{p_i, p_j\}_{q,p}$ For identical function it is zero

$$\{q_i, q_j\}_{q,p} = \sum_k \left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial q_k}{\partial q_j}\frac{\partial p_k}{\partial q_i}\right)$$

Here p and q are treated as independent co-ordinates in phase space. So

$$\frac{\partial p_k}{\partial q_j} = 0 = \frac{\partial p_k}{\partial q_i}$$

Therefore, $\{q_i, q_j\}_{q,p} = 0$

similarly , we can show $\{p_i, p_j\} = 0$

6.4 Lagrange's Bracket

$\{q_i, p_j\} = \delta_{ij}$ $\begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ Equal to Kronecker delta function.

$$\{q_i, p_j\}_{q,p} = \sum_k \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j} \frac{\partial p_k}{\partial q_i} \right)$$

0

Since p, q are independent, we get $\frac{\partial q_k}{\partial p_j} = \frac{\partial p_k}{\partial q_i} = 0$

Therefore, $\{q_i, p_j\}_{q,p} = \sum_k \frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial p_j}$

$$\Rightarrow \{q_i, p_j\}_{q,p} = \sum_k \delta_{ki} \delta_{kj} = \delta_{ij}$$

Where δ_{ij} is a Kronecker delta function

Hence $\{q_i, p_j\} = \delta_{ij}$ $\begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ Equal to Kronecker delta function.

6.4 Lagrange's Bracket

$\{u, v\}_{q,p} = \{u, v\}_{Q,P}$ Invariance of Lagrange Bracket

Or $\sum_i \frac{\partial(q_i,p_i)}{\partial(u,v)} = \sum_j \frac{\partial(Q_j,P_j)}{\partial(u,v)}$

Proof: Let q_i and p_i are function of Q_j and P_j

$$\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u}\frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v}\frac{\partial p_i}{\partial u}\right) \qquad \text{(I)}$$

And for $p_i = p_i(Q_j, P_j) \quad \Rightarrow \frac{\partial p_i}{\partial v} = \sum_j \left(\frac{\partial p_i}{\partial Q_j}\frac{\partial Q_j}{\partial v} + \frac{\partial p_i}{\partial P_j}\frac{\partial P_j}{\partial v}\right)$

Using Maxwell's equations

$$\frac{\partial p_i}{\partial P_j} = \frac{\partial Q_j}{\partial q_i} \quad \& \quad \frac{\partial p_i}{\partial Q_j} = -\frac{\partial P_j}{\partial q_i}$$ putting in above equation

$$\frac{\partial p_i}{\partial v} = \sum_j \left(-\frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v} + \frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v}\right) = \sum_j \left(\frac{\partial Q_j}{\partial q_i}\frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial q_i}\frac{\partial Q_j}{\partial v}\right) = \{q_i, v\}_{Q,P} \qquad \text{(II)}$$

6.4 Lagrange's Bracket

And $q_i = q_i(Q_j, P_j) \Rightarrow \quad \frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial q_i}{\partial Q_j} \frac{\partial Q_j}{\partial v} + \frac{\partial q_i}{\partial P_j} \frac{\partial P_j}{\partial v} \right)$

Using Maxwell's equations

$\frac{\partial q_i}{\partial Q_j} = \frac{\partial P_j}{\partial p_i}$ & $\frac{\partial q_i}{\partial P_j} = -\frac{\partial Q_j}{\partial p_i}$ putting in above equation

$$\frac{\partial q_i}{\partial v} = \sum_j \left(\frac{\partial P_j}{\partial p_i} \frac{\partial Q_j}{\partial v} - \frac{\partial Q_j}{\partial p_i} \frac{\partial P_j}{\partial v} \right) = -\{p_i, v\}_{Q,P} \qquad \text{(III)}$$

Putting Eqs (II) &(III) in Eq (I) $\{u, v\}_{q,p} = \sum_i \left(\frac{\partial q_i}{\partial u} \frac{\partial p_i}{\partial v} - \frac{\partial q_i}{\partial v} \frac{\partial p_i}{\partial u} \right)$

$$\{u, v\}_{q,p} = \sum_{ij} \left(\frac{\partial q_i}{\partial u} \left(\frac{\partial Q_j}{\partial q_i} \frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial q_i} \frac{\partial Q_j}{\partial v} \right) - \left(\frac{\partial P_j}{\partial p_i} \frac{\partial Q_j}{\partial v} - \frac{\partial Q_j}{\partial p_i} \frac{\partial P_j}{\partial v} \right) \frac{\partial p_i}{\partial u} \right)$$

$$\{u, v\}_{q,p} = \sum_{ij} \left(\frac{\partial P_j}{\partial v} \left(\frac{\partial Q_j}{\partial q_i} \frac{\partial q_i}{\partial u} + \frac{\partial Q_j}{\partial p_i} \frac{\partial p_i}{\partial u} \right) - \frac{\partial Q_j}{\partial v} \left(\frac{\partial P_j}{\partial q_i} \frac{\partial q_i}{\partial u} + \frac{\partial P_j}{\partial p_i} \frac{\partial p_i}{\partial u} \right) \right)$$

6.4 Lagrange's Bracket

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial P_j}{\partial v} \sum_i \left(\frac{\partial Q_j}{\partial q_i} \frac{\partial q_i}{\partial u} + \frac{\partial Q_j}{\partial p_i} \frac{\partial p_i}{\partial u} \right) - \frac{\partial Q_j}{\partial v} \sum_i \left(\frac{\partial P_j}{\partial q_i} \frac{\partial q_i}{\partial u} + \frac{\partial P_j}{\partial p_i} \frac{\partial p_i}{\partial u} \right) \right)$$

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial P_j}{\partial v} \frac{\partial Q_j}{\partial u} - \frac{\partial Q_j}{\partial v} \frac{\partial P_j}{\partial u} \right)$$

$$\{u,v\}_{q,p} = \sum_j \left(\frac{\partial Q_j}{\partial u} \frac{\partial P_j}{\partial v} - \frac{\partial P_j}{\partial u} \frac{\partial Q_j}{\partial v} \right)$$

$$\{u,v\}_{q,p} = \{u,v\}_{Q,P}$$

6.5 Poisson's Brackets

Poisson's bracket:

- An important binary operation in Hamiltonian mechanics
- Play a central role in Hamilton's equations of motion.
- Distinguishes a class of coordinate transformations (canonical transformations)
- Very useful tool in quantum mechanics and field theory.

The Poisson bracket are defined as

$$[u, v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

Or

$$[u, v]_{Q,P} = \sum_i \left(\frac{\partial u}{\partial Q_i} \frac{\partial v}{\partial P_i} - \frac{\partial u}{\partial P_i} \frac{\partial v}{\partial Q_i} \right)$$

Where u, v are two functions, $w.r.t$ the canonical variable (q, p) or (Q, P).

6.5 Poisson's Brackets

Consider f is a function such that

$$f = f(q_i, p_i, t)$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t}$$

As we know that $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Eq 1 can be written as $\frac{df}{dt} = \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right] + \frac{\partial f}{\partial t}$

$$\frac{df}{dt} = [f, H]_{q,p} + \frac{\partial f}{\partial t}$$

Where $[f, H]$ is called Poisson's Brackets.

6.5 Poisson's Brackets

Generally, u and v are two function their Poisson's bracket is

$$[u, v] = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

Properties It is obvious from the definition of the Poisson brackets that

i. $[u, v] = -[v, u]$ Poisson bracket are anti-commutative

ii. $[u, u] = 0 = [v, v]$ Poisson bracket of identical functions are zero.

iii. $[u, c] = 0 = [u, c]$ Where c is independent of p or q.

iv. $[u + v, w] = [u, w] + [v, w]$ Poisson brackets obeys the distributive law

v. $[u, vw] = [u, v]w + v[u, w]$

vi. $[q_i, p_j] = \delta_{ij} \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$ Where δ_{ij} is the Kronecker delta function.

6.5 Poisson's Brackets

1)Skew symmetric OR anti commutative

(OR show that Poisson brackets do not obey commutative law)

As we know that $$[u, v]_{q,p} = \sum_{i=1}^{n} \left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

Taking –ve sign out of the bracket

$$[u, v]_{q,p} = -\sum_{i=1}^{n} \left[-\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} + \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right]$$

$$[u, v]_{q,p} = -\sum_{i=1}^{n} \left[\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right]$$

$$[u, v]_{q,p} = -\sum_{i=1}^{n} \left[\frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial u}{\partial q_i}\right]$$

$$[u, v]_{q,p} = -[v, u]_{q,p}$$ **proved**

6.5 Poisson's Brackets

For identical function

$$[u,u]_{q,p} = [v,v]_{q,p} = 0$$

$$[u,u]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial u}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial q_i}\right] = 0$$

Similarly, $[v,v]_{q,p} = 0$

Poisson brackets with a-constant

Let F be function of generalized q_i and p_i and C is any constant.

Then $$[F,C]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial C}{\partial q_i}\right]$$

Where $\frac{\partial C}{\partial q_i} = \frac{\partial C}{\partial p_i} = 0$ where c does not depend on q_i and p_i.

$$[F,C]_{q,p} = 0$$

6.5 Poisson's Brackets

Poisson Brackets obey the distributive law

Consider F_1, F_2 and G are three functions dependently upon q_i andp_i.

Then
$$[(F_1+F_2),G]=\sum_{i=1}^{n}\left[\frac{\partial(F_1+F_2)}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial(F_1+F_2)}{\partial p_i}\frac{\partial G}{\partial q}\right]$$

$$[(F_1+F_2),G]=\sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}+\frac{\partial F_2}{\partial q_i}\right)\frac{\partial G}{\partial p_i}-\left(\frac{\partial F_1}{\partial p_i}+\frac{\partial F_2}{\partial p_i}\right).\frac{\partial G}{\partial q_i}\right]$$

$$[(F_1+F_2),G]=\sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i}+\frac{\partial F_2}{\partial q_i}\frac{\partial G}{\partial p_i}\right)-\left(\frac{\partial F_1}{\partial p_i}\frac{\partial G}{\partial q_i}+\frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$[(F_1+F_2),G]=\sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F_1}{\partial p_i}\frac{\partial G}{\partial q_i}\right)+\left(\frac{\partial F_2}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$[(F_1+F_2),G]=\sum_{i=1}^{n}\left[\left(\frac{\partial F_1}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F_1}{\partial p_i}.\frac{\partial G}{\partial q_i}\right)\right]+\sum_{i=1}^{n}\left[\frac{\partial F_2}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial F_2}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

$$[F_1+F_2,G]_{q,p}=[F_1,G]_{q,p}+[F_2,G]_{q,p}$$

Proved it obeys.

6.5 Poisson's Brackets

Poisson's Brackets are linear

Consider if $F_{(q_i,p_i)}, G_{(q_i,p_i)}$ and $W_{(q_i,p_i)}$ are three functions and a, b are constant, then we here to show that

$$[aF + bG, W] = [aF, W] + [bG, W]$$

$$[aF + bG, W] = a[F, W] + b[G, W]$$

Let $aF = F'$ and $bG = G'$

Then by distributive property we know that

$$[aF + bG, W] = [F' + G', W]_{q,p}$$

$$[F' + G', W] = [F', W] + [G', W]$$

$$[aF + bG, W] = [aF, W] + [bG, W]$$

6.5 Poisson's Brackets

Now consider

$$[aF, W] = \sum_{i=1}^{n} \left[\frac{\partial(aF)}{\partial q_i} \cdot \frac{\partial W}{\partial p_i} - \frac{\partial(aF)}{\partial p_i} \frac{\partial W}{\partial q_i}\right]$$

$$[aF, W] = a \sum_{i=1}^{n} \left[\frac{\partial F}{\partial q_i} \frac{\partial W}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial W}{\partial q_i}\right]$$

$$[aF, W] = a\,[F, W]$$

Similarly $[bG, W] = b\,[G, W]$

Hence

$$[aF + bG, W] = a[F, W] + b[G, W]$$

6.5 Poisson's Brackets

Show that if u, v and w are functions dependent on (q_i, p_i), then show that

$$[u, vw] = [u, v]w + v[u, w]$$

As

$$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\frac{\partial (vw)}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial (vw)}{\partial q_i}\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\frac{\partial u}{\partial q_i}\left(v\frac{\partial w}{\partial p_i} + w\frac{\partial v}{\partial p_i}\right) - \frac{\partial u}{\partial p_i}\left(v\frac{\partial w}{\partial q_i} + w\frac{\partial v}{\partial q_i}\right)\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n}\left[\left(w\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - w\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) + \left(v\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - v\frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right)\right]$$

$$[u, vw]_{q,p} = \sum_{i=1}^{n} w\left[\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right] + v\left[\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i}\right]$$

$$[u, vw]_{p,q} = w[u, v]_{q,p} + v[u, w]_{q,p}$$

6.5 Poisson's Brackets

Proved that $\frac{\partial}{\partial t}[F,G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$

Sol: $$[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \frac{\partial}{\partial t}\sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)\right]$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_i}\right) + \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_i}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial p_i}\right) - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial t}\left(\frac{\partial G}{\partial q_i}\right)\right] + \sum_{i=1}^{n}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_i}\right)\frac{\partial G}{\partial p_i} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial p_i}\right)\frac{\partial G}{\partial q_i}\right]$$

6.5 Poisson's Brackets

$$\frac{\partial}{\partial t}[F,G] == \sum_{i=1}^{n}\left[\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial G}{\partial t}\right) - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial G}{\partial t}\right)\right] + \sum_{i=1}^{n}\left[\frac{\partial}{\partial q_i}\left(\frac{\partial F}{\partial t}\right)\frac{\partial G}{\partial p_i} - \frac{\partial}{\partial p_i}\left(\frac{\partial F}{\partial t}\right)\frac{\partial G}{\partial q_i}\right]$$

$$\frac{\partial}{\partial t}[F,G] = \left[F,\frac{\partial G}{\partial t}\right] + \left[\frac{\partial F}{\partial t},G\right]$$ ***proved***

Assignment **Show that**

i. $[q_i, q_j] = [p_i, p_j] = 0$

$$[q_i, q_j] = \sum_{i=1}^{n}\left[\frac{\partial q_i}{\partial q_i}\frac{\partial q_j}{\partial p_i} - \frac{\partial q_i}{\partial p_i}\frac{\partial q_j}{\partial q_i}\right] = [0 - 0]$$

6.5 Poisson's Brackets

$$[q_i, p_j] = \delta_{ij} \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

0

$$[q_i, p_j] = \sum_{k=1}^{n} \left[\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right]$$

$$where \ \frac{\partial p}{\partial q} = 0 \qquad \frac{\partial q}{\partial p} = 0$$

$$[q_i, p_j] = \sum_{k=1}^{n} \left[\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k}\right] = \delta_{ik}\delta_{jk} = \delta_{ij}$$

6.5 Poisson's Brackets

Show that Poisson's brackets is invariant under canonical transformation

$$[F,G]_{q,p} = [F,G]_{Q,P}$$

Proof: let F and G be two arbitrary function of q and p.Then

$$[F,G]_{q,p} = \sum_i \left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right]$$

Let the Transformation $Q = Q(q,p)$ and $P = P(q,p)$

Inverse transformation $q = q(Q,P)$ and $p = p(Q,P)$

Therefore $G = G(Q,P)$

$$\partial G = \sum_k \frac{\partial G}{\partial Q_k}\partial Q_K + \sum_k \frac{\partial G}{\partial P_k}\partial P_k$$

Then $\frac{\partial G}{\partial q_i} = \sum_k [\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i} + \frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}]$ & $\frac{\partial G}{\partial p_i} = \sum_k [\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial p_i} + \frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial p_i}]$

6.5 Poisson's Brackets

$$\Rightarrow [F,G]_{q,p} = \sum_{i,k}\left[\frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial p_i} + \frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial p_i}\right) - \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i} + \frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{i,k}\left[\frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial p_i}\right) - \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial Q_k}\frac{\partial Q_k}{\partial q_i}\right) + \frac{\partial F}{\partial q_i}\left(\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial p_i}\right) - \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial P_k}\frac{\partial P_k}{\partial q_i}\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}\left(\sum_i\left[\frac{\partial F}{\partial q_i}\frac{\partial Q_k}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial Q_k}{\partial q_i}\right]\right) + \frac{\partial G}{\partial P_k}\left(\sum_i\left[\frac{\partial F}{\partial q_i}\frac{\partial P_k}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial P_k}{\partial q_i}\right]\right)\right]$$

$$\Rightarrow [F,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}[F,Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[F,P_k]_{q,p}\right] \qquad \text{(I)}$$

Replacing F by Q_i

0

$$\Rightarrow [Q_i,G]_{q,p} = \sum_{k}\left[\frac{\partial G}{\partial Q_k}[Q_i,Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[Q_i,P_k]_{q,p}\right]$$

$$\Rightarrow [Q_i,G]_{q,p} = \sum_{k}\frac{\partial G}{\partial P_k}\delta_{ik} = \frac{\partial G}{\partial P_i}$$

6.5 Poisson's Brackets

Now replacing G by F in previous equation

$$\Rightarrow [Q_k, F] = \frac{\partial F}{\partial P_k}$$

Or $$\Rightarrow [F, Q_k]_{q,p} = -\frac{\partial F}{\partial P_k} \qquad \text{(II)}$$

Similarly, replacing F by P_i in eq (I)

$$\Rightarrow [P_i, G]_{q,p} = \sum_k \left[\frac{\partial G}{\partial Q_k}[P_i, Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[P_i, P_k]_{q,p}\right]$$

0

$$\Rightarrow [P_i, G]_{q,p} = \sum_k \frac{\partial G}{\partial Q_k}[P_i, Q_k]$$

$$\Rightarrow [P_i, G]_{q,p} = -\sum_k \frac{\partial G}{\partial Q_k}[Q_k, P_i]$$

6.5 Poisson's Brackets

$$\Rightarrow [P_i, G]_{q,p} = -\sum_k \frac{\partial G}{\partial Q_k}\delta_{ik} = -\frac{\partial G}{\partial Q_k} \qquad \text{for} \qquad i = k$$

And $\Rightarrow [G, P_k]_{q,p} = \frac{\partial G}{\partial Q_k}$

Now replacing G by F, we get $[F, P_k] = \frac{\partial F}{\partial Q_k}$ (III)

Putting (II) and (III) in *eq* (I)

$$\Rightarrow [F, G]_{q,p} = \sum_k \left[\frac{\partial G}{\partial Q_k}[F, Q_k]_{q,p} + \frac{\partial G}{\partial P_k}[F, P_k]_{q,p}\right] = \sum_k \left(-\frac{\partial G}{\partial Q_k}\frac{\partial F}{\partial P_k} + \frac{\partial G}{\partial P_k}\frac{\partial F}{\partial Q_k}\right)$$

$$\Rightarrow [F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k}\right) = [F, G]_{P,Q}$$

$$\Rightarrow [F, G]_{q,p} = [F, G]_{P,Q}$$

6.5 Jacobi Identity

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

Solution Since $[u,v]=\sum_{k=1}^{n}\left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k}-\frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$

$$[u,v]=\sum_{k=1}^{n}\left(\frac{\partial u}{\partial q_k}\frac{\partial}{\partial p_k}-\frac{\partial u}{\partial p_k}\frac{\partial}{\partial q_k}\right)v$$

$$[u,v]=D_u v$$

Where $D_u=\sum_{i}^{2n}\alpha_i\frac{\partial}{\partial \xi_i}$

Similarly $D_v=\sum_{j}^{2n}\beta_j\frac{\partial}{\partial \xi_j}$

Note

For 1 to n	$\alpha_i=\frac{\partial u}{\partial q_k}$	& $\frac{\partial}{\partial \xi_i}=\frac{\partial}{\partial p_k}$
For n+1 to 2n	$\alpha_i=-\frac{\partial u}{\partial p_k}$	& $\frac{\partial}{\partial \xi_i}=\frac{\partial}{\partial q_k}$

Now considering first two terms

$$[u,[v,w]]+[v,[w,u]]=[u,[v,w]]-[v,[u,w]]=[u,D_v w]-[v,D_u w]$$

6.5 Jacobi Identity

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = [u, D_v w] - [v, D_u w] = D_u(D_v w) - D_v(D_u w)$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_i^{2n} \alpha_i \frac{\partial}{\partial \xi_i}\left(\sum_j^{2n} \beta_j \frac{\partial}{\partial \xi_j} w\right) - \sum_j^{2n} \beta_j \frac{\partial}{\partial \xi_j}\left(\sum_i^{2n} \alpha_i \frac{\partial}{\partial \xi_i} w\right)$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \alpha_i \beta_j \frac{\partial^2 w}{\partial \xi_i \partial \xi_j} + \sum_{i,j}^{2n} \alpha_i \frac{\partial \beta_j}{\partial \xi_i}\frac{\partial w}{\partial \xi_j} - \sum_{i,j}^{2n} \alpha_i \beta_j \frac{\partial^2 w}{\partial \xi_j \partial \xi_i} - \sum_{i,j}^{2n} \beta_j \frac{\partial \alpha_i}{\partial \xi_j}\frac{\partial w}{\partial \xi_i}$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \alpha_i \frac{\partial \beta_j}{\partial \xi_i}\frac{\partial w}{\partial \xi_j} - \sum_{i,j}^{2n} \beta_j \frac{\partial \alpha_i}{\partial \xi_j}\frac{\partial w}{\partial \xi_i}$$

$$\Rightarrow [u,[v,w]] + [v,[w,u]] = \sum_{i,j}^{2n} \left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j}$$

By using the property that sum is not effected if the indices are interchanged (dummy indices)

6.5 Jacobi Identity

$$\Rightarrow \sum_{i,j}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} = \sum_{i,j}^{n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} + \sum_{i,j=n+1}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j}$$

$$\Rightarrow \sum_{i,j}^{2n}\left(\alpha_i \frac{\partial \beta_j}{\partial \xi_i} - \beta_i \frac{\partial \alpha_j}{\partial \xi_i}\right)\frac{\partial w}{\partial \xi_j} = \sum_i \left(A_j \frac{\partial w}{\partial p_i} + B_j \frac{\partial w}{\partial q_i}\right)$$

Replacing w by $p_j \Rightarrow \quad \sum_i \left(A_j \frac{\partial p_j}{\partial p_i} + B_j \frac{\partial p_j}{\partial q_i}\right) = \sum_i A_j \frac{\partial p_j}{\partial p_i} = \sum_i A_i \delta_{ij} = A_j$

$$\Rightarrow \left[u,\left[v,p_j\right]\right] - \left[v,\left[u,p_j\right]\right] = A_j$$

$$\Rightarrow A_j = \left[u,\frac{\partial v}{\partial q_j}\right] - \left[v,\frac{\partial u}{\partial q_j}\right]$$

$$\Rightarrow A_j = \left[u,\frac{\partial v}{\partial q_j}\right] + \left[\frac{\partial u}{\partial q_j},v\right] = \frac{\partial}{\partial q_j}[u,v]$$

6.5 Jacobi Identity

Replacing w by q_j

$$\left[u,[v,q_j]\right]-\left[v,[u,q_j]\right]=B_j$$

$$\Rightarrow B_j=-\left[u,\frac{\partial v}{\partial p_j}\right]+\left[v,\frac{\partial u}{\partial p_j}\right]$$

$$\Rightarrow B_j=-\frac{\partial}{\partial p_j}[u,v]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]=\sum_i\left[\frac{\partial w}{\partial p_i}\frac{\partial}{\partial q_j}[u,v]-\frac{\partial w}{\partial q_j}\frac{\partial}{\partial p_j}[u,v]\right]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]=-[w,[u,v]]$$

$$\Rightarrow [u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

6.5 Poisson's Brackets

Evaluate the following Poisson bracket and explain its physical meaning:

$$\left[mr^2\dot{\vartheta}^2,\ \frac{m}{2}\left(p_r^2+\frac{p_\vartheta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\vartheta}\right)+V(r)\right]_{r,\vartheta}$$

Find the value of $\alpha \in \mathbb{R}$ for which the following transformation is canonical:

$$Q(p,q)=\ln\left(\frac{p}{2q}\right);\ P(p,q)=-\frac{1}{2}qp^\alpha$$

$$\begin{aligned}\vec{L} &= \vec{r} \times \vec{p} \\ L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x\end{aligned}$$

Poisson $\Rightarrow [L_x, L_y] =$ some conserved quantity

We will need some partial derivatives to compute the PB of L_x and L_y.

$$\begin{aligned}\frac{\partial L_x}{\partial \vec{r}} &= \{0, p_z, -p_y\} \quad , \quad \frac{\partial L_x}{\partial \vec{p}} = \{0, -z, y\} \\ \frac{\partial L_y}{\partial \vec{r}} &= \{-p_z, 0, p_x\} \quad , \quad \frac{\partial L_y}{\partial \vec{p}} = \{z, 0, -x\}\end{aligned}$$

$$[L_x, L_y] = \frac{\partial L_x}{\partial \vec{p}} \cdot \frac{\partial L_y}{\partial \vec{r}} - \frac{\partial L_x}{\partial \vec{r}} \cdot \frac{\partial L_y}{\partial \vec{p}} = yp_x - xp_y = -L_z$$

$$\text{if} \quad [H, L_z] = 0 \Rightarrow L_z = \text{const}$$
$$[T + U, L_z] = [T, L_z] + [U, L_z]$$

Let's do these one at a time...

$$\begin{aligned}[T, L_z] &= [T, xp_y - yp_x] = [T, xp_y] - [T, yp_x] \\ &= ([T, x]p_y + [T, p_y]x) - ([T, y]p_x + [T, p_x]y)\end{aligned}$$

It doesn't look like we are winning, but what I am doing is breaking this down into small enough parts that I can use my identities. For instance,

$$\begin{aligned}[T, p_x] &= \frac{1}{2m} [p_x^2 + p_y^2 + p_z^2, p_x] \\ [p_x^2, p_x] &= 2p_x [p_x, p_x] = 0 \\ \Rightarrow & \quad [T, p_i] = 0 \quad \forall i\end{aligned}$$

$$[T, x] = \frac{1}{2m} [p_x^2, x] = \frac{p_x}{m} [p_x, x] = \frac{p_x}{m}$$

note that $[p_y, x] = [p_z, x] = 0$. Finally,

$$[T, L_z] = \frac{p_x p_y}{m} - \frac{p_y p_x}{m} = 0$$

$\Rightarrow L_z$ conserved for free particle! (and L_x and L_y)

Well, I guess we knew that. Let's do U...

$$\begin{aligned} [U, L_z] &= ([U, x]p_y + [U, p_y]x) - ([U, y]p_x + [U, p_x]y) \\ &= [U, p_y]x - [U, p_x]y \end{aligned}$$

where I have dropped 2 terms since $[q_j, q_k] = 0$ and $U(r)$ has no p_i in it.

$$\begin{aligned} [U, p_y] &= -\frac{\partial U}{\partial y} = -\frac{\partial U}{\partial r}\frac{\partial r}{\partial y} \\ &= -\frac{\partial U}{\partial r}\frac{y}{r} \end{aligned}$$

where $\quad r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial y} = \frac{1}{2r} 2y = \frac{y}{r}$

Putting these together to find the PB of L_z with U,

$$\begin{aligned} [U, L_z] &= [U, p_y]x - [U, p_x]y \\ &= -\frac{\partial U}{\partial r}\left(\frac{y}{r}x - \frac{x}{r}y\right) = 0 \end{aligned}$$