

Gauge Transformations:

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In electrodynamics Maxwell's equations are expressed as

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \bar{A}) = -\frac{\rho}{\epsilon_0} \quad (1)$$

$$\nabla^2 \bar{A} - \epsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} - \nabla (\nabla \cdot \bar{A} + \epsilon_0 \mu_0 \frac{\partial \phi}{\partial t}) = -\mu_0 \bar{J} \quad (2)$$

$$\text{As } \bar{B} = \nabla \times \bar{A} \quad (3)$$

$$\bar{E} = -\nabla \phi - \frac{\partial \bar{A}}{\partial t} \quad (4)$$

The equations (1) & (2) seem to be very complex & looks not suitable for potential formulation. However, we have succeeded in reducing six problems - i.e. finding \bar{E} & \bar{B} (three components each) down to four

ϕ & \bar{A} (i.e. ϕ one and three components of \bar{A}). Moreover equations

(1) & (2) do not uniquely define the potentials, we are free to impose extra conditions on " ϕ " & " \bar{A} " as long as original fields \bar{E} & \bar{B} are not affected.

Contd.

Let us work out precisely what the freedom demands.

Suppose we have two sets of potentials (U, \bar{A}) & (U', \bar{A}') which corresponds to the same \bar{E} & \bar{B} . By how much they can differ? let us say

$$\bar{A}' = \bar{A} + \bar{\lambda} \quad \& \quad U' = U + \beta$$

Since \bar{A}' give the same \bar{B} their curl must be equal to $\text{curl } \bar{B}$. But for that

We should have $\nabla \times \bar{\lambda} = 0$ — (5)

$$\begin{aligned} \text{Hence } \bar{B}' &= \nabla \times \bar{A}' = \nabla \times (\bar{A} + \bar{\lambda}) \\ &= \nabla \times \bar{A} + \nabla \times \bar{\lambda} \quad \text{Since } \nabla \times \bar{\lambda} = 0 \\ \bar{B}' &= \nabla \times \bar{A} = \bar{B} \end{aligned}$$

We can therefore write or predict that

$$\bar{\lambda} = \nabla \lambda \quad \text{as } \nabla \times \nabla \lambda = 0 \\ \nabla \times \bar{\lambda} = 0$$

Where λ is a scalar potential for the newly imposed condition that $\bar{\lambda} = \nabla \lambda$

Now for $\bar{E}' = -\nabla U' - \frac{\partial \bar{A}'}{\partial t}$

Sim. $U' = U + \beta$

Contd.

$$\vec{E}' = -\left\{ \vec{\nabla}(u + \beta) + \frac{\partial}{\partial t}(\vec{A} + \vec{\lambda}) \right\}$$

$$= -\vec{\nabla}u - \vec{\nabla}\beta - \frac{\partial \vec{A}}{\partial t} - \frac{\partial \vec{\lambda}}{\partial t}$$

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$$\vec{E}' = -\vec{\nabla}u - \frac{\partial \vec{A}}{\partial t} - (\vec{\nabla}\beta + \frac{\partial \vec{\lambda}}{\partial t}) \quad \text{--- (6)}$$

The equation (6) will lead to the original field \vec{E} if

$$\vec{\nabla}\beta + \frac{\partial \vec{\lambda}}{\partial t} = 0 \quad \text{--- (7)}$$

Hence

$$\vec{E}' = -\vec{\nabla}u - \frac{\partial \vec{A}}{\partial t} - (0) = \vec{E} \quad \text{--- (8)}$$

Since $d = \vec{\nabla}\lambda$

$$\vec{\nabla}\beta + \vec{\nabla}\left(\frac{\partial \lambda}{\partial t}\right) = 0 \quad \text{--- (9)}$$

Integrating both sides of eq (9) w.r.t. 'x'

We get $\int d\left(\beta + \frac{\partial \lambda}{\partial t}\right) = \int (0) dx$

$$\beta + \frac{\partial \lambda}{\partial t} = K(t),$$

$$\beta = -\frac{\partial \lambda}{\partial t}$$

Where $K(t)$ is a constant & may be taken as $K(t) = 0$

$$\beta = -\frac{\partial \lambda}{\partial t}$$

Contd:

The term $\frac{\partial \lambda}{\partial t}$ is therefore independent of position it could however depend on time. Actually we may also absorb k_{\pm} the constant in λ_{\pm} as $\lambda'_{\pm} = \int k_{\pm} dt'$

Now defining a new λ'_{\pm} as above & adding to old one this will not affect the gradient of λ_{\pm} . it just adds

k_{\pm} to $\frac{\partial \lambda}{\partial t}$. So with all this done

$$\text{we have } \bar{A}' = \bar{A} + \bar{\nabla} \lambda = \bar{A} + \bar{\alpha}$$

$$U' = U - \frac{\partial \lambda}{\partial t} = U + \beta$$

$$\text{When } \bar{\alpha} = \bar{\nabla} \lambda \quad \& \quad \beta = -\frac{\partial \lambda}{\partial t}$$

Conclusion: For any old scalar function λ we can with impunity (exemption) add

$\bar{\nabla} \lambda$ to \bar{A} provided we simultaneously subtract $\left(\frac{\partial \lambda}{\partial t}\right)$ from U . None of these changes

will affect the original physical quantities \bar{E} & \bar{B} such changes in

U & \bar{A} are called Gauge Transformations.