Potential formulation (u, $\bar{A}$):

In Electrostatics $\bar{\nabla} \times \bar{E} = 0$ allowed us to write $\bar{E}$ as a gradient of a Scalar potential

ie $\bar{E} = -\bar{\nabla} u$

In electrodynamics this is no longer possible, because the $\bar{\nabla} \times \bar{E}$ is non-zero. But $\bar{B}$ remains divergence less. So we can still write

$$\bar{B} = \bar{\nabla} \times \bar{A} \quad \text{------} (1)$$

as in magnetostatics putting it in to the Faraday's Law field we get

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\frac{\partial}{\partial t}(\bar{\nabla} \times \bar{A}).$$

$$\bar{\nabla} \times \bar{E} + \bar{\nabla} \times \frac{\partial \bar{A}}{\partial t} = 0$$

$$\bar{\nabla} \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0$$

Here is a quantity unlike $\bar{E}$ alone, whose curl does not vanish it can

Contd,

be written as the gradient of a Scalar potential

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla} u$$

$$\bar{E} = -\bar{\nabla} u - \frac{\partial \bar{A}}{\partial t} \quad \text{——} (2)$$

This reduces the above equation (2) to old form of course in electrostatics as we put $\frac{\partial \bar{A}}{\partial t} = 0$

The potential formulation of equation (1) & (2) automatically full fills the two homogeneous Maxwell's equations

i.e $\bar{\nabla} \cdot B = 0$ & $\bar{\nabla} \times \bar{E} = -\frac{\partial B}{\partial t}$

But what about the Gauss's law in electrostatics & Ampere's Law with Maxwell's equation.

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0} \quad \& \quad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \frac{\partial \bar{D}}{\partial t}$$

for free space.

Putting equation (2), the $\bar{E}$ value in Gauss's law for electrostatics

Contd.

We have

$$\bar\nabla\cdot\left(-\bar\nabla u-\frac{\partial\bar A}{\partial t}\right)=\rho/\epsilon_0$$

or

$$\nabla^2 u+\frac{\partial}{\partial t}(\bar\nabla\times\bar A)=-\frac{1}{\epsilon_0}\rho \quad\text{—— (3)}$$

Equation (3) replaces Poisson's Equation.

Let us put the relevant terms from (1) & (2) in Maxwell's 4th equation

i.e $\quad \bar\nabla\times\bar B=\mu_0\bar J+\epsilon_0\mu_0\dfrac{\partial \bar E}{\partial t}$

putting $\bar B=\bar\nabla\times A$ & $\bar E=-\bar\nabla u-\dfrac{\partial\bar A}{\partial t}$ we get

$$\bar\nabla\times\bar\nabla\times\bar A=\mu_0\bar J-\mu_0\epsilon_0\bar\nabla\left(\frac{\partial u}{\partial t}\right)-\epsilon_0\mu_0\frac{\partial^2\bar A}{\partial t^2}$$

Using Vector Identity

$$\bar\nabla\times\bar\nabla\times\bar A=\bar\nabla(\bar\nabla\cdot\bar A)-\nabla^2 A \quad\text{—— (4)}$$

We have

$$\nabla^2\bar A-\epsilon_0\mu_0\frac{\partial^2\bar A}{\partial t^2}-\bar\nabla\left(\bar\nabla\cdot\bar A+\mu_0\epsilon_0\frac{\partial u}{\partial t}\right)=-\mu_0\bar J \quad\text{—— (5)}$$

Equations (3) & (5) carry all the information in Maxwell's equations in electrodynamics & hence termed as potential formulation of u & $\bar A$.