Potential formulation (u, $\bar{A}$):

In Electrostatics $\bar{\nabla}\times\bar{E}=0$ allowed us to write $\bar{E}$ as a gradient of a Scalar potential

i.e $\bar{E} = -\bar{\nabla}u$

In electrodynamics this is no longer possible, because the $\bar{\nabla}\times\bar{E}$ is non-zero. But $\bar{B}$ remains divergence less. So we can still write

$$\bar{B} = \bar{\nabla}\times\bar{A} \quad \text{------} (1)$$

as in magnetostatics putting it in to the Faraday's Law field we get

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{A})$$

$$\bar{\nabla}\times\bar{E} + \bar{\nabla}\times\frac{\partial\bar{A}}{\partial t} = 0$$

$$\bar{\nabla}\times\left(\bar{E}+\frac{\partial\bar{A}}{\partial t}\right) = 0$$

Here is a quantity unlike $\bar{E}$ alone, whose curl does not vanish it can

Contd,

be written as the gradient of a scalar potential

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla} u$$

$$\bar{E} = -\bar{\nabla} u - \frac{\partial \bar{A}}{\partial t} \quad \text{——} (2)$$

This reduces the above equation(2) to old form of course in electrostatics as we put $\frac{\partial \bar{A}}{\partial t} = 0$

The potential formulation of equation (1) & (2) automatically fullfills the two homogeneous Maxwell's equations

i.e $\bar{\nabla} \cdot \bar{B} = 0$ & $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

But what about the Gauss's law in electrostatics & Ampere's Law with Maxwell's equation.

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0} \quad \& \quad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \frac{\partial \bar{D}}{\partial t}$$

for free space.

Putting equation(2), the $\bar{E}$ value in Gauss's law for electrostatics

Contd.

We have

$$\bar\nabla \cdot \left(-\bar\nabla u - \frac{\partial \bar A}{\partial t}\right) = \rho/\epsilon_0$$

or

$$\nabla^2 u + \frac{\partial}{\partial t}(\bar\nabla \cdot \bar A) = -\frac{1}{\epsilon_0}\rho \quad \text{——(3)}$$

Equation (3) replaces Poisson's Equation.

Let us put the relevant terms from (1) & (2) in Maxwell's 4th equation

i.e $\quad \bar\nabla \times \bar B = \mu_0 \bar J + \epsilon_0 \mu_0 \frac{\partial \bar E}{\partial t}$

putting $\bar B = \bar\nabla \times A$ & $\bar E = -\bar\nabla u - \frac{\partial \bar A}{\partial t}$ we get

$$\bar\nabla \times \bar\nabla \times \bar A = \mu_0 \bar J - \mu_0 \epsilon_0 \bar\nabla\left(\frac{\partial u}{\partial t}\right) - \epsilon_0 \mu_0 \frac{\partial^2 \bar A}{\partial t^2}$$

Using Vector Identity

$$\bar\nabla \times \bar\nabla \times \bar A = \bar\nabla(\bar\nabla \cdot \bar A) - \nabla^2 \bar A \quad \text{——(4)}$$

We have

$$\nabla^2 \bar A - \epsilon_0 \mu_0 \frac{\partial^2 \bar A}{\partial t^2} - \bar\nabla\left(\bar\nabla \cdot \bar A + \mu_0 \epsilon_0 \frac{\partial u}{\partial t}\right) = -\mu_0 \bar J \quad \text{——(5)}$$

Equations (3) & (5) carry all the information in Maxwell's equations in electrodynamics & hence termed as potential formulation of u & $\bar A$.