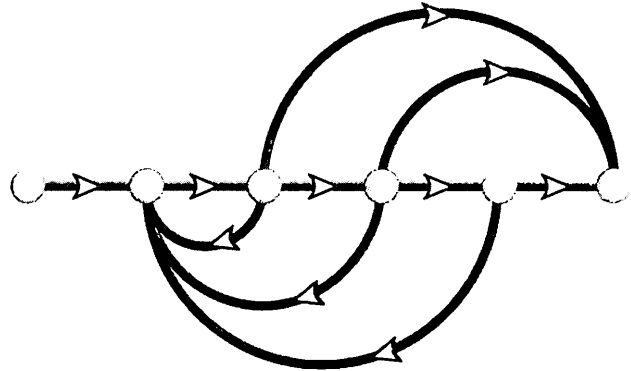


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Root Locus Techniques

8



Chapter Learning Outcomes

After completing this chapter the student will be able to:

- Define a root locus (Sections 8.1–8.2)
- State the properties of a root locus (Section 8.3)
- Sketch a root locus (Section 8.4)
- Find the coordinates of points on the root locus and their associated gains (Sections 8.5–8.6)
- Use the root locus to design a parameter value to meet a transient response specification for systems of order 2 and higher (Sections 8.7–8.8)
- Sketch the root locus for positive-feedback systems (Section 8.9)
- Find the root sensitivity for points along the root locus (Section 8.10)

Case Study Learning Outcomes

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to find the preamplifier gain to meet a transient response specification.
- Given the pitch or heading control system for the Unmanned Free-Swimming Submersible vehicle shown on the back endpapers, you will be able to plot the root locus and design the gain to meet a transient response specification. You will then be able to evaluate other performance characteristics.

8.1 Introduction

Root locus, a graphical presentation of the closed-loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response (Evans, 1948; 1950). Feedback control systems are difficult to comprehend from a qualitative point of view, and hence they rely heavily upon mathematics. The root locus covered in this chapter is a graphical technique that gives us the qualitative description of a control system's performance that we are looking for and also serves as a powerful quantitative tool that yields more information than the methods already discussed.

Up to this point, gains and other system parameters were designed to yield a desired transient response for only first- and second-order systems. Even though the root locus can be used to solve the same kind of problem, its real power lies in its ability to provide solutions for systems of order higher than 2. For example, under the right conditions, a fourth-order system's parameters can be designed to yield a given percent overshoot and settling time using the concepts learned in Chapter 4.

The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed. The qualitative description can then be verified with quantitative analysis.

Besides transient response, the root locus also gives a graphical representation of a system's stability. We can clearly see ranges of stability, ranges of instability, and the conditions that cause a system to break into oscillation.

Before presenting root locus, let us review two concepts that we need for the ensuing discussion: (1) the control system problem and (2) complex numbers and their representation as vectors.

The Control System Problem

We have previously encountered the control system problem in Chapter 6: Whereas the poles of the open-loop transfer function are easily found (typically, they are known by inspection and do not change with changes in system gain), the poles of the closed-loop transfer function are more difficult to find (typically, they cannot be found without factoring the closed-loop system's characteristic polynomial, the denominator of the closed-loop transfer function), and further, the closed-loop poles change with changes in system gain.

A typical closed-loop feedback control system is shown in Figure 8.1(a). The open-loop transfer function was defined in Chapter 5 as $KG(s)H(s)$. Ordinarily, we

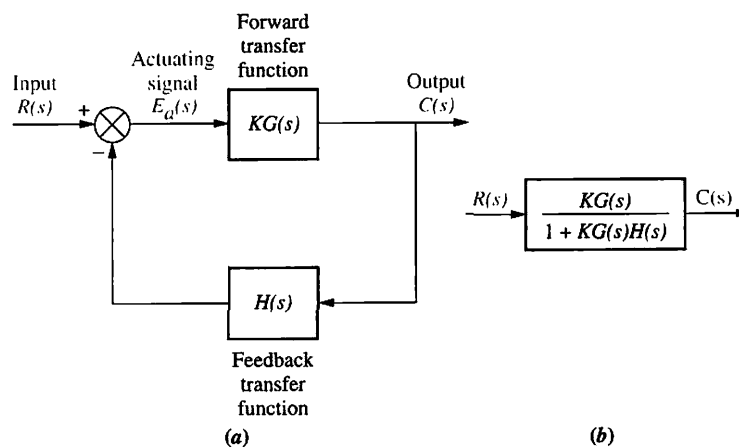


FIGURE 8.1 a. Closed-loop system; b. equivalent transfer function

can determine the poles of $KG(s)H(s)$, since these poles arise from simple cascaded first- or second-order subsystems. Further, variations in K do not affect the location of any pole of this function. On the other hand, we cannot determine the poles of $T(s) = KG(s)/[1 + KG(s)H(s)]$ unless we factor the denominator. Also, the poles of $T(s)$ change with K .

Let us demonstrate. Letting

$$G(s) = \frac{N_G(s)}{D_G(s)} \quad (8.1)$$

and

$$H(s) = \frac{N_H(s)}{D_H(s)} \quad (8.2)$$

then

$$T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)} \quad (8.3)$$

where N and D are factored polynomials and signify numerator and denominator terms, respectively. We observe the following: Typically, we know the factors of the numerators and denominators of $G(s)$ and $H(s)$. Also, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known and in fact can change with K . For example, if $G(s) = (s+1)/[s(s+2)]$ and $H(s) = (s+3)/(s+4)$, the poles of $KG(s)H(s)$ are 0, -2 , and -4 . The zeros of $KG(s)H(s)$ are -1 and -3 . Now, $T(s) = K(s+1)(s+4)/[s^3 + (6+K)s^2 + (8+4K)s + 3K]$. Thus, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of K . Since the system's transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system's performance unless we factor the denominator for specific values of K . The root locus will be used to give us a vivid picture of the poles of $T(s)$ as K varies.

Vector Representation of Complex Numbers

Any *complex number*, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in Figure 8.2(a). The complex number also can be described in polar form with magnitude M and angle θ , as $M\angle\theta$. If the complex number is substituted into a complex function, $F(s)$, another complex number will result. For example, if $F(s) = (s+a)$, then substituting the complex number $s = \sigma + j\omega$ yields $F(s) = (\sigma+a) + j\omega$, another complex number. This number is shown in Figure 8.2(b). Notice that $F(s)$ has a zero at $-a$. If we translate the vector a units to the left, as in Figure 8.2(c), we have an alternate representation of the complex number that originates at the zero of $F(s)$ and terminates on the point $s = \sigma + j\omega$.

We conclude that $(s+a)$ is a complex number and can be represented by a vector drawn from the zero of the function to the point s . For example, $(s+7)|_{s=5+j2}$ is a complex number drawn from the zero of the function, -7 , to the point s , which is $5 + j2$, as shown in Figure 8.2(d).

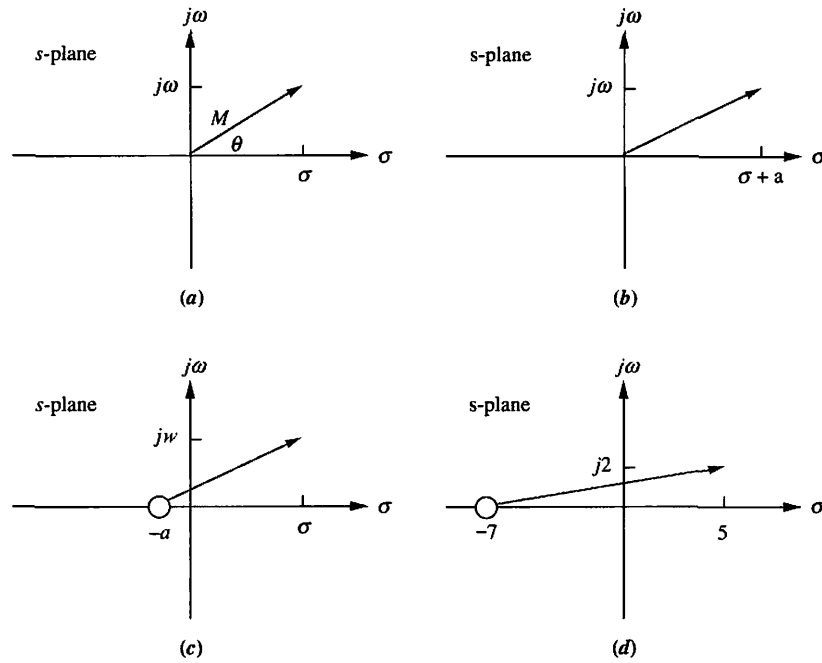


FIGURE 8.2 Vector representation of complex numbers: **a.** $s = \sigma + j\omega$; **b.** $(s + a)$; **c.** alternate representation of $(s + a)$; **d.** $(s + 7)|_{s=5+j2}$

Now let us apply the concepts to a complicated function. Assume a function

$$F(s) = \frac{\prod_{i=1}^m (s + z_i)}{\prod_{j=1}^n (s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}} \quad (8.4)$$

where the symbol \prod means “product,” m = number of zeros, and n = number of poles. Each factor in the numerator and each factor in the denominator is a complex number that can be represented as a vector. The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point, s . Since each complex factor can be thought of as a vector, the magnitude, M , of $F(s)$ at any point, s , is

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^m |(s + z_i)|}{\prod_{j=1}^n |(s + p_j)|} \quad (8.5)$$

where a zero length, $|(s + z_i)|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point s , and a pole length, $|(s + p_j)|$, is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s . The angle, θ , of $F(s)$ at any point, s , is

$$\begin{aligned} \theta &= \sum \text{zero angles} - \sum \text{pole angles} \\ &= \sum_{i=1}^m \angle(s + z_i) - \sum_{j=1}^n \angle(s + p_j) \end{aligned} \quad (8.6)$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$ to the point s , and a pole angle is the

angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_i$ to the point s .

As a demonstration of the above concept, consider the following example.

Example 8.1

Evaluation of a Complex Function via Vectors

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)} \quad (8.7)$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION: The problem is graphically depicted in Figure 8.3, where each vector, $(s + \alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20} \angle 116.6^\circ \quad (8.8)$$

The vector originating at the pole at the origin is

$$5 \angle 126.9^\circ \quad (8.9)$$

The vector originating at the pole at -2 is

$$\sqrt{17} \angle 104.0^\circ \quad (8.10)$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M \angle \theta = \frac{\sqrt{20}}{5\sqrt{17}} \angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217 \angle -114.3^\circ \quad (8.11)$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

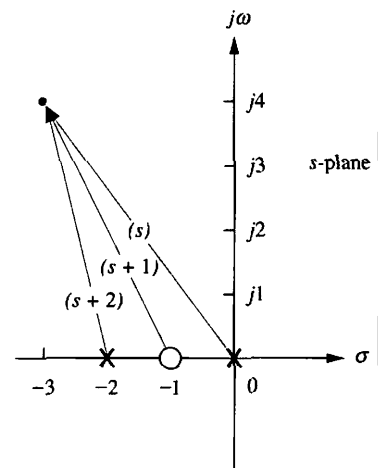


FIGURE 8.3 Vector representation of Eq. (8.7)

Skill-Assessment Exercise 8.1

PROBLEM: Given

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

find $F(s)$ at the point $s = -7 + j9$ the following ways:

- Directly substituting the point into $F(s)$
- Calculating the result using vectors

ANSWER:

$$-0.0339 - j0.0899 = 0.096 \angle -110.7^\circ$$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.1

Use the following MATLAB statements to solve the problem given in Skill-Assessment Exercise 8.1.

```
s=-7+9j;
G=(s+2)*(s+4)/...
(s*(s+3)*(s+6));
Theta=(180/pi)*...
angle(G)
M=abs(G)
```

We are now ready to begin our discussion of the root locus.

8.2 Defining the Root Locus

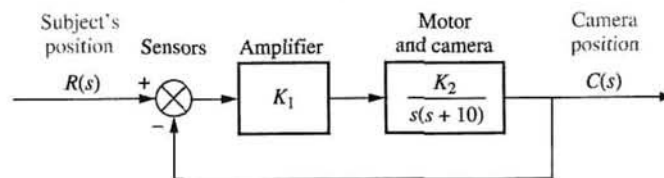
A security camera system similar to that shown in Figure 8.4(a) can automatically follow a subject. The tracking system monitors pixel changes and positions the camera to center the changes.

The root locus technique can be used to analyze and design the effect of loop gain upon the system's transient response and stability. Assume the block diagram representation of a tracking system as shown in Figure 8.4(b), where the closed-loop poles of the system change location as the gain, K , is varied. Table 8.1, which was formed by applying the quadratic formula to the denominator of the transfer function in Figure 8.4(c), shows the variation of pole location for different values of gain, K . The data of Table 8.1 is graphically displayed in Figure 8.5(a), which shows each pole and its gain.

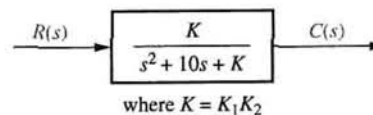
As the gain, K , increases in Table 8.1 and Figure 8.5(a), the closed-loop pole, which is at -10 for $K = 0$, moves toward the right, and the closed-loop pole, which is at 0 for $K = 0$, moves toward the left. They meet at -5 , break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward. We cannot tell which pole moves up or which moves down. In Figure 8.5(b), the individual closed-loop pole locations are removed and their paths are represented with solid lines. It is this *representation of the paths of the*



(a)



(b)

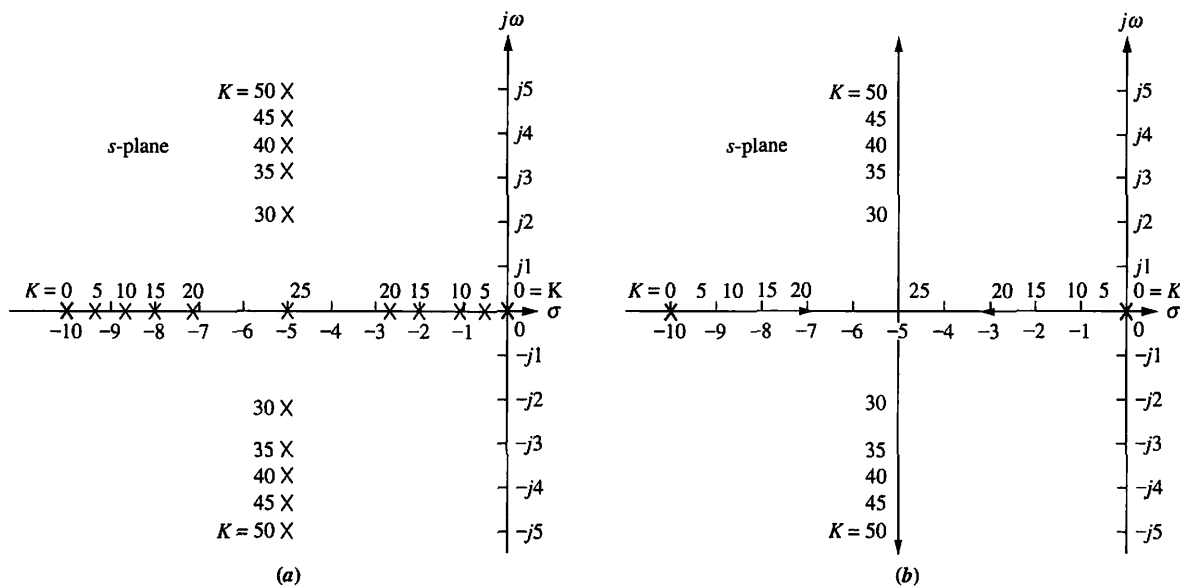


(c)

FIGURE 8.4 a. Security cameras with auto tracking can be used to follow moving objects automatically; b. block diagram; c. closed-loop transfer function

TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

**FIGURE 8.5** a. Pole plot from Table 8.1; b. root locus

closed-loop poles as the gain is varied that we call a *root locus*. For most of our work, the discussion will be limited to positive gain, or $K \geq 0$.

The root locus shows the changes in the transient response as the gain, K , varies. First of all, the poles are real for gains less than 25. Thus, the system is overdamped. At a gain of 25, the poles are real and multiple and hence critically damped. For gains above 25, the system is underdamped. Even though these preceding conclusions were available through the analytical techniques covered in Chapter 4, the following conclusions are graphically demonstrated by the root locus.

Directing our attention to the underdamped portion of the root locus, we see that regardless of the value of gain, the real parts of the complex poles are always the same.

Since the settling time is inversely proportional to the real part of the complex poles for this second-order system, the conclusion is that regardless of the value of gain, the settling time for the system remains the same under all conditions of underdamped responses.

Also, as we increase the gain, the damping ratio diminishes, and the percent overshoot increases. The damped frequency of oscillation, which is equal to the imaginary part of the pole, also increases with an increase in gain, resulting in a reduction of the peak time. Finally, since the root locus never crosses over into the right half-plane, the system is always stable, regardless of the value of gain, and can never break into a sinusoidal oscillation.

These conclusions for such a simple system may appear to be trivial. What we are about to see is that the analysis is applicable to systems of order higher than 2. For these systems, it is difficult to tie transient response characteristics to the pole location. The root locus will allow us to make that association and will become an important technique in the analysis and design of higher-order systems.

8.3 Properties of the Root Locus

In Section 8.2, we arrived at the root locus by factoring the second-order polynomial in the denominator of the transfer function. Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain.

We are about to examine the properties of the root locus. From these properties we will be able to make a *rapid sketch* of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.

The properties of the root locus can be derived from the general control system of Figure 8.1(a). The closed-loop transfer function for the system is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} \quad (8.12)$$

From Eq. (8.12), a pole, s , exists when the characteristic polynomial in the denominator becomes zero, or

$$KG(s)H(s) = -1 = 1 \angle (2k + 1)180^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \quad (8.13)$$

where -1 is represented in polar form as $1 \angle (2k + 1)180^\circ$. Alternately, a value of s is a closed-loop pole if

$$|KG(s)H(s)| = 1 \quad (8.14)$$

and

$$\angle KG(s)H(s) = (2k + 1)180^\circ \quad (8.15)$$

Equation (8.13) implies that if a value of s is substituted into the function $KG(s)H(s)$, a complex number results. If the angle of the complex number is an odd multiple of 180° , that value of s is a system pole for some particular value of K . What

value of K ? Since the angle criterion of Eq. (8.15) is satisfied, all that remains is to satisfy the magnitude criterion, Eq. (8.14). Thus,

$$K = \frac{1}{|G(s)||H(s)|} \quad (8.16)$$

We have just found that a pole of the closed-loop system causes the angle of $KG(s)H(s)$, or simply $G(s)H(s)$ since K is a scalar, to be an odd multiple of 180° . Furthermore, the magnitude of $KG(s)H(s)$ must be unity, implying that the value of K is the reciprocal of the magnitude of $G(s)H(s)$ when the pole value is substituted for s .

Let us demonstrate this relationship for the second-order system of Figure 8.4. The fact that closed-loop poles exist at -9.47 and -0.53 when the gain is 5 has already been established in Table 8.1. For this system,

$$KG(s)H(s) = \frac{K}{s(s+10)} \quad (8.17)$$

Substituting the pole at -9.47 for s and 5 for K yields $KG(s)H(s) = -1$. The student can repeat the exercise for other points in Table 8.1 and show that each case yields $KG(s)H(s) = -1$.

It is helpful to visualize graphically the meaning of Eq. (8.15). Let us apply the complex number concepts reviewed in Section 8.1 to the root locus of the system shown in Figure 8.6. For this system the open-loop transfer function is

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \quad (8.18)$$

The closed-loop transfer function, $T(s)$, is

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \quad (8.19)$$

If point s is a closed-loop system pole for some value of gain, K , then s must satisfy Eqs. (8.14) and (8.15).

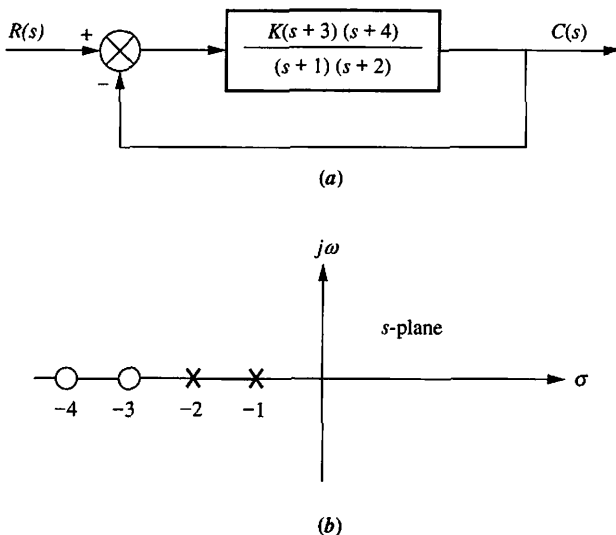


FIGURE 8.6 a. Example system; b. pole-zero plot of $G(s)$

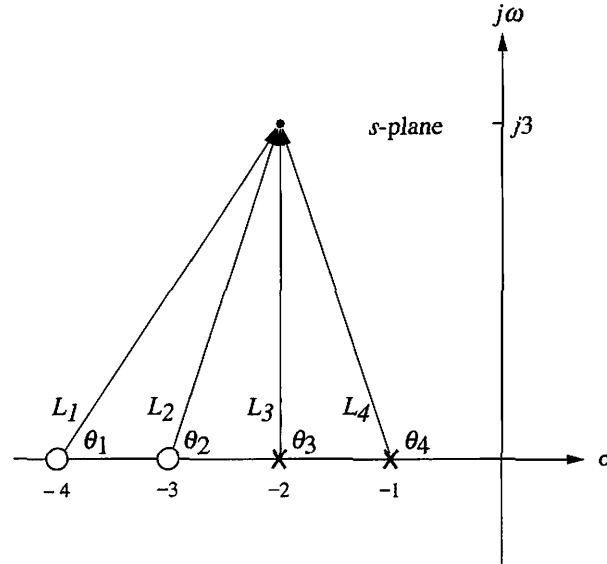


FIGURE 8.7 Vector representation of $G(s)$ from Figure 8.6(a) at $-2 + j3$

Consider the point $-2 + j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180° . From Figure 8.7,

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ \quad (8.20)$$

Therefore, $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain.

If these calculations are repeated for the point $-2 + j(\sqrt{2}/2)$, the angles do add up to 180° . That is, $-2 + j(\sqrt{2}/2)$ is a point on the root locus for some value of gain. We now proceed to evaluate that value of gain.

From Eqs. (8.5) and (8.16),

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}} \quad (8.21)$$

Looking at Figure 8.7 with the point $-2 + j3$ replaced by $-2 + j(\sqrt{2}/2)$, the gain, K , is calculated as

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2} (1.22)}{(2.12)(1.22)} = 0.33 \quad (8.22)$$

Thus, the point $-2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.

We summarize what we have found as follows: Given the poles and zeros of the open-loop transfer function, $KG(s)H(s)$, a point in the s -plane is on the root locus for a particular value of gain, K , if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the s -plane, add up to $(2k + 1)180^\circ$. Furthermore, gain K at that point for which the angles add up to $(2k + 1)180^\circ$ is found by dividing the product of the pole lengths by the product of the zero lengths.

Skill-Assessment Exercise 8.2

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 + 4s + 13)}$$

do the following:

- Calculate the angle of $G(s)$ at the point $(-3 + j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point.
- Determine if the point specified in **a** is on the root locus.
- If the point specified in **a** is on the root locus, find the gain, K , using the lengths of the vectors.

ANSWERS:

- Sum of angles = 180°
- Point is on the root locus
- $K = 10$

The complete solution is at www.wiley.com/college/nise.

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Control Solutions

TryIt 8.2

Use MATLAB and the following statements to solve Skill-Assessment Exercise 8.2.

```
s=-3+0j;
G=(s+2)/(s^2+4*s+13);
Theta=(180/pi)*...
angle(G)
M=abs(G);
K=1/M
```

8.4 Sketching the Root Locus

It appears from our previous discussion that the root locus can be obtained by sweeping through every point in the s -plane to locate those points for which the angles, as previously described, add up to an odd multiple of 180° . Although this task is tedious without the aid of a computer, the concept can be used to develop rules that can be used to *sketch* the root locus without the effort required to *plot* the locus. Once a sketch is obtained, it is possible to accurately plot just those points that are of interest to us for a particular problem.

The following five rules allow us to sketch the root locus using minimal calculations. The rules yield a sketch that gives intuitive insight into the behavior of a control system. In the next section, we refine the sketch by finding actual points or angles on the root locus. These refinements, however, require some calculations or the use of computer programs, such as MATLAB.

- Number of branches.** Each closed-loop pole moves as the gain is varied. If we define a *branch* as the path that one pole traverses, then there will be one branch for each closed-loop pole. Our first rule, then, defines the number of branches of the root locus:

The number of branches of the root locus equals the number of closed-loop poles.

As an example, look at Figure 8.5(b), where the two branches are shown. One originates at the origin, the other at -10 .

- Symmetry.** If complex closed-loop poles do not exist in conjugate pairs, the resulting polynomial, formed by multiplying the factors containing the closed-loop poles,

would have complex coefficients. Physically realizable systems cannot have complex coefficients in their transfer functions. Thus, we conclude:

The root locus is symmetrical about the real axis.

An example of symmetry about the real axis is shown in Figure 8.5(b).

- 3. Real-axis segments.** Let us make use of the angle property, Eq. (8.15), of the points on the root locus to determine where the real-axis segments of the root locus exist. Figure 8.8 shows the poles and zeros of a general open-loop system. If an attempt is made to calculate the angular contribution of the poles and zeros at each point, P_1 , P_2 , P_3 , and P_4 , along the real axis, we observe the following: (1) At each point the angular contribution of a pair of open-loop complex poles or zeros is zero, and (2) the contribution of the open-loop poles and open-loop zeros to the left of the respective point is zero. The conclusion is that the only contribution to the angle at any of the points comes from the open-loop, real-axis poles and zeros that exist to the right of the respective point. If we calculate the angle at each point using only the open-loop, real-axis poles and zeros to the right of each point, we note the following: (1) The angles on the real axis alternate between 0° and 180° , and (2) the angle

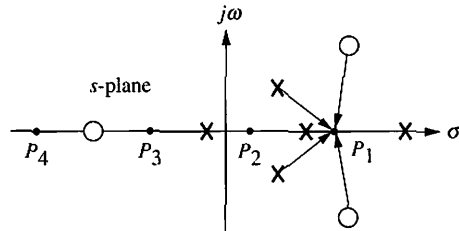


FIGURE 8.8 Poles and zeros of a general open-loop system with test points, P_i , on the real axis

is 180° for regions of the real axis that exist to the left of an odd number of poles and/or zeros. The following rule summarizes the findings:

On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Examine Figure 8.6(b). According to the rule just developed, the real-axis segments of the root locus are between -1 and -2 and between -3 and -4 as shown in Figure 8.9.

- 4. Starting and ending points.** Where does the root locus begin (zero gain) and end (infinite gain)? The answer to this question will enable us to expand the sketch of the root locus beyond the real-axis segments. Consider the closed-loop transfer function, $T(s)$, described by Eq. (8.3). $T(s)$ can now be evaluated for both large and small gains, K . As K approaches zero (small gain),

$$T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + \epsilon} \quad (8.23)$$

From Eq. (8.23) we see that the closed-loop system poles at small gains approach the combined poles of $G(s)$ and $H(s)$. We conclude that the root locus begins at the poles of $G(s)H(s)$, the open-loop transfer function.

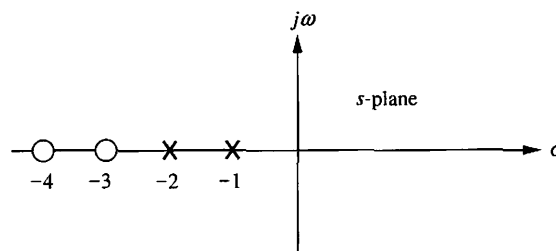


FIGURE 8.9 Real-axis segments of the root locus for the system of Figure 8.6

At high gains, where K is approaching infinity,

$$T(s) \approx \frac{KN_G(s)D_H(s)}{\epsilon + KN_G(s)N_H(s)} \quad (8.24)$$

From Eq. (8.24) we see that the closed-loop system poles at large gains approach the combined zeros of $G(s)$ and $H(s)$. Now we conclude that the root locus ends at the zeros of $G(s)H(s)$, the open-loop transfer function.

Summarizing what we have found:

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Remember that these poles and zeros are the open-loop poles and zeros.

In order to demonstrate this rule, look at the system in Figure 8.6(a), whose real-axis segments have been sketched in Figure 8.9. Using the rule just derived, we find that the root locus begins at the poles at -1 and -2 and ends at the zeros at -3 and -4 (see Figure 8.10). Thus, the poles start out at -1 and -2 and move through the real-axis space between the two poles. They meet somewhere between the two poles and break out into the complex plane, moving as complex conjugates. The poles return to the real axis somewhere between the zeros at -3 and -4 , where their path is completed as they move away from each other, and end up, respectively, at the two zeros of the open-loop system at -3 and -4 .

5. **Behavior at infinity.** Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \quad (8.25)$$

There are three finite poles, at $s = 0, -1$, and -2 , and no finite zeros.

A function can also have *infinite* poles and zeros. If the function approaches infinity as s approaches infinity, then the function has a pole at infinity. If the function approaches zero as s approaches infinity, then the function has a zero at infinity. For example, the function $G(s) = s$ has a pole at infinity, since $G(s)$ approaches infinity as s approaches infinity. On the other hand, $G(s) = 1/s$ has a zero at infinity, since $G(s)$ approaches zero as s approaches infinity.

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros. In this example,

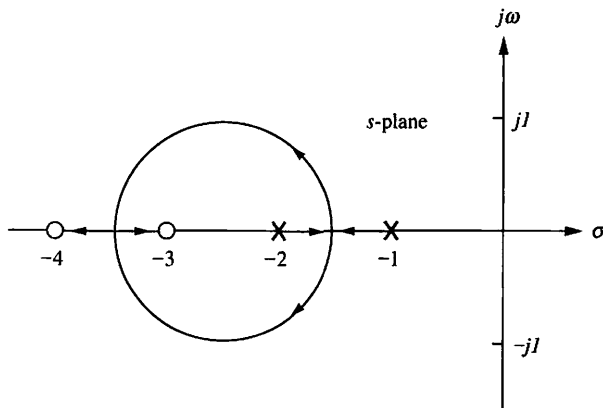


FIGURE 8.10 Complete root locus for the system of Figure 8.6

Eq. (8.25) contains three finite poles and three infinite zeros. To illustrate, let s approach infinity. The open-loop transfer function becomes

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s} \quad (8.26)$$

Each s in the denominator causes the open-loop function, $KG(s)H(s)$, to become zero as that s approaches infinity. Hence, Eq. (8.26) has three zeros at infinity.

Thus, for Eq. (8.25), the root locus begins at the finite poles of $KG(s)H(s)$ and ends at the infinite zeros. The question remains: Where are the infinite zeros? We must know where these zeros are in order to show the locus moving from the three finite poles to the three infinite zeros. Rule 5 helps us locate these zeros at infinity. Rule 5 also helps us locate poles at infinity for functions containing more finite zeros than finite poles.¹

We now state Rule 5, which will tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity. The derivation can be found in Appendix M.1 at www.wiley.com/college/nise.

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.27)$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.28)$$

where $k = 0, \pm 1, \pm 2, \pm 3$ and the angle is given in radians with respect to the positive extension of the real axis.

Notice that the running index, k , in Eq. (8.28) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity. Let us demonstrate the concepts with an example.

Example 8.2

Sketching a Root Locus with Asymptotes

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

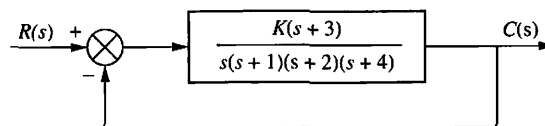


FIGURE 8.11 System for Example 8.2.

¹ Physical systems, however, have more finite poles than finite zeros, since the implied differentiation yields infinite output for discontinuous input functions, such as step inputs.

SOLUTION: Let us begin by calculating the asymptotes. Using Eq. (8.27), the real-axis intercept is evaluated as

$$\sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3} \quad (8.29)$$

The angles of the lines that intersect at $-4/3$, given by Eq. (8.28), are

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.30a)$$

$$= \pi/3 \quad \text{for } k = 0 \quad (8.30b)$$

$$= \pi \quad \text{for } k = 1 \quad (8.30c)$$

$$= 5\pi/3 \quad \text{for } k = 2 \quad (8.30d)$$

If the value for k continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

Rule 4 states that the locus begins at the open-loop poles and ends at the open-loop zeros. For the example there are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.

Figure 8.12 shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real-axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For the example there is only one open-loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

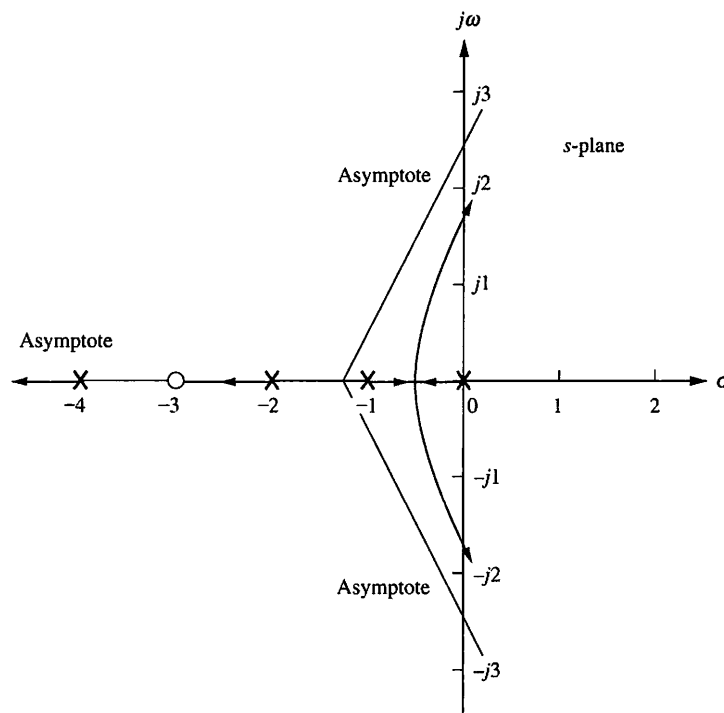


FIGURE 8.12 Root locus and asymptotes for the system of Figure 8.11

Skill-Assessment Exercise 8.3

PROBLEM: Sketch the root locus and its asymptotes for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.5 Refining the Sketch

The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain. Points on the real axis where the root locus enters or leaves the complex plane—real-axis breakaway and break-in points—and the $j\omega$ -axis crossings are candidates. We can also derive a better picture of the root locus by finding the angles of departure and arrival from complex poles and zeros, respectively.

In this section, we discuss the calculations required to obtain specific points on the root locus. Some of these calculations can be made using the basic root locus relationship that the sum of the zero angles minus the sum of the pole angles equals an odd multiple of 180° , and the gain at a point on the root locus is found as the ratio of (1) the product of pole lengths drawn to that point to (2) the product of zero lengths drawn to that point. We have yet to address how to implement this task. In the past, an inexpensive tool called a SpiruleTM added the angles together rapidly and then quickly multiplied and divided the lengths to obtain the gain. Today we can rely on hand-held or programmable calculators as well as personal computers.

Students pursuing MATLAB will learn how to apply it to the root locus at the end of Section 8.6. Other alternatives are discussed in Appendix H.2 at www.wiley.com/college/nise. The discussion can be adapted to programmable hand-held calculators. All readers are encouraged to select a computational aid at this point. Root locus calculations can be labor intensive if hand calculations are used.

We now discuss how to refine our root locus sketch by calculating real-axis breakaway and break-in points, $j\omega$ -axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We conclude by showing how to find accurately any point on the root locus and calculate the gain.

Real-Axis Breakaway and Break-In Points

Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. We illustrate this in Figure 8.13. This locus is sketched using the first four rules: (1) number of branches, (2) symmetry, (3) real-axis segments, and (4) starting and ending points. The figure shows a root locus leaving the real axis between -1 and -2 and returning to the real axis between $+3$ and $+5$. The point where the locus leaves the real axis, $-\sigma_1$, is called the *breakaway point*, and the point where the locus returns to the real axis, σ_2 , is called the *break-in point*.

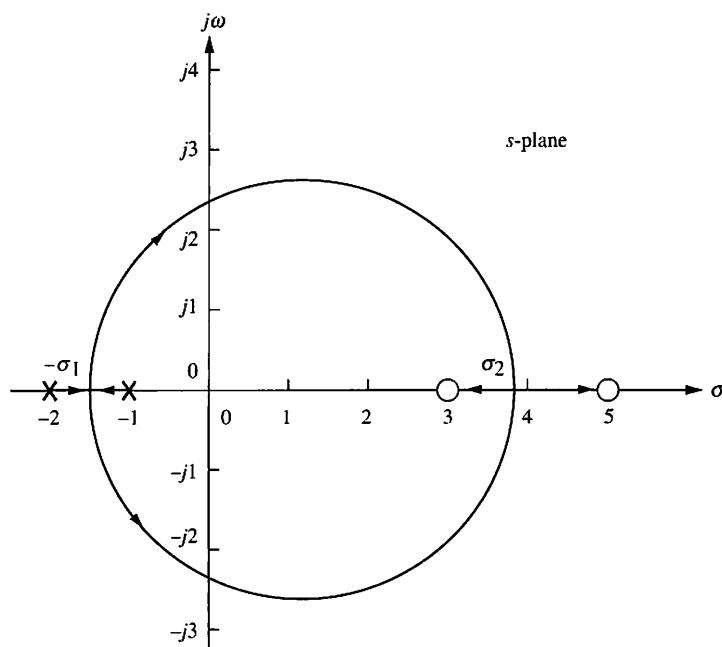


FIGURE 8.13 Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

At the breakaway or break-in point, the branches of the root locus form an angle of $180^\circ/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis (*Kuo, 1991*). Thus, for the two poles shown in Figure 8.13, the branches at the breakaway point form 90° angles with the real axis.

We now show how to find the breakaway and break-in points. As the two closed-loop poles, which are at -1 and -2 when $K = 0$, move toward each other, the gain increases from a value of zero. We conclude that the gain must be maximum along the real axis at the point where the breakaway occurs, somewhere between -1 and -2 . Naturally, the gain increases above this value as the poles move into the complex plane. We conclude that the breakaway point occurs at a point of maximum gain on the real axis between the open-loop poles.

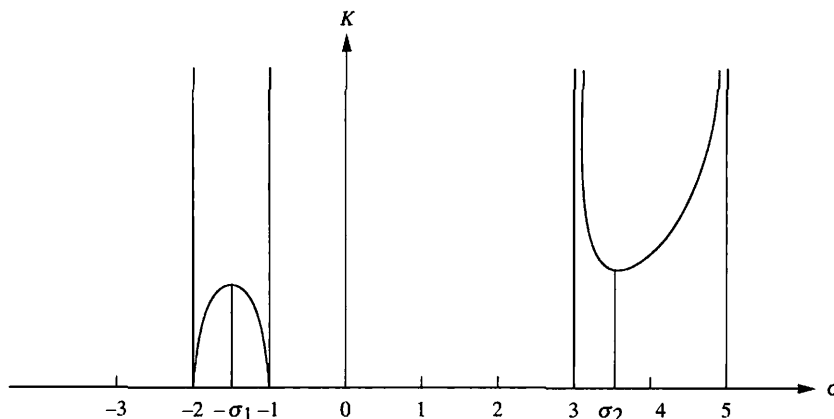
Now let us turn our attention to the break-in point somewhere between $+3$ and $+5$ on the real axis. When the closed-loop complex pair returns to the real axis, the gain will continue to increase to infinity as the closed-loop poles move toward the open-loop zeros. It must be true, then, that the gain at the break-in point is the minimum gain found along the real axis between the two zeros.

The sketch in Figure 8.14 shows the variation of real-axis gain. The breakaway point is found at the maximum gain between -1 and -2 , and the break-in point is found at the minimum gain between $+3$ and $+5$.

There are three methods for finding the points at which the root locus breaks away from and breaks into the real axis. The first method is to maximize and minimize the gain, K , using differential calculus. For all points on the root locus, Eq. (8.13) yields

$$K = -\frac{1}{G(s)H(s)} \quad (8.31)$$

FIGURE 8.14 Variation of gain along the real axis for the root locus of Figure 8.13



For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence, along the real axis Eq. (8.31) becomes

$$K = -\frac{1}{G(\sigma)H(\sigma)} \quad (8.32)$$

This equation then represents a curve of K versus σ similar to that shown in Figure 8.14. Hence, if we differentiate Eq. (8.32) with respect to σ and set the derivative equal to zero, we can find the points of maximum and minimum gain and hence the breakaway and break-in points. Let us demonstrate.

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2 - 8s + 15)}{(s^2 + 3s + 2)} \quad (8.33)$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2 - 8\sigma + 15)}{(\sigma^2 + 3\sigma + 2)} = -1 \quad (8.34)$$

Solving for K , we find

$$K = \frac{-(\sigma^2 + 3\sigma + 2)}{(\sigma^2 - 8\sigma + 15)} \quad (8.35)$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2 - 26\sigma - 61)}{(\sigma^2 - 8\sigma + 15)^2} = 0 \quad (8.36)$$

Solving for σ , we find $\sigma = -1.45$ and 3.82 , which are the breakaway and break-in points.

The second method is a variation on the differential calculus method. Called the *transition method*, it eliminates the step of differentiation (Franklin, 1991). This method, derived in Appendix M.2 at www.wiley.com/college/nise, is now stated:

Breakaway and break-in points satisfy the relationship

$$\sum_{i=1}^m \frac{1}{\sigma + z_i} = \sum_{i=1}^n \frac{1}{\sigma + p_i} \quad (8.37)$$

where z_i and p_i are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

Solving Eq. (8.37) for σ , the real-axis values that minimize or maximize K , yields the breakaway and break-in points without differentiating. Let us look at an example.

Example 8.4

Breakaway and Break-in Points Without Differentiation

PROBLEM: Repeat Example 8.3 without differentiating.

SOLUTION: Using Eq. (8.37),

$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2} \quad (8.38)$$

Simplifying,

$$11\sigma^2 - 26\sigma - 61 = 0 \quad (8.39)$$

Hence, $\sigma = -1.45$ and 3.82 , which agrees with Example 8.3.

For the third method, the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise can be used to find the breakaway and break-in points. Simply use the program to search for the point of maximum gain between -1 and -2 and to search for the point of minimum gain between $+3$ and $+5$. Table 8.2 shows the results of the search. The locus leaves the axis at -1.45 , the point of maximum gain between -1 and -2 , and reenters the real axis at $+3.8$, the point of minimum gain between $+3$ and $+5$. These results are the same as those obtained using the first two methods. MATLAB also has the capability of finding breakaway and break-in points.

The $j\omega$ -Axis Crossings

We now further refine the root locus by finding the imaginary-axis crossings. The importance of the $j\omega$ -axis crossings should be readily apparent. Looking at Figure 8.12, we see that the system's poles are in the left half-plane up to a particular value of gain. Above this value of gain, two of the closed-loop system's poles move into the right half-plane, signifying that the system is unstable. The $j\omega$ -axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation, while the gain at the $j\omega$ -axis crossing yields, for this example, the maximum positive gain for system stability. We should note here that other examples

TABLE 8.2 Data for breakaway and break-in points for the root locus of Figure 8.13

Real-axis value	Gain		Comment
-1.41	0.008557		
-1.42	0.008585		
-1.43	0.008605		
-1.44	0.008617		
-1.45	0.008623	←	Max. gain: breakaway
-1.46	0.008622		
3.3	44.686		
3.4	37.125		
3.5	33.000		
3.6	30.667		
3.7	29.440		
3.8	29.000	←	Min. gain: break-in
3.9	29.202		

illustrate instability at small values of gain and stability at large values of gain. These systems have a root locus starting in the right-half-plane (unstable at small values of gain) and ending in the left-half-plane (stable for high values of gain).

To find the $j\omega$ -axis crossing, we can use the Routh-Hurwitz criterion, covered in Chapter 6, as follows: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

Example 8.5

Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, K , for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function for the system of Figure 8.11 is

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K} \quad (8.40)$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3.

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0 \quad (8.41)$$

From this equation K is evaluated as

$$K = 9.65 \quad (8.42)$$

TABLE 8.3 Routh table for Eq. (8.40)

s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\frac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Forming the even polynomial by using the s^2 row with $K = 9.65$, we obtain

$$(90 - K)s^2 + 21K = 80.35s^2 + 202.7 = 0 \quad (8.43)$$

and s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$ -axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \leq K < 9.65$.

Another method for finding the $j\omega$ -axis crossing (or any point on the root locus, for that matter) uses the fact that at the $j\omega$ -axis crossing, the sum of angles from the finite open-loop poles and zeros must add to $(2k + 1)180^\circ$. Thus, we can search $j\omega$ -axis until we find the point that meets this angle condition. A computer program, such as the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise or MATLAB, can be used for this purpose. Subsequent examples in this chapter use this method to determine the $j\omega$ -axis crossing.

Angles of Departure and Arrival

In this subsection, we further refine our sketch of the root locus by finding angles of departure and arrival from complex poles and zeros. Consider Figure 8.15, which shows the open-loop poles and zeros, some of which are complex. The root locus starts at the open-loop poles and ends at the open-loop zeros. In order to sketch the root locus more accurately, we want to calculate the root locus departure angle from the complex poles and the arrival angle to the complex zeros.

If we assume a point on the root locus ϵ close to a complex pole, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180° . Except for the pole that is ϵ close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the pole that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the pole that is ϵ close. We can solve for this unknown angle, which is also the angle of departure from this complex pole. Hence, from Figure 8.15(a),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ \quad (8.44a)$$

or

$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k + 1)180^\circ \quad (8.44b)$$

If we assume a point on the root locus ϵ close to a complex zero, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180° . Except for the zero that is ϵ close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the zero that is near the point. Thus,

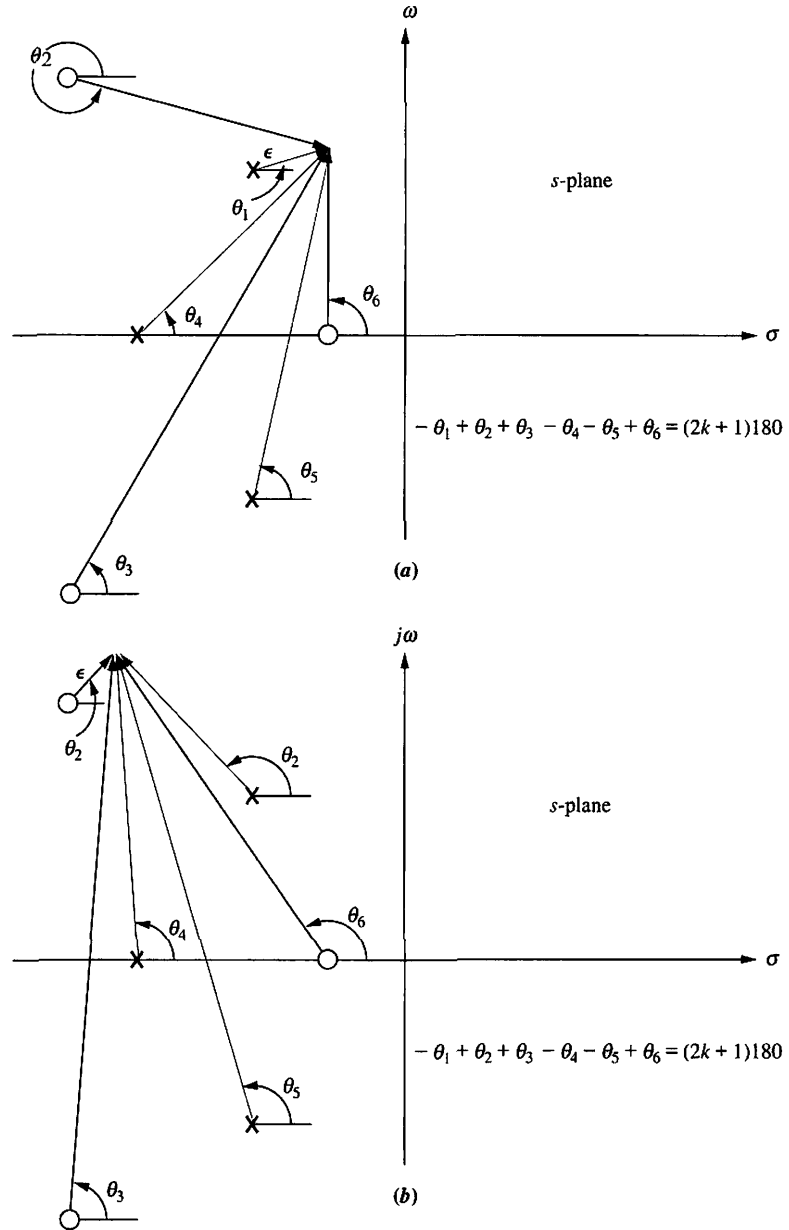


FIGURE 8.15 Open-loop poles and zeros and calculation of **a.** angle of departure; **b.** angle of arrival

the only unknown angle in the sum is the angle drawn from the zero that is ϵ close. We can solve for this unknown angle, which is also the angle of arrival to this complex zero. Hence, from Figure 8.15(b),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ \quad (8.45a)$$

or

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k + 1)180^\circ \quad (8.45b)$$

Let us look at an example.

Example 8.6

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

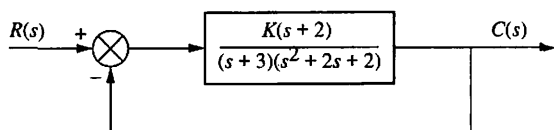


FIGURE 8.16 Unity feedback system with complex poles

SOLUTION: Using the poles and zeros of $G(s) = (s+2)/[(s+3)(s^2+2s+2)]$ as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1 + j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ \quad (8.46)$$

from which $\theta = -251.6^\circ = 108.4^\circ$. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

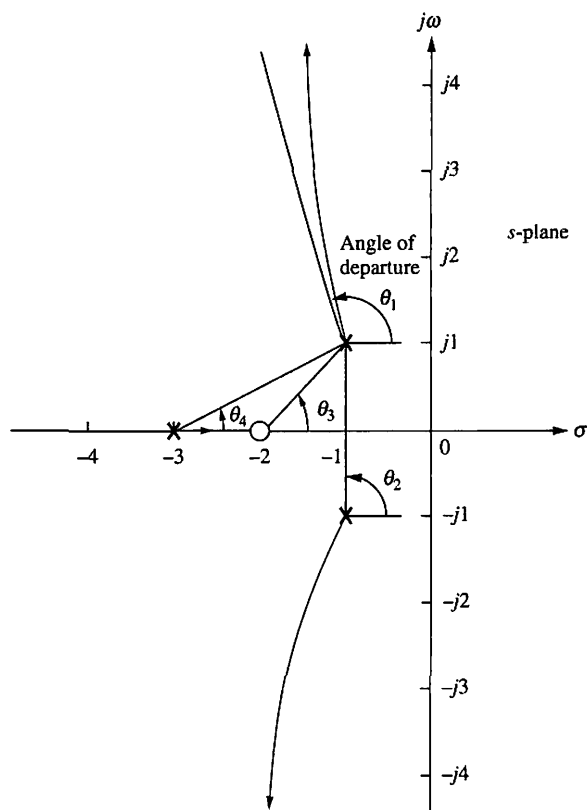


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

Plotting and Calibrating the Root Locus

Once we sketch the root locus using the rules from Section 8.4, we may want to accurately locate points on the root locus as well as find their associated gain. For example, we might want to know the exact coordinates of the root locus as it crosses the radial line representing 20% overshoot. Further, we also may want the value of gain at that point.

Consider the root locus shown in Figure 8.12. Let us assume we want to find the exact point at which the locus crosses the 0.45 damping ratio line and the gain at that point. Figure 8.18 shows the system's open-loop poles and zeros along with the $\zeta = 0.45$ line. If a few test points along the $\zeta = 0.45$ line are selected, we can evaluate their angular sum and locate that point where the angles add up to an odd multiple of 180° . It is at this point that the root locus exists. Equation (8.20) can then be used to evaluate the gain, K , at that point.

Selecting the point at radius 2 ($r = 2$) on the $\zeta = 0.45$ line, we add the angles of the zeros and subtract the angles of the poles, obtaining

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -251.5^\circ \quad (8.47)$$

Since the sum is not equal to an odd multiple of 180° , the point at radius = 2 is not on the root locus. Proceeding similarly for the points at radius = 1.5, 1, 0.747, and 0.5, we obtain the table shown in Figure 8.18. This table lists the points, giving their radius, r , and the sum of angles indicated by the symbol \angle . From the table we see that the point at radius 0.747 is on the root locus, since the angles add up to -180° . Using Eq. (8.21), the gain, K , at this point is

$$K = \frac{|A||C||D||E|}{|B|} = 1.71 \quad (8.48)$$

In summary, we search a given line for the point yielding a summation of angles (zero angles–pole angles) equal to an odd multiple of 180° . We conclude that the point is on the root locus. The gain at that point is then found by multiplying the pole lengths drawn to that point and dividing by the product of the zero lengths drawn to that point. A computer program, such as that discussed in Appendix H.2 at www.wiley.com/college/nise or MATLAB, can be used.

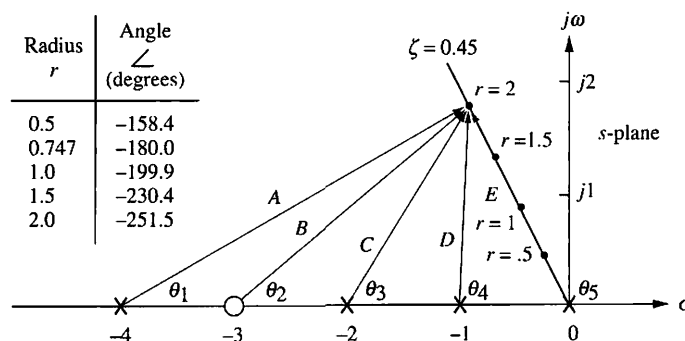


FIGURE 8.18 Finding and calibrating exact points on the root locus of Figure 8.12

Skill-Assessment Exercise 8.4

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

do the following:

- Sketch the root locus.
- Find the imaginary-axis crossing.
- Find the gain, K , at the $j\omega$ -axis crossing.
- Find the break-in point.
- Find the angle of departure from the complex poles.

ANSWERS:

- See solution at www.wiley.com/college/nise.
- $s = \pm j\sqrt{21}$
- $K = 4$
- Break-in point = -7
- Angle of departure = -233.1°

The complete solution is at www.wiley.com/college/nise.

8.6 An Example

We now review the rules for sketching and finding points on the root locus, as well as present an example. The root locus is the path of the closed-loop poles of a system as a parameter of the system is varied. Each point on the root locus satisfies the angle condition, $\angle G(s)H(s) = (2k+1)180^\circ$. Using this relationship, rules for sketching and finding points on the root locus were developed and are now summarized:

Basic Rules for Sketching the Root Locus

Number of branches The number of branches of the root locus equals the number of closed-loop poles.

Symmetry The root locus is symmetrical about the real axis.

Real-axis segments On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Starting and ending points The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Behavior at infinity The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes are given by

the real-axis intercept and angle in radians as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.49)$$

$$\theta_a = \frac{(2k + 1)\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.50)$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$

Additional Rules for Refining the Sketch

Real-axis breakaway and break-in points The root locus breaks away from the real axis at a point where the gain is maximum and breaks into the real axis at a point where the gain is minimum.

Calculation of $j\omega$ -axis crossings The root locus crosses the $j\omega$ -axis at the point where $\angle G(s)H(s) = (2k + 1)180^\circ$. Routh-Hurwitz or a search of the $j\omega$ -axis for $(2k + 1)180^\circ$ can be used to find the $j\omega$ -axis crossing.

Angles of departure and arrival The root locus departs from complex, open-loop poles and arrives at complex, open-loop zeros at angles that can be calculated as follows. Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all open-loop poles and zeros to this point. The sum equals $(2k + 1)180^\circ$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Plotting and calibrating the root locus All points on the root locus satisfy the relationship $\angle G(s)H(s) = (2k + 1)180^\circ$. The gain, K , at any point on the root locus is given by

$$K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} - \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}} \quad (8.51)$$

Let us now look at a summary example.

Example 8.7

Sketching a Root Locus and Finding Critical Points

PROBLEM: Sketch the root locus for the system shown in Figure 8.19(a) and find the following:

- The exact point and gain where the locus crosses the 0.45 damping ratio line
- The exact point and gain where the locus crosses the $j\omega$ -axis
- The breakaway point on the real axis
- The range of K within which the system is stable

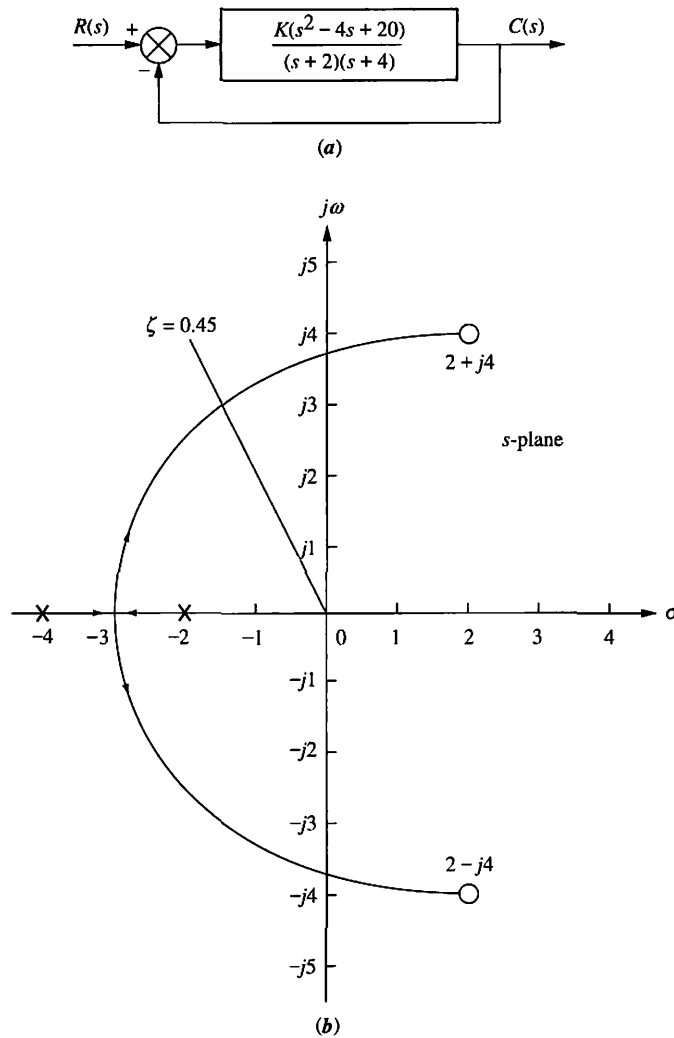


FIGURE 8.19 a. System for Example 8.7; b. root locus sketch.

SOLUTION: The problem solution is shown, in part, in Figure 8.19(b). First sketch the root locus. Using Rule 3, the real-axis segment is found to be between -2 and -4 . Rule 4 tells us that the root locus starts at the open-loop poles and ends at the open-loop zeros. These two rules alone give us the general shape of the root locus.

- a. To find the exact point where the locus crosses the $\zeta = 0.45$ line, we can use the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise to search along the line

$$\theta = 180^\circ - \cos^{-1} 0.45 = 116.7^\circ \quad (8.52)$$

for the point where the angles add up to an odd multiple of 180° . Searching in polar coordinates, we find that the root locus crosses the $\zeta = 0.45$ line at $3.4 \angle 116.7^\circ$ with a gain, K , of 0.417 .

- b. To find the exact point where the locus crosses the $j\omega$ -axis, use the root locus program to search along the line

$$\theta = 90^\circ \quad (8.53)$$

for the point where the angles add up to an odd multiple of 180° . Searching in polar coordinates, we find that the root locus crosses the $j\omega$ -axis at $\pm j3.9$ with a gain of $K = 1.5$.

- c. To find the breakaway point, use the root locus program to search the real axis between -2 and -4 for the point that yields maximum gain. Naturally, all points will have the sum of their angles equal to an odd multiple of 180° . A maximum gain of 0.0248 is found at the point -2.88 . Therefore, the breakaway point is between the open-loop poles on the real axis at -2.88 .
- d. From the answer to **b**, the system is stable for K between 0 and 1.5 .

MATLAB
ML

Students who are using MATLAB should now run `ch8p1` in Appendix B. You will learn how to use MATLAB to plot and title a root locus, overlay constant ζ and ω_n curves, zoom into and zoom out from a root locus, and interact with the root locus to find critical points as well as gains at those points. This exercise solves Example 8.7 using MATLAB.

Skill-Assessment Exercise 8.5

WileyPLUS

WPCS

Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s-2)(s-4)}{(s^2+6s+25)}$$

do the following:

- a. Sketch the root locus.
- b. Find the imaginary-axis crossing.
- c. Find the gain, K , at the $j\omega$ -axis crossing.
- d. Find the break-in point.
- e. Find the point where the locus crosses the 0.5 damping ratio line.
- f. Find the gain at the point where the locus crosses the 0.5 damping ratio line.
- g. Find the range of gain, K , for which the system is stable.

ANSWERS:

- a. See solution at www.wiley.com/college/nise.
- b. $s = \pm j4.06$
- c. $K = 1$
- d. Break-in point = $+2.89$
- e. $s = -2.42 + j4.18$
- f. $K = 0.108$
- g. $K < 1$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.3

Use MATLAB, the Control System Toolbox, and the following statements to plot the root locus for Skill-Assessment Exercise 8.5. Solve the remaining parts of the problem by clicking on the appropriate points on the plotted root locus.

```
numg=poly([2 4]);
deng=[1 6 25];
G=tf(numg,deng)
rlocus(G)
z=0.5
sgrid(z,0)
```

8.7 Transient Response Design via Gain Adjustment

Now that we know how to sketch a root locus, we show how to use it for the design of transient response. In the last section we found that the root locus crossed the 0.45 damping ratio line with a gain of 0.417. Does this mean that the system will respond with 20.5% overshoot, the equivalent to a damping ratio of 0.45? It must be emphasized that the formulas describing percent overshoot, settling time, and peak time were derived only for a system with two closed-loop complex poles and no closed-loop zeros. The effect of additional poles and zeros and the conditions for justifying an approximation of a two-pole system were discussed in Sections 4.7 and 4.8 and apply here to closed-loop systems and their root loci. The conditions justifying a second-order approximation are restated here:

1. Higher-order poles are much farther into the left half of the s -plane than the dominant second-order pair of poles. The response that results from a higher-order pole does not appreciably change the transient response expected from the dominant second-order poles.
2. Closed-loop zeros near the closed-loop second-order pole pair are nearly canceled by the close proximity of higher-order closed-loop poles.
3. Closed-loop zeros not canceled by the close proximity of higher-order closed-loop poles are far removed from the closed-loop second-order pole pair.

The first condition as it applies to the root locus is shown graphically in Figure 8.20(a) and (b). Figure 8.20(b) would yield a much better second-order approximation than Figure 8.20(a), since closed-loop pole p_3 is farther from the dominant, closed-loop second-order pair, p_1 and p_2 .

The second condition is shown graphically in Figure 8.20(c) and (d). Figure 8.20(d) would yield a much better second-order approximation than Figure 8.20(c), since closed-loop pole p_3 is closer to canceling the closed-loop zero.

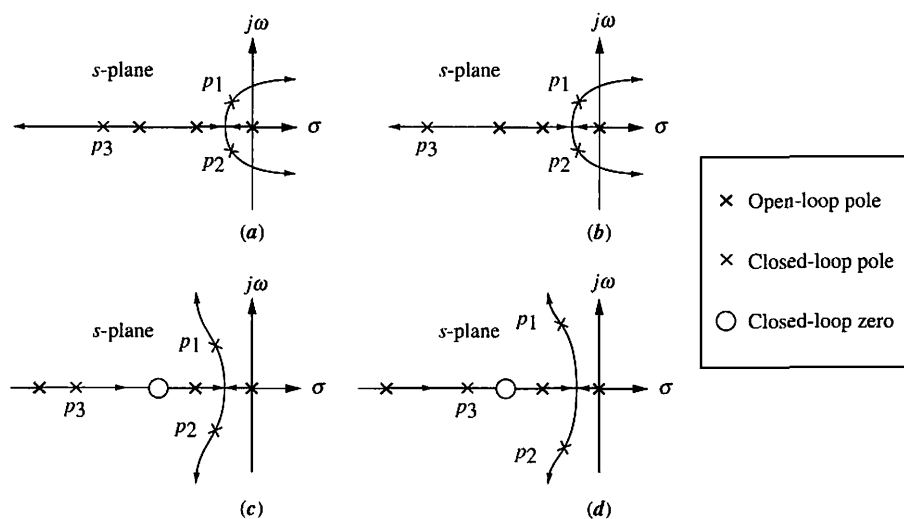


FIGURE 8.20 Making second-order approximations

Summarizing the design procedure for higher-order systems, we arrive at the following:

1. Sketch the root locus for the given system.
2. Assume the system is a second-order system without any zeros and then find the gain to meet the transient response specification.
3. Justify your second-order assumption by finding the location of all higher-order poles and evaluating the fact that they are much farther from the $j\omega$ -axis than the dominant second-order pair. As a rule of thumb, this textbook assumes a factor of five times farther. Also, verify that closed-loop zeros are approximately canceled by higher-order poles. If closed-loop zeros are not canceled by higher-order poles, be sure that the zero is far removed from the dominant second-order pole pair to yield approximately the same response obtained without the finite zero.
4. If the assumptions cannot be justified, your solution will have to be simulated in order to be sure it meets the transient response specification. It is a good idea to simulate all solutions, anyway.

We now look at a design example to show how to make a second-order approximation and then verify whether or not the approximation is valid.

Example 8.8

Third-Order System Gain Design

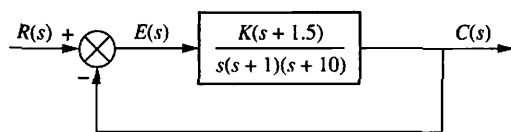


FIGURE 8.21 System for Example 8.8

PROBLEM: Consider the system shown in Figure 8.21. Design the value of gain, K , to yield 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.

SOLUTION: The root locus is shown in Figure 8.22. Notice that this is a third-order system with one zero. Breakaway points on the real

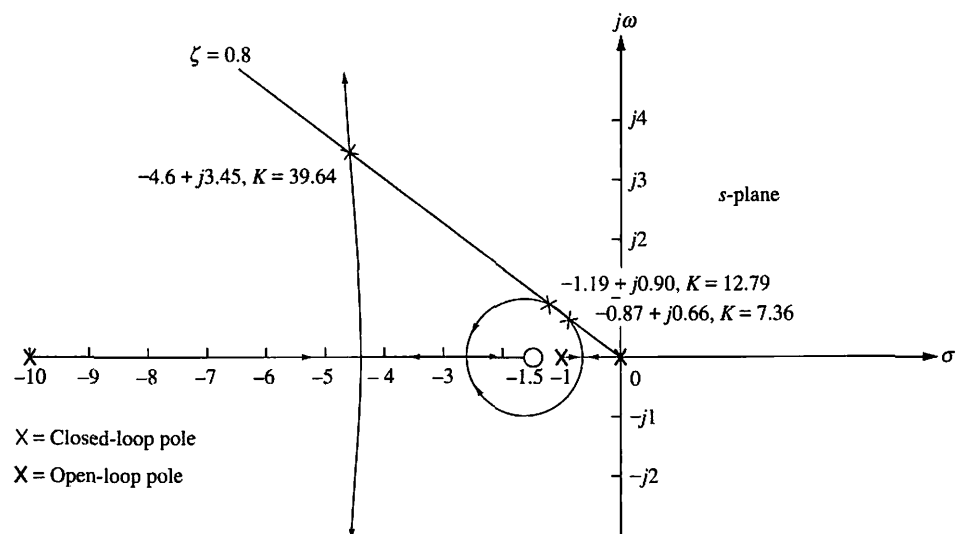


FIGURE 8.22 Root locus for Example 8.8

axis can occur between 0 and -1 and between -1.5 and -10 , where the gain reaches a peak. Using the root locus program and searching in these regions for the peaks in gain, breakaway points are found at -0.62 with a gain of 2.511 and at -4.4 with a gain of 28.89. A break-in point on the real axis can occur between -1.5 and -10 , where the gain reaches a local minimum. Using the root locus program and searching in these regions for the local minimum gain, a break-in point is found at -2.8 with a gain of 27.91.

Next assume that the system can be approximated by a second-order, underdamped system without any zeros. A 1.52% overshoot corresponds to a damping ratio of 0.8. Sketch this damping ratio line on the root locus, as shown in Figure 8.22.

Use the root locus program to search along the 0.8 damping ratio line for the point where the angles from the open-loop poles and zeros add up to an odd multiple of 180° . This is the point where the root locus crosses the 0.8 damping ratio or 1.52 percent overshoot line. Three points satisfy this criterion: $-0.87 \pm j0.66$, $-1.19 \pm j0.90$, and $-4.6 \pm j3.45$ with respective gains of 7.36, 12.79, and 39.64. For each point the settling time and peak time are evaluated using

$$T_s = \frac{4}{\zeta\omega_n} \quad (8.54)$$

where $\zeta\omega_n$ is the real part of the closed-loop pole, and also using

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \quad (8.55)$$

where $\omega_n \sqrt{1 - \zeta^2}$ is the imaginary part of the closed-loop pole.

To test our assumption of a second-order system, we must calculate the location of the third pole. Using the root locus program, search along the negative extension of the real axis between the zero at -1.5 and the pole at -10 for points that match the value of gain found at the second-order dominant poles. For each of the three crossings of the 0.8 damping ratio line, the third closed-loop pole is at -9.25 , -8.6 , and -1.8 , respectively. The results are summarized in Table 8.4.

Finally, let us examine the steady-state error produced in each case. Note that we have little control over the steady-state error at this point. When the gain is set to meet the transient response, we have also designed the steady-state error. For the example, the steady-state error specification is given by K_v and is calculated as

$$K_v = \lim_{s \rightarrow 0} sG(s) = \frac{K(1.5)}{(1)(10)} \quad (8.56)$$

The results for each case are shown in Table 8.4.

How valid are the second-order assumptions? From Table 8.4, Cases 1 and 2 yield third closed-loop poles that are relatively far from the closed-loop zero. For these two cases there is no pole-zero cancellation, and a second-order system

TABLE 8.4 Characteristics of the system of Example 8.8

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9

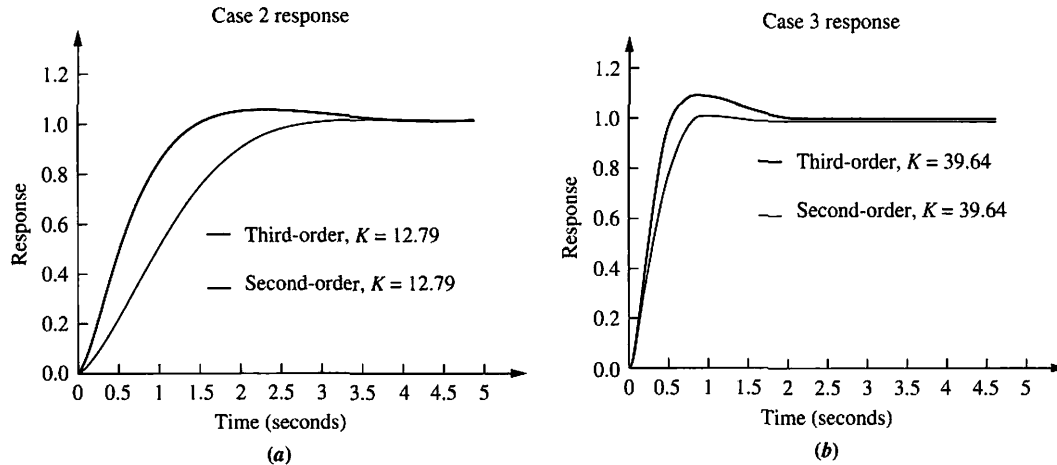


FIGURE 8.23 Second- and third-order responses for Example 8.8: **a.** Case 2; **b.** Case 3

approximation is not valid. In Case 3, the third closed-loop pole and the closed-loop zero are relatively close to each other, and a second-order system approximation can be considered valid. In order to show this, let us make a partial-fraction expansion of the closed-loop step response of Case 3 and see that the amplitude of the exponential decay is much less than the amplitude of the underdamped sinusoid. The closed-loop step response, $C_3(s)$, formed from the closed-loop poles and zeros of Case 3 is

$$\begin{aligned}
 C_3(s) &= \frac{39.64(s + 1.5)}{s(s + 1.8)(s + 4.6 + j3.45)(s + 4.6 - j3.45)} \\
 &= \frac{39.64(s + 1.5)}{s(s + 1.8)(s^2 + 9.2s + 33.06)} \\
 &= \frac{1}{s} + \frac{0.3}{s + 1.8} - \frac{1.3(s + 4.6) + 1.6(3.45)}{(s + 4.6)^2 + 3.45^2}
 \end{aligned} \tag{8.57}$$

Thus, the amplitude of the exponential decay from the third pole is 0.3, and the amplitude of the underdamped response from the dominant poles is $\sqrt{1.3^2 + 1.6^2} = 2.06$. Hence, the dominant pole response is 6.9 times as large as the nondominant exponential response, and we assume that a second-order approximation is valid.

Using a simulation program, we obtain Figure 8.23, which shows comparisons of step responses for the problem we have just solved. Cases 2 and 3 are plotted for both the third-order response and a second-order response, assuming just the dominant pair of poles calculated in the design problem. Again, the second-order approximation was justified for Case 3, where there is a small difference in percent overshoot. The second-order approximation is not valid for Case 2. Other than the excess overshoot, Case 3 responses are similar.

Students who are using MATLAB should now run ch8p2 in Appendix B. You will learn how to use MATLAB to enter a value of percent overshoot from the keyboard. MATLAB will then draw the root locus and overlay the percent overshoot line requested. You will then interact with MATLAB and select the point of intersection of the

root locus with the requested percent overshoot line. MATLAB will respond with the value of gain, all closed-loop poles at that gain, and a closed-loop step response plot corresponding to the selected point. This exercise solves Example 8.8 using MATLAB.

Students who are using MATLAB may want to explore the SISO Design Tool described in Appendix E at www.wiley.com/college/nise. The SISO Design Tool is a convenient and intuitive way to obtain, view, and interact with a system's root locus. Section D.7 describes the advantages of using the tool, while Section D.8 describes how to use it. For practice, you may want to apply the SISO Design Tool to some of the problems at the end of this chapter.

Gui Tool

GUIT

Skill-Assessment Exercise 8.6

PROBLEM: Given a unity feedback system that has the forward-path transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

do the following:

- Sketch the root locus.
- Using a second-order approximation, design the value of K to yield 10% overshoot for a unit-step input.
- Estimate the settling time, peak time, rise time, and steady-state error for the value of K designed in (b).
- Determine the validity of your second-order approximation.

ANSWERS:

- See solution located at www.wiley.com/college/nise.
- $K = 45.55$
- $T_s = 1.97 \text{ s}$, $T_p = 1.13 \text{ s}$, $T_r = 0.53 \text{ s}$, and $e_{\text{step}}(\infty) = 0.51$
- Second-order approximation is not valid.

The complete solution is located at www.wiley.com/college/nise.

8.8 Generalized Root Locus

Up to this point we have always drawn the root locus as a function of the forward-path gain, K . The control system designer must often know how the closed-loop poles change as a function of another parameter. For example, in Figure 8.24, the parameter of interest is the open-loop pole at $-p_1$. How can we obtain a root locus for variations of the value of p_1 ?

If the function $KG(s)H(s)$ is formed as

$$KG(s)H(s) = \frac{10}{(s+2)(s+p_1)} \quad (8.58)$$

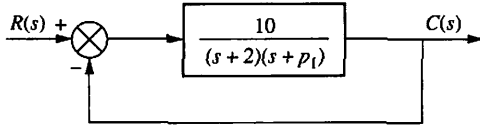


FIGURE 8.24 System requiring a root locus calibrated with p_1 as a parameter

the problem is that p_1 is not a multiplying factor of the function, as the gain, K , was in all of the previous problems. The solution to this dilemma is to create an equivalent system where p_1 appears as the forward-path gain. Since the closed-loop transfer function's denominator is $1 + KG(s)H(s)$, we effectively want to create an equivalent system whose denominator is $1 + p_1 G(s)H(s)$.

For the system of Figure 8.24, the closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{10}{s^2 + (p_1 + 2)s + 2p_1 + 10} \quad (8.59)$$

Isolating p_1 , we have

$$T(s) = \frac{10}{s^2 + 2s + 10 + p_1(s+2)} \quad (8.60)$$

Converting the denominator to the form $[1 + p_1 G(s)H(s)]$ by dividing numerator and denominator by the term not included with p_1 , $s^2 + 2s + 10$, we obtain

$$T(s) = \frac{\frac{10}{s^2 + 2s + 10}}{1 + \frac{p_1(s+2)}{s^2 + 2s + 10}} \quad (8.61)$$

Conceptually, Eq. (8.61) implies that we have a system for which

$$KG(s)H(s) = \frac{p_1(s+2)}{s^2 + 2s + 10} \quad (8.62)$$

The root locus can now be sketched as a function of p_1 , assuming the open-loop system of Eq. (8.62). The final result is shown in Figure 8.25.

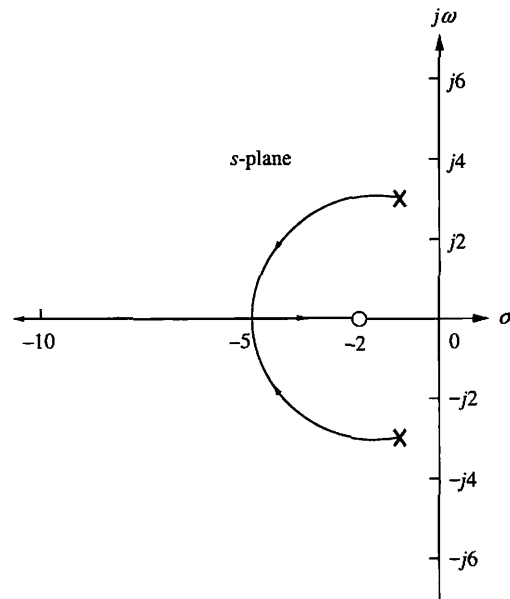


FIGURE 8.25 Root locus for the system of Figure 8.24, with p_1 as a parameter

Skill-Assessment Exercise 8.7

PROBLEM: Sketch the root locus for variations in the value of p_1 , for a unity feedback system that has the following forward transfer function:

$$G(s) = \frac{100}{s(s + p_1)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

WileyPLUS
WPCS
 Control Solutions

In this section, we learned to plot the root locus as a function of any system parameter. In the next section we will learn how to plot root loci for positive-feedback systems.

8.9 Root Locus for Positive-Feedback Systems

The properties of the root locus were derived from the system of Figure 8.1. This is a negative-feedback system because of the negative summing of the feedback signal to the input signal. The properties of the root locus change dramatically if the feedback signal is added to the input rather than subtracted. A positive-feedback system can be thought of as a negative-feedback system with a negative value of $H(s)$. Using this concept, we find that the transfer function for the positive-feedback system shown in Figure 8.26 is

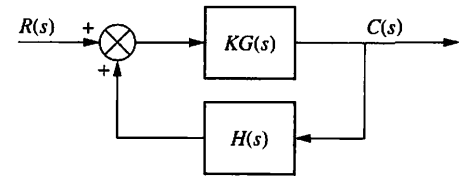


FIGURE 8.26 Positive-feedback system

$$T(s) = \frac{KG(s)}{1 - KG(s)H(s)} \quad (8.63)$$

We now retrace the development of the root locus for the denominator of Eq. (8.63). Obviously, a pole, s , exists when

$$KG(s)H(s) = 1 = 1 \angle k360^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \dots \quad (8.64)$$

Therefore, the root locus for positive-feedback systems consists of all points on the s -plane where the angle of $KG(s)H(s) = k360^\circ$. How does this relationship change the rules for sketching the root locus presented in Section 8.4?

- 1. Number of branches.** The same arguments as for negative feedback apply to this rule. There is no change.
- 2. Symmetry.** The same arguments as for negative feedback apply to this rule. There is no change.
- 3. Real-axis segments.** The development in Section 8.4 for the real-axis segments led to the fact that the angles of $G(s)H(s)$ along the real axis added up to either an odd multiple of 180° or a multiple of 360° . Thus, for positive-feedback systems the

root locus exists on the real axis along sections where the locus for negative-feedback systems does not exist. The rule follows:

Real-axis segments: On the real axis, the root locus for positive-feedback systems exists to the left of an even number of real-axis, finite open-loop poles and/or finite open-loop zeros.

The change in the rule is the word *even*; for negative-feedback systems the locus existed to the left of an *odd* number of real-axis, finite open-loop poles and/or zeros.

4. **Starting and ending points.** You will find no change in the development in Section 8.4 if Eq. (8.63) is used instead of Eq. (8.12). Therefore, we have the following rule.

Starting and ending points: The root locus for positive-feedback systems begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

5. **Behavior at infinity.** The changes in the development of the asymptotes begin at Eq. (M.4) in Appendix M at www.wiley.com/college/nise since positive-feedback systems follow the relationship in Eq. (8.64). That change yields a different slope for the asymptotes. The value of the real-axis intercept for the asymptotes remains unchanged. The student is encouraged to go through the development in detail and show that the behavior at infinity for positive-feedback systems is given by the following rule:

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes for positive-feedback systems are given by the real-axis intercept, σ_a , and angle, θ_a , as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.65)$$

$$\theta_a = \frac{k2\pi}{\# \text{finite poles} - \# \text{finite zeros}} \quad (8.66)$$

where $k = 0, \pm 1, \pm 2, \pm 3, \dots$, and the angle is given in radians with respect to the positive extension of the real axis.

The change we see is that the numerator of Eq. (8.66) is $k2\pi$ instead of $(2k + 1)\pi$.

What about other calculations? The imaginary-axis crossing can be found using the root locus program. In a search of the $j\omega$ -axis, you are looking for the point where the angles add up to a multiple of 360° instead of an odd multiple of 180° . The breakaway points are found by looking for the maximum value of K . The break-in points are found by looking for the minimum value of K .

When we were discussing *negative-feedback* systems, we always made the root locus plot for positive values of gain. Since *positive-feedback* systems can also be thought of as *negative-feedback* systems with negative gain, the rules developed in this section apply equally to *negative-feedback* systems with *negative gain*. Let us look at an example.

Example 8.9

Root Locus for a Positive-Feedback System

PROBLEM: Sketch the root locus as a function of negative gain, K , for the system shown in Figure 8.11.

SOLUTION: The equivalent positive-feedback system found by pushing -1 , associated with K , to the right past the pickoff point is shown in Figure 8.27(a).

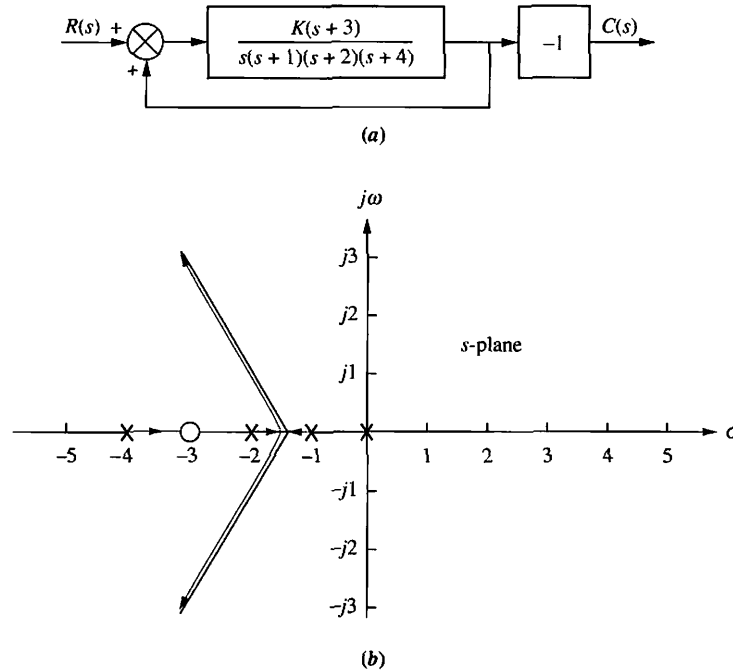


FIGURE 8.27 a. Equivalent positive-feedback system for Example 8.9; b. root locus

Therefore, as the gain of the equivalent system goes through positive values of K , the root locus will be equivalent to that generated by the gain, K , of the original system in Figure 8.11 as it goes through negative values.

The root locus exists on the real axis to the left of an even number of real, finite open-loop poles and/or zeros. Therefore, the locus exists on the entire positive extension of the real axis, between -1 and -2 and between -3 and -4 . Using Eq. (8.27), the σ_a intercept is found to be

$$\sigma_a = \frac{(-1 - 2 - 4) - (-3)}{4 - 1} = -\frac{4}{3} \quad (8.67)$$

The angles of the lines that intersect at $-4/3$ are given by

$$\theta_a = \frac{k2\pi}{\# \text{ finite poles} - \# \text{ finite zeros}} \quad (8.68a)$$

$$= 0 \quad \text{for } k = 0 \quad (8.68b)$$

$$= 2\pi/3 \quad \text{for } k = 1 \quad (8.68c)$$

$$= 4\pi/3 \quad \text{for } k = 2 \quad (8.68d)$$

The final root locus sketch is shown in Figure 8.27(b)

Skill-Assessment Exercise 8.8

PROBLEM: Sketch the root locus for the positive-feedback system whose forward transfer function is

$$G(s) = \frac{K(s+4)}{(s+1)(s+2)(s+3)}$$

The system has unity feedback.

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.10 Pole Sensitivity

The root locus is a plot of the closed-loop poles as a system parameter is varied. Typically, that system parameter is gain. Any change in the parameter changes the closed-loop poles and, subsequently, the performance of the system. Many times the parameter changes against our wishes, due to heat or other environmental conditions. We would like to find out the extent to which changes in parameter values affect the performance of our system.

The root locus exhibits a nonlinear relationship between gain and pole location. Along some sections of the root locus, (1) very small changes in gain yield very large changes in pole location and hence performance; along other sections of the root locus, (2) very large changes in gain yield very small changes in pole location. In the first case we say that the system has a high sensitivity to changes in gain. In the second case, the system has a low sensitivity to changes in gain. We prefer systems with low sensitivity to changes in gain.

In Section 7.7, we defined sensitivity as the ratio of the fractional change in a function to the fractional change in a parameter as the change in the parameter approaches zero. Applying the same definition to the closed-loop poles of a system that vary with a parameter, we define *root sensitivity* as the ratio of the fractional change in a closed-loop pole to the fractional change in a system parameter, such as gain. Using Eq. (7.75), we calculate the sensitivity of a closed-loop pole, s , to gain, K :

$$S_{s:K} = \frac{K}{s} \frac{\delta s}{\delta K} \quad (8.69)$$

where s is the current pole location, and K is the current gain. Using Eq. (8.69) and converting the partials to finite increments, the actual change in the closed-loop poles can be approximated as

$$\Delta s = s(S_{s:K}) \frac{\Delta K}{K} \quad (8.70)$$

where Δs is the change in pole location, and $\Delta K/K$ is the fractional change in the gain, K . Let us demonstrate with an example. We begin with the characteristic equation from which $\delta s/\delta K$ can be found. Then, using Eq. (8.69) with the current closed-loop pole, s , and its associated gain, K , we can find the sensitivity.