Gauge Transformations:

In electrodynamics Maxwell's equations are expressed as

$$\nabla^2 u + \frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A}) = -\frac{\rho}{\epsilon_0} \quad \text{---(1)}$$

$$\nabla^2\bar{A} - \epsilon_0\mu_0\frac{\partial^2\bar{A}}{\partial t^2} - \bar{\nabla}\left(\bar{\nabla}\cdot\bar{A} + \epsilon_0\mu_0\frac{\partial u}{\partial t}\right) = -\mu_0\bar{J} \quad \text{---(2)}$$

As
$$\bar{B} = \bar{\nabla}\times\bar{A} \quad \text{---(3)}$$

$$\bar{E} = -\bar{\nabla}u - \frac{\partial\bar{A}}{\partial t} \quad \text{---(4)}$$

The equations (1) & (2) seem to be very complex & looks not suitable for potential formulation. However, we have succeeded in reducing six problems - i.e Finding $\bar{E}$ & $\bar{B}$ (three components each) down to four u & $\bar{A}$ i.e (u one and three compo... of $\bar{A}$). ...... More over equations (1) & (2) donot uniquely define the potentials, we are free to impose extra conditions on "u" & "$\bar{A}$" as long as original fields $\bar{E}$ & $\bar{B}$ are not affected.

Contd.

Let us workout precisely what the freedom demands.

Suppose we have two sets of potentials $(U, \bar{A})$ & $(U', \bar{A}')$ which corresponds to the same $\bar{E}$ & $\bar{B}$. By how much they can differ? let us say

$$\bar{A}' = \bar{A} + \bar{\alpha} \quad \& \quad U' = U + \beta$$

Since $\bar{A}'s$ give the same $\bar{B}$ their curl must be equal to eqn (3), But for that we should have $\bar{\nabla} \times \bar{\alpha} = 0$ ——— (5)

Hence $\bar{B}' = \bar{\nabla} \times \bar{A}' = \bar{\nabla} \times (\bar{A} + \bar{\alpha})$

$= \bar{\nabla} \times \bar{A} + \bar{\nabla} \times \bar{\alpha}$ Since $\bar{\nabla} \times \bar{\alpha} = 0$

$\bar{B}' = \bar{\nabla} \times \bar{A} = \bar{B}$

We can therefore write or predict that

$$\bar{\alpha} = \bar{\nabla} \lambda \quad \text{as} \quad \bar{\nabla} \times \bar{\nabla} \lambda = 0$$

$$\bar{\nabla} \times \bar{\alpha} = 0.$$

Where λ is a scalar potential for the newly imposed condition that $\bar{\alpha} = \bar{\nabla} \lambda$

Now for $\bar{E}' = -\bar{\nabla} U' - \frac{\partial \bar{A}'}{\partial t}$

Since $U' = U + \beta$

Contd.

$$\bar{E}' = -\left\{ \nabla(u+\beta) + \frac{\partial}{\partial t}(\bar{A}+\bar{\alpha}) \right\}$$

$$= -\nabla u - \nabla\beta - \frac{\partial \bar{A}}{\partial t} - \frac{\partial \bar{\alpha}}{\partial t}$$

$$\bar{E}' = -\bar{\nabla} u - \frac{\partial \bar{A}}{\partial t} - \left(\nabla\beta + \frac{\partial \bar{\alpha}}{\partial t}\right) \quad \text{—— (6)}$$

The equation (6) will lead to the original field $\bar{E}$ if

$$\nabla\beta + \frac{\partial \bar{\alpha}}{\partial t} = 0 \quad \text{—— (7)}$$

Hence

$$\bar{E}' = -\bar{\nabla} u - \frac{\partial \bar{A}}{\partial t} - (0) = \bar{E}.$$

Since $\alpha = \nabla\lambda$

$$\nabla\beta + \bar{\nabla}\left(\frac{\partial \lambda}{\partial t}\right) = 0 \quad \text{—— (8)}$$

Integrating both sides of eq (8) w.r.t. 'x'

We get $\int d\left(\beta + \frac{\partial \lambda}{\partial t}\right) = \int (0)\, dx$

$$\beta + \frac{\partial \lambda}{\partial t} = K(t)$$

$$\beta = -\frac{\partial \lambda}{\partial t}$$

where K_{t} is a constant & may be taken as $K_{t} = 0$

$$\beta = -\frac{\partial \lambda}{\partial t}$$

Contd:

The term '$\frac{\partial \lambda}{\partial t}$' is therefor independent of position it could however depend on time. Actually we may also absorb K_{t_1} the constant in to λ_{t_1}' as $\lambda_{t_1}' = \int_0^t K_{t_1} dt'$

Now defining a new λ_{t_1} as above & adding to old one this will not affect the gradient of λ_{t_1}. It just adds K_{t_1} to $\frac{\partial \lambda}{\partial t}$. So with all this done

we have
$$\bar{A}' = \bar{A} + \bar{\nabla}\lambda = \bar{A} + \bar{\alpha}$$
$$U' = U - \frac{\partial \lambda}{\partial t} = U + \beta$$

When $\bar{\alpha} = \bar{\nabla}\lambda$ & $\beta = -\frac{\partial \lambda}{\partial t}$

Conclusion: For any old scaler function λ we can with impunity (exemption) add $\bar{\nabla}\lambda$ to $\bar{A}$ provided we simultaneously subtract $\left(\frac{\partial \lambda}{\partial t}\right)$ from U. None of these changes will affact the original physical quantities $\bar{E}$ & $\bar{B}$ such changes in U & $\bar{A}$ are called Gauge Transformations.