Modeling in the Time Domain

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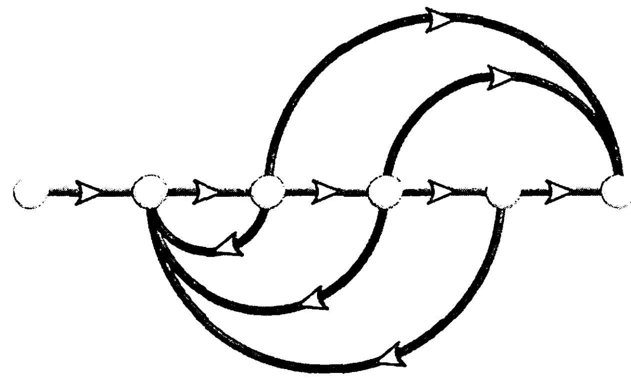

This chapter covers only state-space methods.

State Space
SS

Chapter Learning Outcomes

After completing this chapter, the student will be able to:

- Find a mathematical model, called a *state-space* representation, for a linear, time-invariant system (Sections 3.1–3.3)
- Model electrical and mechanical systems in state space (Section 3.4)
- Convert a transfer function to state space (Section 3.5)
- Convert a state-space representation to a transfer function (Section 3.6)
- Linearize a state-space representation (Section 3.7)

Case Study Learning Outcomes

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to find the state-space representation of each subsystem.
- Given a description of the way a pharmaceutical drug flows through a human being, you will be able to find the state-space representation to determine drug concentrations in specified compartmentalized blocks of the process and of the human body. You will also be able to apply the same concepts to an aquifer to find water level.

3.1 Introduction

Two approaches are available for the analysis and design of feedback control systems. The first, which we began to study in Chapter 2, is known as the *classical*, or *frequency-domain*, technique. This approach is based on converting a system's differential equation to a transfer function, thus generating a mathematical model of the system that *algebraically* relates a representation of the output to a representation of the input. Replacing a differential equation with an algebraic equation not only simplifies the representation of individual subsystems but also simplifies modeling interconnected subsystems.

The primary disadvantage of the classical approach is its limited applicability: It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

A major advantage of frequency-domain techniques is that they rapidly provide stability and transient response information. Thus, we can immediately see the effects of varying system parameters until an acceptable design is met.

With the arrival of space exploration, requirements for control systems increased in scope. Modeling systems by using linear, time-invariant differential equations and subsequent transfer functions became inadequate. The *state-space* approach (also referred to as the *modern*, or *time-domain*, approach) is a unified method for modeling, analyzing, and designing a wide range of systems. For example, the state-space approach can be used to represent nonlinear systems that have backlash, saturation, and dead zone. Also, it can handle, conveniently, systems with nonzero initial conditions. Time-varying systems, (for example, missiles with varying fuel levels or lift in an aircraft flying through a wide range of altitudes) can be represented in state space. Many systems do not have just a single input and a single output. Multiple-input, multiple-output systems (such as a vehicle with input direction and input velocity yielding an output direction and an output velocity) can be compactly represented in state space with a model similar in form and complexity to that used for single-input, single-output systems. The time-domain approach can be used to represent systems with a digital computer in the loop or to model systems for digital simulation. With a simulated system, system response can be obtained for changes in system parameters—an important design tool. The state-space approach is also attractive because of the availability of numerous state-space software packages for the personal computer.

The time-domain approach can also be used for the same class of systems modeled by the classical approach. This alternate model gives the control systems designer another perspective from which to create a design. While the state-space approach can be applied to a wide range of systems, it is not as intuitive as the classical approach. The designer has to engage in several calculations before the physical interpretation of the model is apparent, whereas in classical control a few quick calculations or a graphic presentation of data rapidly yields the physical interpretation.

In this book, the coverage of state-space techniques is to be regarded as an introduction to the subject, a springboard to advanced studies, and an alternate approach to frequency-domain techniques. We will limit the state-space approach to linear, time-invariant systems or systems that can be linearized by the methods of Chapter 2. The study of other classes of systems is beyond the scope of this book. Since state-space analysis and design rely on matrices and matrix operations, you may want to review this topic in Appendix G, located at www.wiley.com/college/nise, before continuing.

3.2 Some Observations

We proceed now to establish the state-space approach as an alternate method for representing physical systems. This section sets the stage for the formal definition of the state-space representation by making some observations about systems and their variables. In the discussion that follows, some of the development has been placed in footnotes to avoid clouding the main issues with an excess of equations and to ensure that the concept is clear. Although we use two electrical networks to illustrate the concepts, we could just as easily have used a mechanical or any other physical system.

We now demonstrate that for a system with many variables, such as inductor voltage, resistor voltage, and capacitor charge, we need to use differential equations only to solve for a selected subset of system variables because all other remaining system variables can be evaluated algebraically from the variables in the subset. Our examples take the following approach:

1. We select a particular *subset* of all possible system variables and call the variables in this subset *state variables.*
2. For an nth-order system, we write *n simultaneous, first-order differential equations* in terms of the state variables. We call this system of simultaneous differential equations *state equations.*
3. If we know the initial condition of all of the state variables at t_0 as well as the system input for $t \geq t_0$, we can solve the simultaneous differential equations for the state variables for $t \geq t_0$.
4. We *algebraically* combine the state variables with the system's input and find all of the other system variables for $t \geq t_0$. We call this algebraic equation the *output equation.*
5. We consider the state equations and the output equations a viable representation of the system. We call this representation of the system a *state-space representation.*

Let us now follow these steps through an example. Consider the RL network shown in Figure 3.1 with an initial current of $i(0)$.

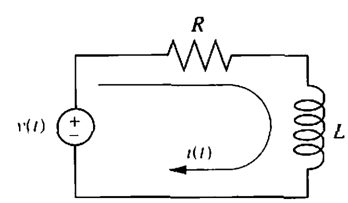

FIGURE 3.1 RL network

1. We select the current, $i(t)$, for which we will write and solve a differential equation using Laplace transforms.
2. We write the loop equation,

$$L\frac{di}{dt} + Ri = v(t) \tag{3.1}$$

3. Taking the Laplace transform, using Table 2.2, Item 7, and including the initial conditions, yields

$$L[sI(s) - i(0)] + RI(s) = V(s) \tag{3.2}$$

Assuming the input, $v(t)$, to be a unit step, $u(t)$, whose Laplace transform is $V(s) = 1/s$, we solve for $I(s)$ and get

$$I(s) = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0)}{s+\frac{R}{L}} \tag{3.3}$$

from which

$$i(t) = \frac{1}{R}\left(1 - e^{-(R/L)t}\right) + i(0)e^{-(R/L)t} \tag{3.4}$$

The function $i(t)$ is a subset of all possible network variables that we are able to find from Eq. (3.4) if we know its initial condition, $i(0)$, and the input, $v(t)$. Thus, $i(t)$ is a state variable, and the differential equation (3.1) is a *state equation.*

4. We can now solve for all of the other network variables *algebraically* in terms of $i(t)$ and the applied voltage, $v(t)$. For example, the voltage across the resistor is

$$v_R(t) = Ri(t) \tag{3.5}$$

The voltage across the inductor is

$$v_L(t) = v(t) - Ri(t) \qquad (3.6)^1$$

The derivative of the current is

$$\frac{di}{dt} = \frac{1}{L}[v(t) - Ri(t)] \qquad (3.7)^2$$

Thus, knowing the state variable, $i(t)$, and the input, $v(t)$, we can find the value, or *state*, of any network variable at any time, $t \geq t_0$. Hence, the algebraic equations, Eqs. (3.5) through (3.7), are *output equations.*

5. Since the variables of interest are completely described by Eq. (3.1) and Eqs. (3.5) through (3.7), we say that the combined state equation (3.1) and the output equations (3.5 through 3.7) form a viable representation of the network, which we call a *state-space representation.*

Equation (3.1), which describes the dynamics of the network, is not unique. This equation could be written in terms of any other network variable. For example, substituting $i = v_R/R$ into Eq. (3.1) yields

$$\frac{L}{R}\frac{dv_R}{dt} + v_R = v(t) \tag{3.8}$$

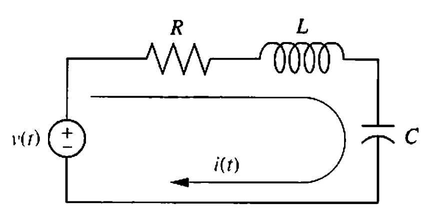

FIGURE 3.2 *RLC* network

which can be solved knowing that the initial condition $v_R(0) = Ri(0)$ and knowing $v(t)$. In this case, the state variable is $v_R(t)$. Similarly, all other network variables can now be written in terms of the state variable, $v_R(t)$, and the input, $v(t)$. Let us now extend our observations to a second-order system, such as that shown in Figure 3.2.

1. Since the network is of second order, two simultaneous, first-order differential equations are needed to solve for two state variables. We select $i(t)$ and $q(t)$, the charge on the capacitor, as the two state variables.
2. Writing the loop equation yields

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t) \tag{3.9}$$

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[1] Since $v_L(t) = v(t) - v_R(t) = v(t) - Ri(t)$.

[2] Since $\frac{di}{dt} = \frac{1}{L}v_L(t) = \frac{1}{L}[v(t) - Ri(t)]$.

Converting to charge, using $i(t) = dq/dt$, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t) \tag{3.10}$$

But an nth-order differential equation can be converted to n simultaneous first-order differential equations, with each equation of the form

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i f(t) \tag{3.11}$$

where each x_i is a state variable, and the a_{ij}'s and b_i are constants for linear, time-invariant systems. We say that the right-hand side of Eq. (3.11) is a *linear combination* of the state variables and the input, $f(t)$.

We can convert Eq. (3.10) into two simultaneous, first-order differential equations in terms of $i(t)$ and $q(t)$. The first equation can be $dq/dt = i$. The second equation can be formed by substituting $\int i\,dt = q$ into Eq. (3.9) and solving for di/dt. Summarizing the two resulting equations, we get

$$\frac{dq}{dt} = i \tag{3.12a}$$

$$\frac{di}{dt} = -\frac{1}{LC}q - \frac{R}{L}i + \frac{1}{L}v(t) \tag{3.12b}$$

3. These equations are the state equations and can be solved simultaneously for the state variables, $q(t)$ and $i(t)$, using the Laplace transform and the methods of Chapter 2, if we know the initial conditions for $q(t)$ and $i(t)$ and if we know $v(t)$, the input.
4. From these two state variables, we can solve for all other network variables. For example, the voltage across the inductor can be written in terms of the solved state variables and the input as

$$v_L(t) = -\frac{1}{C}q(t) - Ri(t) + v(t) \qquad (3.13)^3$$

Equation (3.13) is an *output equation*; we say that $v_L(t)$ is a *linear combination* of the state variables, $q(t)$ and $i(t)$, and the input, $v(t)$.

5. The combined state equations (3.12) and the output equation (3.13) form a viable representation of the network, which we call a *state-space representation*.

Another choice of two state variables can be made, for example, $v_R(t)$ and $v_C(t)$, the resistor and capacitor voltage, respectively. The resulting set of simultaneous, first-order differential equations follows:

$$\frac{dv_R}{dt} = -\frac{R}{L}v_R - \frac{R}{L}v_C + \frac{R}{L}v(t) \qquad (3.14a)^4$$

$$\frac{dv_C}{dt} = \frac{1}{RC}v_R \tag{3.14b}$$

[3] Since $v_L(t) = L(di/dt) = -(1/C)q - Ri + v(t)$, where di/dt can be found from Eq. (3.9), and $\int i\,dt = q$.

[4] Since $v_R(t) = i(t)R$, and $v_C(t) = (1/C)\int i\,dt$, differentiating $v_R(t)$ yields $dv_R/dt = R(di/dt) = (R/L)v_L = (R/L)[v(t) - v_R - v_C]$, and differentiating $v_C(t)$ yields $dv_C/dt = (1/C)i = (1/RC)v_R$.

Again, these differential equations can be solved for the state variables if we know the initial conditions along with $v(t)$. Further, all other network variables can be found as a linear combination of these state variables.

Is there a restriction on the choice of state variables? Yes! Typically, the minimum number of state variables required to describe a system equals the order of the differential equation. Thus, a second-order system requires a minimum of two state variables to describe it. We can define more state variables than the minimal set; however, within this minimal set the state variables must be linearly independent. For example, if $v_R(t)$ is chosen as a state variable, then $i(t)$ cannot be chosen, because $v_R(t)$ can be written as a linear combination of $i(t)$, namely $v_R(t) = Ri(t)$. Under these circumstances we say that the state variables are *linearly dependent*. State variables must be *linearly independent*; that is, no state variable can be written as a linear combination of the other state variables, or else we would not have enough information to solve for all other system variables, and we could even have trouble writing the simultaneous equations themselves.

The state and output equations can be written in vector-matrix form if the system is linear. Thus, Eq. (3.12), the state equations, can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{3.15}$$

where

$$\dot{\mathbf{x}} = \begin{bmatrix} dq/dt \\ di/dt \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}; \quad u = v(t)$$

Equation (3.13), the output equation, can be written as

$$y = \mathbf{C}\mathbf{x} + Du \tag{3.16}$$

where

$$y = v_L(t); \quad \mathbf{C} = [-1/C \quad -R]; \quad \mathbf{x} = \begin{bmatrix} q \\ i \end{bmatrix}; \quad D = 1; \quad u = v(t)$$

We call the combination of Eqs. (3.15) and (3.16) a *state-space representation* of the network of Figure 3.2. A state-space representation, therefore, consists of (1) the simultaneous, first-order differential equations from which the state variables can be solved and (2) the algebraic output equation from which all other system variables can be found. A state-space representation is not unique, since a different choice of state variables leads to a different representation of the same system.

In this section, we used two electrical networks to demonstrate some principles that are the foundation of the state-space representation. The representations developed in this section were for single-input, single-output systems, where y, D, and u in Eqs. (3.15) and (3.16) are scalar quantities. In general, systems have multiple inputs and multiple outputs. For these cases, y and u become vector quantities, and D becomes a matrix. In Section 3.3 we will generalize the representation for multiple-input, multiple-output systems and summarize the concept of the state-space representation.

3.3 The General State-Space Representation

Now that we have represented a physical network in state space and have a good idea of the terminology and the concept, let us summarize and generalize the representation for linear differential equations. First, we formalize some of the definitions that we came across in the last section.

Linear combination. A linear combination of n variables, x_i, for $i = 1$ to n, is given by the following sum, S:

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1 \tag{3.17}$$

where each K_i is a constant.

Linear independence. A set of variables is said to be linearly independent if none of the variables can be written as a linear combination of the others. For example, given x_1, x_2, and x_3, if $x_2 = 5x_1 + 6x_3$, then the variables are not linearly independent, since one of them can be written as a linear combination of the other two. Now, what must be true so that one variable cannot be written as a linear combination of the other variables? Consider the example $K_2x_2 = K_1x_1 + K_3x_3$. If no $x_i = 0$, then any x_i can be written as a linear combination of other variables, unless all $K_i = 0$. Formally, then, variables x_i, for $i = 1$ to n, are said to be linearly independent if their linear combination, S, equals zero *only* if every $K_i = 0$ and *no* $x_i = 0$ for all $t \geq 0$.

System variable. Any variable that responds to an input or initial conditions in a system.

State variables. The smallest set of linearly independent system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t \geq t_0$.

State vector. A vector whose elements are the state variables.

State space. The n-dimensional space whose axes are the state variables. This is a new term and is illustrated in Figure 3.3, where the state variables are assumed to be a resistor voltage, v_R, and a capacitor voltage, v_C. These variables form the axes of the *state space*. A trajectory can be thought of as being mapped out by the state vector, $\mathbf{x}(t)$, for a range of t. Also shown is the state vector at the particular time $t = 4$.

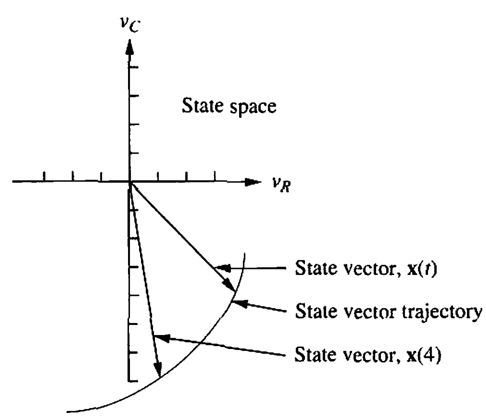

FIGURE 3.3 Graphic representation of state space and a state vector

State equations. A set of n simultaneous, first-order differential equations with n variables, where the n variables to be solved are the state variables.

Output equation. The algebraic equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

Now that the definitions have been formally stated, we define the state-space representation of a system. A system is represented in state space by the following equations:

$$\boxed{\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}} \tag{3.18}$$

$$\boxed{\mathbf{y} = \mathbf{Cx} + \mathbf{Du}} \tag{3.19}$$

for $t \geq t_0$ and initial conditions, $\mathbf{x}(t_0)$, where

$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

Equation (3.18) is called the *state equation*, and the vector $\mathbf{x}$, the *state vector*, contains the state variables. Equation (3.18) can be solved for the state variables, which we demonstrate in Chapter 4. Equation (3.19) is called the *output equation*. This equation is used to calculate any other system variables. This representation of a system provides complete knowledge of all variables of the system at any $t \geq t_0$.

As an example, for a linear, time-invariant, second-order system with a single input $v(t)$, the state equations could take on the following form:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1 v(t) \tag{3.20a}$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2 v(t) \tag{3.20b}$$

where x_1 and x_2 are the state variables. If there is a single output, the output equation could take on the following form:

$$y = c_1 x_1 + c_2 x_2 + d_1 v(t) \tag{3.21}$$

The choice of state variables for a given system is not unique. The requirement in choosing the state variables is that they be linearly independent and that a minimum number of them be chosen.

3.4 Applying the State-Space Representation

In this section, we apply the state-space formulation to the representation of more complicated physical systems. The first step in representing a system is to select the state vector, which must be chosen according to the following considerations:

1. A minimum number of state variables must be selected as components of the state vector. This minimum number of state variables is sufficient to describe completely the state of the system.
2. The components of the state vector (that is, this minimum number of state variables) must be linearly independent.

Let us review and clarify these statements.

Linearly Independent State Variables

The components of the state vector must be linearly independent. For example, following the definition of linear independence in Section 3.3, if x_1, x_2, and x_3 are chosen as state variables, but $x_3 = 5x_1 + 4x_2$, then x_3 is not linearly independent of x_1

and x_2, since knowledge of the values of x_1 and x_2 will yield the value of x_3. Variables and their successive derivatives are linearly independent. For example, the voltage across an inductor, v_L, is linearly independent of the current through the inductor, i_L, since $v_L = L di_L/dt$. Thus, v_L cannot be evaluated as a linear combination of the current, i_L.

Minimum Number of State Variables

How do we know the minimum number of state variables to select? Typically, the minimum number required equals the order of the differential equation describing the system. For example, if a third-order differential equation describes the system, then three simultaneous, first-order differential equations are required along with three state variables. From the perspective of the transfer function, the order of the differential equation is the order of the denominator of the transfer function after canceling common factors in the numerator and denominator.

In most cases, another way to determine the number of state variables is to count the number of independent energy-storage elements in the system.[5] The number of these energy-storage elements equals the order of the differential equation and the number of state variables. In Figure 3.2 there are two energy-storage elements, the capacitor and the inductor. Hence, two state variables and two state equations are required for the system.

If too few state variables are selected, it may be impossible to write particular output equations, since some system variables cannot be written as a linear combination of the reduced number of state variables. In many cases, it may be impossible even to complete the writing of the state equations, since the derivatives of the state variables cannot be expressed as linear combinations of the reduced number of state variables.

If you select the minimum number of state variables but they are not linearly independent, at best you may not be able to solve for all other system variables. At worst you may not be able to complete the writing of the state equations.

Often the state vector includes more than the minimum number of state variables required. Two possible cases exist. Often state variables are chosen to be physical variables of a system, such as position and velocity in a mechanical system. Cases arise where these variables, although linearly independent, are also *decoupled*. That is, some linearly independent variables are not required in order to solve for any of the other linearly independent variables or any other dependent system variable. Consider the case of a mass and viscous damper whose differential equation is $M\,dv/dt + Dv = f(t)$, where v is the velocity of the mass. Since this is a first-order equation, one state equation is all that is required to define this system in state space with velocity as the state variable. Also, since there is only one energy-storage element, mass, only one state variable is required to represent this system in state space. However, the mass also has an associated position, which is linearly independent of velocity. If we want to include position in the state vector along with velocity, then we add position as a state variable that is linearly independent of the other state variable, velocity. Figure 3.4 illustrates what is happening. The first block is the transfer

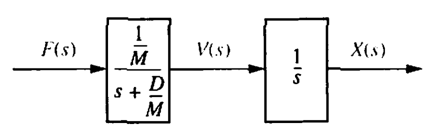

FIGURE 3.4 Block diagram of a mass and damper

[5] Sometimes it is not apparent in a schematic how many independent energy-storage elements there are. It is possible that more than the minimum number of energy-storage elements could be selected, leading to a state vector whose components number more than the minimum required and are not linearly independent. Selecting additional dependent energy-storage elements results in a system matrix of higher order and more complexity than required for the solution of the state equations.

function equivalent to $M\,dv(t)/dt + Dv(t) = f(t)$. The second block shows that we integrate the output velocity to yield output displacement (see Table 2.2, Item 10). Thus, if we want displacement as an output, the denominator, or characteristic equation, has increased in order to 2, the product of the two transfer functions. Many times, the writing of the state equations is simplified by including additional state variables.

Another case that increases the size of the state vector arises when the added variable is not linearly independent of the other members of the state vector. This usually occurs when a variable is selected as a state variable but its dependence on the other state variables is not immediately apparent. For example, energy-storage elements may be used to select the state variables, and the dependence of the variable associated with one energy-storage element on the variables of other energy-storage elements may not be recognized. Thus, the dimension of the system matrix is increased unnecessarily, and the solution for the state vector, which we cover in Chapter 4, is more difficult. Also, adding dependent state variables affects the designer's ability to use state-space methods for design.[6]

We saw in Section 3.2 that the state-space representation is not unique. The following example demonstrates one technique for selecting state variables and representing a system in state space. Our approach is to write the simple derivative equation for each energy-storage element and solve for each derivative term as a linear combination of any of the system variables and the input that are present in the equation. Next we select each differentiated variable as a state variable. Then we express all other system variables in the equations in terms of the state variables and the input. Finally, we write the output variables as linear combinations of the state variables and the input.

Example 3.1

Representing an Electrical Network

PROBLEM: Given the electrical network of Figure 3.5, find a state-space representation if the output is the current through the resistor.

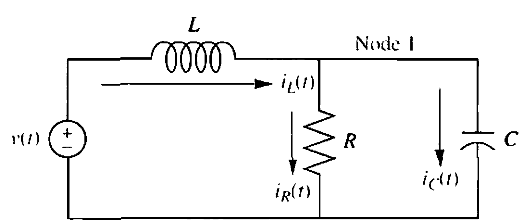

FIGURE 3.5 Electrical network for representation in state space

SOLUTION: The following steps will yield a viable representation of the network in state space.

Step 1 Label all of the branch currents in the network. These include i_L, i_R, and i_C, as shown in Figure 3.5.

[6] See Chapter 12 for state-space design techniques.

Step 2 Select the state variables by writing the derivative equation for all energy-storage elements, that is, the inductor and the capacitor. Thus,

$$C\frac{dv_C}{dt} = i_C \tag{3.22}$$

$$L\frac{di_L}{dt} = v_L \tag{3.23}$$

From Eqs. (3.22) and (3.23), choose the state variables as the quantities that are differentiated, namely v_C and i_L. Using Eq. (3.20) as a guide, we see that the state-space representation is complete if the right-hand sides of Eqs. (3.22) and (3.23) can be written as linear combinations of the state variables and the input.

Since i_C and v_L are not state variables, our next step is to express i_C and v_L as linear combinations of the state variables, v_C and i_L, and the input, $v(t)$.

Step 3 Apply network theory, such as Kirchhoff's voltage and current laws, to obtain i_C and v_L in terms of the state variables, v_C and i_L. At Node 1,

$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R}v_C + i_L \end{aligned} \tag{3.24}$$

which yields i_C in terms of the state variables, v_C and i_L.

Around the outer loop,

$$v_L = -v_C + v(t) \tag{3.25}$$

which yields v_L in terms of the state variable, v_C, and the source, $v(t)$.

Step 4 Substitute the results of Eqs. (3.24) and (3.25) into Eqs. (3.22) and (3.23) to obtain the following state equations:

$$C\frac{dv_C}{dt} = -\frac{1}{R}v_C + i_L \tag{3.26a}$$

$$L\frac{di_L}{dt} = \quad -v_C + v(t) \tag{3.26b}$$

or

$$\frac{dv_C}{dt} = -\frac{1}{RC}v_C + \frac{1}{C}i_L \tag{3.27a}$$

$$\frac{di_L}{dt} = \quad -\frac{1}{L}v_C + \frac{1}{L}v(t) \tag{3.27b}$$

Step 5 Find the output equation. Since the output is $i_R(t)$,

$$i_R = \frac{1}{R}v_C \tag{3.28}$$

The final result for the state-space representation is found by representing Eqs. (3.27) and (3.28) in vector-matrix form as follows:

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t) \tag{3.29a}$$

$$i_R = [1/R \quad 0]\begin{bmatrix} v_C \\ i_L \end{bmatrix} \tag{3.29b}$$

where the dot indicates differentiation with respect to time.

In order to clarify the representation of physical systems in state space, we will look at two more examples. The first is an electrical network with a dependent source. Although we will follow the same procedure as in the previous problem, this problem will yield increased complexity in applying network analysis to find the state equations. For the second example, we find the state-space representation of a mechanical system.

Example 3.2

Representing an Electrical Network with a Dependent Source

PROBLEM: Find the state and output equations for the electrical network shown in Figure 3.6 if the output vector is $\mathbf{y} = [v_{R_2} \quad i_{R_2}]^T$, where T means transpose.[7]

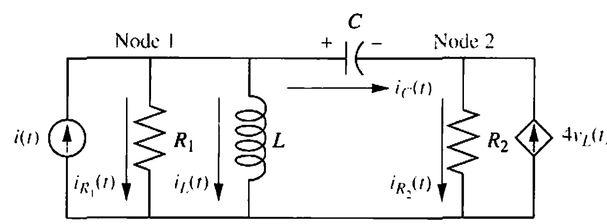

FIGURE 3.6 Electrical network for Example 3.2

SOLUTION: Immediately notice that this network has a voltage-dependent current source.

Step 1 Label all of the branch currents on the network, as shown in Figure 3.6.

Step 2 Select the state variables by listing the voltage-current relationships for all of the energy-storage elements:

$$L\frac{di_L}{dt} = v_L \tag{3.30a}$$

$$C\frac{dv_C}{dt} = i_C \tag{3.30b}$$

From Eqs. (3.30) select the state variables to be the differentiated variables. Thus, the state variables, x_1 and x_2, are

$$x_1 = i_L; \quad x_2 = v_C \tag{3.31}$$

Step 3 Remembering that the form of the state equation is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{3.32}$$

we see that the remaining task is to transform the right-hand side of Eq. (3.30) into linear combinations of the state variables and input source current. Using Kirchhoff's voltage and current laws, we find v_L and i_C in terms of the state variables and the input current source.

[7] See Appendix G for a discussion of the transpose. Appendix G is located at www.wiley.com/college/nise.

Around the mesh containing L and C,

$$v_L = v_C + v_{R_2} = v_C + i_{R_2}R_2 \tag{3.33}$$

But at Node 2, $i_{R_2} = i_C + 4v_L$. Substituting this relationship for i_{R_2} into Eq. (3.33) yields

$$v_L = v_C + (i_C + 4v_L)R_2 \tag{3.34}$$

Solving for v_L, we get

$$v_L = \frac{1}{1-4R_2}(v_C + i_C R_2) \tag{3.35}$$

Notice that since v_C is a state variable, we only need to find i_C in terms of the state variables. We will then have obtained v_L in terms of the state variables.

Thus, at Node 1 we can write the sum of the currents as

$$\begin{aligned} i_C &= i(t) - i_{R_1} - i_L \\ &= i(t) - \frac{v_{R_1}}{R_1} - i_L \\ &= i(t) - \frac{v_L}{R_1} - i_L \end{aligned} \tag{3.36}$$

where $v_{R_1} = v_L$. Equations (3.35) and (3.36) are two equations relating v_L and i_C in terms of the state variables i_L and v_C. Rewriting Eqs. (3.35) and (3.36), we obtain two simultaneous equations yielding v_L and i_C as linear combinations of the state variables i_L and v_C:

$$(1-4R_2)v_L - R_2 i_C = v_C \tag{3.37a}$$

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t) \tag{3.37b}$$

Solving Eq. (3.37) simultaneously for v_L and i_C yields

$$v_L = \frac{1}{\Delta}[R_2 i_L - v_C - R_2 i(t)] \tag{3.38}$$

and

$$i_C = \frac{1}{\Delta}\left[(1-4R_2)i_L + \frac{1}{R_1}v_C - (1-4R_2)i(t)\right] \tag{3.39}$$

where

$$\Delta = -\left[(1-4R_2) + \frac{R_2}{R_1}\right] \tag{3.40}$$

Substituting Eqs. (3.38) and (3.39) into (3.30), simplifying, and writing the result in vector-matrix form renders the following state equation:

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} R_2/(L\Delta) & -1/(L\Delta) \\ (1-4R_2)/(C\Delta) & 1/(R_1 C\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/(L\Delta) \\ -(1-4R_2)/(C\Delta) \end{bmatrix} i(t) \tag{3.41}$$

Step 4 Derive the output equation. Since the specified output variables are v_{R_2} and i_{R_2}, we note that around the mesh containing C, L, and R_2,

$$v_{R_2} = -v_C + v_L \tag{3.42a}$$

$$i_{R_2} = i_C + 4v_L \tag{3.42b}$$

Substituting Eqs. (3.38) and (3.39) into Eq. (3.42), v_{R_2} and i_{R_2} are obtained as linear combinations of the state variables, i_L and v_C. In vector-matrix form, the output equation is

$$\begin{bmatrix} v_{R_2} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1+1/\Delta) \\ 1/\Delta & (1-4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t) \tag{3.43}$$

In the next example, we find the state-space representation for a mechanical system. It is more convenient when working with mechanical systems to obtain the state equations directly from the equations of motion rather than from the energy-storage elements. For example, consider an energy-storage element such as a spring, where $F = Kx$. This relationship does not contain the derivative of a physical variable as in the case of electrical networks, where $i = C\,dv/dt$ for capacitors, and $v = L\,di/dt$ for inductors. Thus, in mechanical systems we change our selection of state variables to be the position and velocity of each point of linearly independent motion. In the example, we will see that although there are three energy-storage elements, there will be four state variables; an additional linearly independent state variable is included for the convenience of writing the state equations. It is left to the student to show that this system yields a fourth-order transfer function if we relate the displacement of either mass to the applied force, and a third-order transfer function if we relate the velocity of either mass to the applied force.

Example 3.3

Representing a Translational Mechanical System

PROBLEM: Find the state equations for the translational mechanical system shown in Figure 3.7.

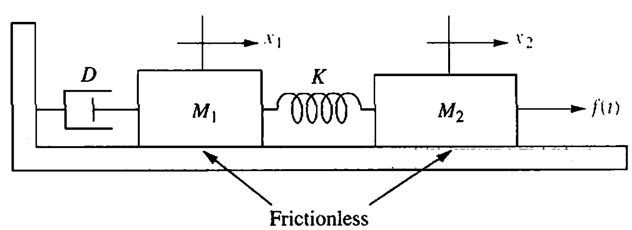

FIGURE 3.7 Translational mechanical system

SOLUTION: First write the differential equations for the network in Figure 3.7, using the methods of Chapter 2 to find the Laplace-transformed equations of motion. Next take the inverse Laplace transform of these equations, assuming zero initial conditions, and obtain

$$M_1 \frac{d^2x_1}{dt^2} + D\frac{dx_1}{dt} + Kx_1 - Kx_2 = 0 \tag{3.44}$$

$$-Kx_1 + M_2\frac{d^2x_2}{dt^2} + Kx_2 = f(t) \tag{3.45}$$

Now let $d^2x_1/dt^2 = dv_1/dt$, and $d^2x_2/dt^2 = dv_2/dt$, and then select x_1, v_1, x_2, and v_2 as state variables. Next form two of the state equations by solving Eq. (3.44) for dv_1/dt and Eq. (3.45) for dv_2/dt. Finally, add $dx_1/dt = v_1$ and $dx_2/dt = v_2$ to complete the set of state equations. Hence,

$$\frac{dx_1}{dt} = +v_1 \tag{3.46a}$$

$$\frac{dv_1}{dt} = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2 \tag{3.46b}$$

$$\frac{dx_2}{dt} = +v_2 \tag{3.46c}$$

$$\frac{dv_2}{dt} = +\frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{1}{M_2}f(t) \tag{3.46d}$$

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t) \tag{3.47}$$

where the dot indicates differentiation with respect to time. What is the output equation if the output is $x(t)$?

Skill-Assessment Exercise 3.1

PROBLEM: Find the state-space representation of the electrical network shown in Figure 3.8. The output is $v_o(t)$.

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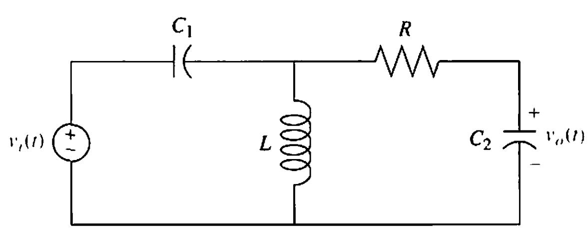

FIGURE 3.8 Electric circuit for Skill-Assessment Exercise 3.1

ANSWER:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1/C_1 & 1/C_1 & -1/C_1 \\ -1/L & 0 & 0 \\ 1/C_2 & 0 & -1/C_2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v_i(t)$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 3.2

PROBLEM: Represent the translational mechanical system shown in Figure 3.9 in state space, where $x_3(t)$ is the output.

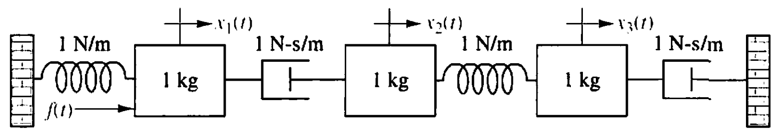

FIGURE 3.9 Translational mechanical system for Skill-Assessment Exercise 3.2

ANSWER:

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{z}$$

where

$$\mathbf{z} = \begin{bmatrix} x_1 & \dot{x}_1 & x_2 & \dot{x}_2 & x_3 & \dot{x}_3 \end{bmatrix}^T$$

The complete solution is at www.wiley.com/college/nise.

3.5 Converting a Transfer Function to State Space

In the last section, we applied the state-space representation to electrical and mechanical systems. We learn how to convert a transfer function representation to a state-space representation in this section. One advantage of the state-space representation is that it can be used for the simulation of physical systems on the digital computer. Thus, if we want to simulate a system that is represented by a transfer function, we must first convert the transfer function representation to state space.

At first we select a set of state variables, called *phase variables*, where each subsequent state variable is defined to be the derivative of the previous state variable. In Chapter 5 we show how to make other choices for the state variables.

Let us begin by showing how to represent a general, nth-order, linear differential equation with constant coefficients in state space in the phase-variable form. We will then show how to apply this representation to transfer functions.

Consider the differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u \tag{3.48}$$

A convenient way to choose state variables is to choose the output, $y(t)$, and its $(n-1)$ derivatives as the state variables. This choice is called the *phase-variable choice*. Choosing the state variables, x_i, we get

$$x_1 = y \tag{3.49a}$$

$$x_2 = \frac{dy}{dt} \tag{3.49b}$$

$$x_3 = \frac{d^2 y}{dt^2} \tag{3.49c}$$

$$\vdots$$

$$x_n = \frac{d^{n-1} y}{dt^{n-1}} \tag{3.49d}$$

and differentiating both sides yields

$$\dot{x}_1 = \frac{dy}{dt} \tag{3.50a}$$

$$\dot{x}_2 = \frac{d^2 y}{dt^2} \tag{3.50b}$$

$$\dot{x}_3 = \frac{d^3 y}{dt^3} \tag{3.50c}$$

$$\vdots$$

$$\dot{x}_n = \frac{d^n y}{dt^n} \tag{3.50d}$$

where the dot above the x signifies differentiation with respect to time.

Substituting the definitions of Eq. (3.49) into Eq. (3.50), the state equations are evaluated as

$$\dot{x}_1 = x_2 \tag{3.51a}$$

$$\dot{x}_2 = x_3 \tag{3.51b}$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n \tag{3.51c}$$

$$\dot{x}_n = -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u \tag{3.51d}$$

where Eq. (3.51d) was obtained from Eq. (3.48) by solving for $d^n y/dt^n$ and using Eq. (3.49). In vector-matrix form, Eq. (3.51) become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u \tag{3.52}$$

Equation (3.52) is the phase-variable form of the state equations. This form is easily recognized by the unique pattern of 1's and 0's and the negative of the coefficients of the differential equation written in reverse order in the last row of the system matrix.

Finally, since the solution to the differential equation is $y(t)$, or x_1, the output equation is

$$y = [1 \quad 0 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \tag{3.53}$$

In summary, then, to convert a transfer function into state equations in phase-variable form, we first convert the transfer function to a differential equation by cross-multiplying and taking the inverse Laplace transform, assuming zero initial conditions. Then we represent the differential equation in state space in phase-variable form. An example illustrates the process.

Example 3.4

Converting a Transfer Function with Constant Term in Numerator

PROBLEM: Find the state-space representation in phase-variable form for the transfer function shown in Figure 3.10(a).

SOLUTION:

Step 1 Find the associated differential equation. Since

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \tag{3.54}$$

cross-multiplying yields

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s) \tag{3.55}$$

The corresponding differential equation is found by taking the inverse Laplace transform, assuming zero initial conditions:

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r \tag{3.56}$$

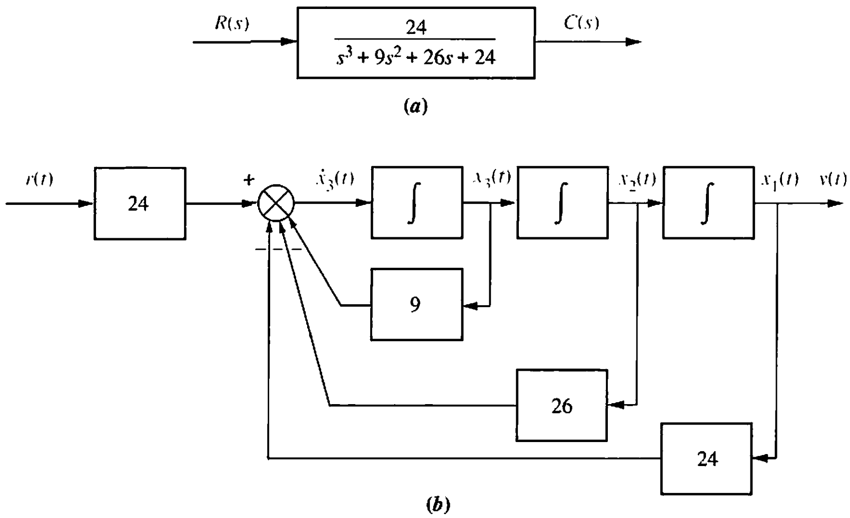

FIGURE 3.10 **a.** Transfer function; **b.** equivalent block diagram showing phase variables. Note: $y(t) = c(t)$.

Step 2 Select the state variables.
Choosing the state variables as successive derivatives, we get

$$x_1 = c \tag{3.57a}$$

$$x_2 = \dot{c} \tag{3.57b}$$

$$x_3 = \ddot{c} \tag{3.57c}$$

Differentiating both sides and making use of Eq. (3.57) to find $\dot{x}_1$ and $\dot{x}_2$, and Eq. (3.56) to find $\dddot{c} = \dot{x}_3$, we obtain the state equations. Since the output is $c = x_1$, the combined state and output equations are

$$\dot{x}_1 = \quad x_2 \tag{3.58a}$$

$$\dot{x}_2 = \quad x_3 \tag{3.58b}$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r \tag{3.58c}$$

$$y = c = x_1 \tag{3.58d}$$

In vector-matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r \tag{3.59a}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.59b}$$

Notice that the third row of the system matrix has the same coefficients as the denominator of the transfer function but negative and in reverse order.

At this point, we can create an equivalent block diagram of the system of Figure 3.10(a) to help visualize the state variables. We draw three integral blocks as shown in Figure 3.10(b) and label each output as one of the state variables, $x_i(t)$, as shown. Since the input to each integrator is $\dot{x}_i(t)$, use Eqs. (3.58a), (3.58b), and (3.58c) to determine the combination of input signals to each integrator. Form and label each input. Finally, use Eq. (3.58d) to form and label the output, $y(t) = c(t)$. The final result of Figure 3.10 (*b*) is a system equivalent to Figure 3.10(*a*) that explicitly shows the state variables and gives a vivid picture of the state-space representation.

MATLAB ML

Students who are using MATLAB should now run ch3p1 through ch3p4 in Appendix B. You will learn how to represent the system matrix **A**, the input matrix **B**, and the output matrix **C** using MATLAB. You will learn how to convert a transfer function to the state-space representation in phase-variable form. Finally, Example 3.4 will be solved using MATLAB.

The transfer function of Example 3.4 has a constant term in the numerator. If a transfer function has a polynomial in *s* in the numerator that is of order less than the polynomial in the denominator, as shown in Figure 3.11(*a*), the numerator and denominator can be handled separately. First separate the transfer function into two cascaded transfer functions, as shown in Figure 3.11(*b*); the first is the denominator, and the second is just the numerator. The first transfer function with just the denominator is converted to the phase-variable representation in state space as demonstrated in the last example. Hence, phase variable x_1 is the output, and the rest of the phase variables are the internal variables of the first block, as shown in Figure 3.11(*b*). The second transfer function with just the numerator yields

$$Y(s) = C(s) = (b_2 s^2 + b_1 s + b_0)X_1(s) \tag{3.60}$$

where, after taking the inverse Laplace transform with zero initial conditions,

$$y(t) = b_2 \frac{d^2 x_1}{dt^2} + b_1 \frac{dx_1}{dt} + b_0 x_1 \tag{3.61}$$

But the derivative terms are the definitions of the phase variables obtained in the first block. Thus, writing the terms in reverse order to conform to an output equation,

$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3 \tag{3.62}$$

Hence, the second block simply forms a specified linear combination of the state variables developed in the first block.

From another perspective, the denominator of the transfer function yields the state equations, while the numerator yields the output equation. The next example demonstrates the process.

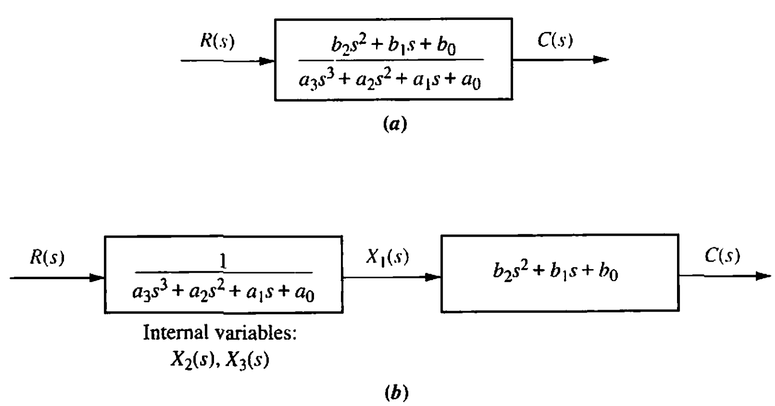

FIGURE 3.11 Decomposing a transfer function

Example 3.5

Converting a Transfer Function with Polynomial in Numerator

PROBLEM: Find the state-space representation of the transfer function shown in Figure 3.12(*a*).

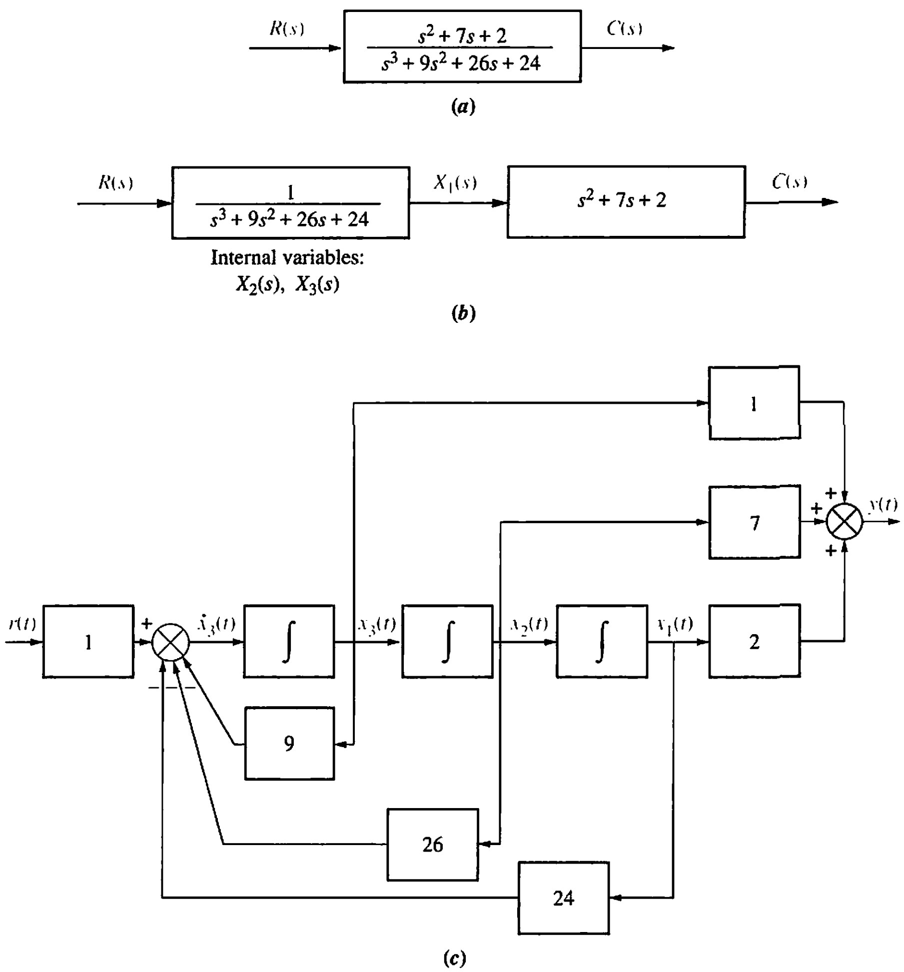

FIGURE 3.12 **a.** Transfer function; **b.** decomposed transfer function; **c.** equivalent block diagram Note: $y(t) = c(t)$.

SOLUTION: This problem differs from Example 3.4 since the numerator has a polynomial in s instead of just a constant term.

Step 1 Separate the system into two cascaded blocks, as shown in Figure 3.12(*b*). The first block contains the denominator and the second block contains the numerator.

Step 2 Find the state equations for the block containing the denominator. We notice that the first block's numerator is 1/24 that of Example 3.4. Thus, the state equations are the same except that this system's input matrix is 1/24 that of Example 3.4. Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \tag{3.63}$$

Step 3 Introduce the effect of the block with the numerator. The second block of Figure 3.12(*b*), where $b_2 = 1$, $b_1 = 7$, and $b_0 = 2$, states that

$$C(s) = (b_2s^2 + b_1s + b_0)X_1(s) = (s^2 + 7s + 2)X_1(s) \tag{3.64}$$

Taking the inverse Laplace transform with zero initial conditions, we get

$$c = \ddot{x}_1 + 7\dot{x}_1 + 2x_1 \tag{3.65}$$

But

$$x_1 = x_1$$
$$\dot{x}_1 = x_2$$
$$\ddot{x}_1 = x_3$$

Hence,

$$y = c(t) = b_2x_3 + b_1x_2 + b_0x_1 = x_3 + x_2 + 2x_1 \tag{3.66}$$

Thus, the last box of Figure 3.11(*b*) "collects" the states and generates the output equation. From Eq. (3.66),

$$y = [b_0 \quad b_1 \quad b_2]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [2 \quad 7 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{3.67}$$

Although the second block of Figure 3.12(*b*) shows differentiation, this block was implemented without differentiation because of the partitioning that was applied to the transfer function. The last block simply collected derivatives that were already formed by the first block.

Once again we can produce an equivalent block diagram that vividly represents our state-space model. The first block of Figure 3.12(*b*) is the same as Figure 3.10(*a*) except for the different constant in the numerator. Thus, in Figure 3.12(*c*) we reproduce Figure 3.10(*b*) except for the change in the numerator constant, which appears as a change in the input multiplying factor. The second block of Figure 3.12(*b*) is represented using Eq. (3.66), which forms the output from a linear combination of the state variables, as shown in Figure 3.12(*c*).

TryIt 3.1

Use the following MATLAB statements to form an LTI state-space representation from the transfer function shown in Figure 3.12(a). The **A** matrix and **B** vector are shown in Eq. (3.63). The **C** vector is shown in Eq. (3.67).

```
num=[1 7 2];
den=[1 9 26 24];
[A,B,C,D]=tf2ss...
 (num,den);
P=[0 0 1;0 1 0;1 0 0];
A=inv(P)*A*P
B=inv(P)*B
C=C*P
```

Skill-Assessment Exercise 3.3

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PROBLEM: Find the state equations and output equation for the phase-variable representation of the transfer function $G(s) = \dfrac{2s + 1}{s^2 + 7s + 9}$.

ANSWER:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$
$$y = [1 \quad 2]\mathbf{x}$$

The complete solution is at www.wiley.com/college.nise.

3.6 Converting from State Space to a Transfer Function

In Chapters 2 and 3, we have explored two methods of representing systems: the transfer function representation and the state-space representation. In the last section, we united the two representations by converting transfer functions into state-space representations. Now we move in the opposite direction and convert the state-space representation into a transfer function.

Given the state and output equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.68a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{3.68b}$$

take the Laplace transform assuming zero initial conditions:[8]

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{3.69a}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{3.69b}$$

Solving for $\mathbf{X}(s)$ in Eq. (3.69a),

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) \tag{3.70}$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \tag{3.71}$$

where $\mathbf{I}$ is the identity matrix.

Substituting Eq. (3.71) into Eq. (3.69b) yields

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s) \tag{3.72}$$

We call the matrix $[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]$ the transfer function matrix, since it relates the output vector, $\mathbf{Y}(s)$, to the input vector, $\mathbf{U}(s)$. However, if $\mathbf{U}(s) = U(s)$ and $\mathbf{Y}(s) = Y(s)$ are scalars, we can find the transfer function,

$$\boxed{T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}} \tag{3.73}$$

Let us look at an example.

---

[8] The Laplace transform of a vector is found by taking the Laplace transform of each component. Since $\dot{\mathbf{x}}$ consists of the derivatives of the state variables, the Laplace transform of $\dot{\mathbf{x}}$ with zero initial conditions yields each component with the form $sX_i(s)$, where $X_i(s)$ is the Laplace transform of the state variable. Factoring out the complex variable, s, in each component yields the Laplace transform of $\dot{\mathbf{x}}$ as $s\,\mathbf{X}(s)$, where $\mathbf{X}(s)$ is a column vector with components $X_i(s)$.

Example 3.6

State-Space Representation to Transfer Function

PROBLEM: Given the system defined by Eq. (3.74), find the transfer function, $T(s) = Y(s)/U(s)$, where $U(s)$ is the input and $Y(s)$ is the output.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u \tag{3.74a}$$

$$y = [1 \quad 0 \quad 0]\mathbf{x} \tag{3.74b}$$

SOLUTION: The solution revolves around finding the term $(s\mathbf{I} - \mathbf{A})^{-1}$ in Eq. (3.73).[9] All other terms are already defined. Hence, first find $(s\mathbf{I} - \mathbf{A})$:

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix} \tag{3.75}$$

Now form $(s\mathbf{I} - \mathbf{A})^{-1}$:

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2+3s+2) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1} \tag{3.76}$$

Substituting $(s\mathbf{I} - \mathbf{A})^{-1}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ into Eq. (3.73), where

$$\mathbf{B} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0]$$

$$\mathbf{D} = 0$$

we obtain the final result for the transfer function:

$$T(s) = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1} \tag{3.77}$$

MATLAB **ML**

Students who are using MATLAB should now run ch3p5 in Appendix B. You will learn how to convert a state-space representation to a transfer function using MATLAB. You can practice by writing a MATLAB program to solve Example 3.6.

Symbolic Math **SM**

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch3sp1 in Appendix F located at www.wiley.com/college/nise. You will learn how to use the Symbolic Math Toolbox to write matrices and vectors. You will see that the Symbolic Math Toolbox yields an alternative way to use MATLAB to solve Example 3.6.

[9] See Appendix G. *It is located at* www.wiley.com/college/nise *and discusses the evaluation of the matrix inverse.*

Skill-Assessment Exercise 3.4

PROBLEM: Convert the state and output equations shown in Eq. (3.78) to a transfer function.

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}u(t) \quad (3.78a)$$

$$y = [1.5 \quad 0.625]\mathbf{x} \quad (3.78b)$$

ANSWER:

$$G(s) = \frac{3s+5}{s^2+4s+6}$$

The complete solution is located at www.wiley.com/college/nise.

TryIt 3.2

Use the following MATLAB and the Control System Toolbox statements to obtain the transfer function shown in Skill-Assessment Exercise 3.4 from the state-space representation of Eq. (3.78).

```
A=[-4 -1.5;4 0];
B=[2 0]';
C=[1.5 0.625];
D=0;
T=ss(A,B,C,D);
T=tf(T)
```

In Example 3.6, the state equations in phase-variable form were converted to transfer functions. In Chapter 5, we will see that other forms besides the phase-variable form can be used to represent a system in state space. The method of finding the transfer function representation for these other forms is the same as that presented in this section.

3.7 Linearization

A prime advantage of the state-space representation over the transfer function representation is the ability to represent systems with nonlinearities, such as the one shown in Figure 3.13. The ability to represent nonlinear systems does not imply the ability to solve their state equations for the state variables and the output. Techniques do exist for the solution of some nonlinear state equations, but this study is beyond the scope of this course. However, in Appendix H, located at www.wiley.com/college/nise,

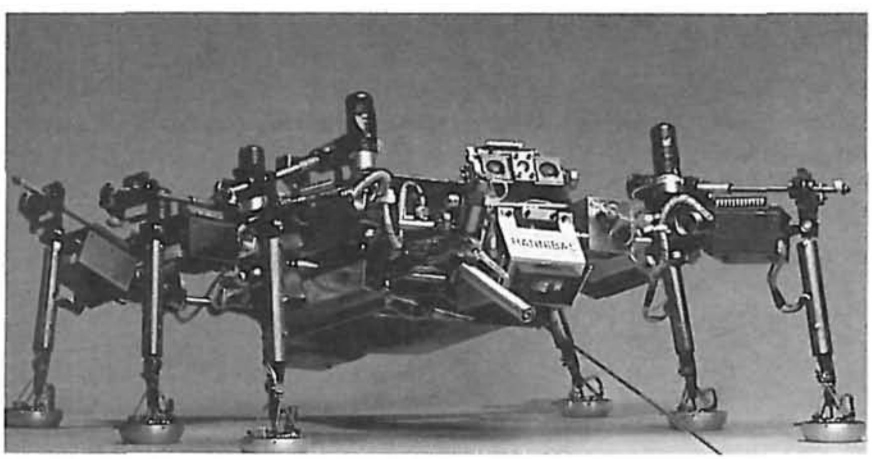

FIGURE 3.13 Walking robots, such as *Hannibal* shown here, can be used to explore hostile environments and rough terrain, such as that found on other planets or inside volcanoes.