Electromagnetic Theory II (EMT II); Online Unit 1.

REFLECTION AND TRANSMISSION AT OBLIQUE INCIDENCE

(Laws of Reflection and Refraction and Total Internal Reflection)

(Introduction to Electrodynamics Chap 9)

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Suppose an incident wave makes an angle θ_I with the normal to the xy-plane at z=0 (in medium 1) as shown in Figure 1. Suppose the wave splits into parts partially reflecting back in medium 1 and partially transmitting into medium 2 making angles θ_R and θ_T, respectively, with the normal.

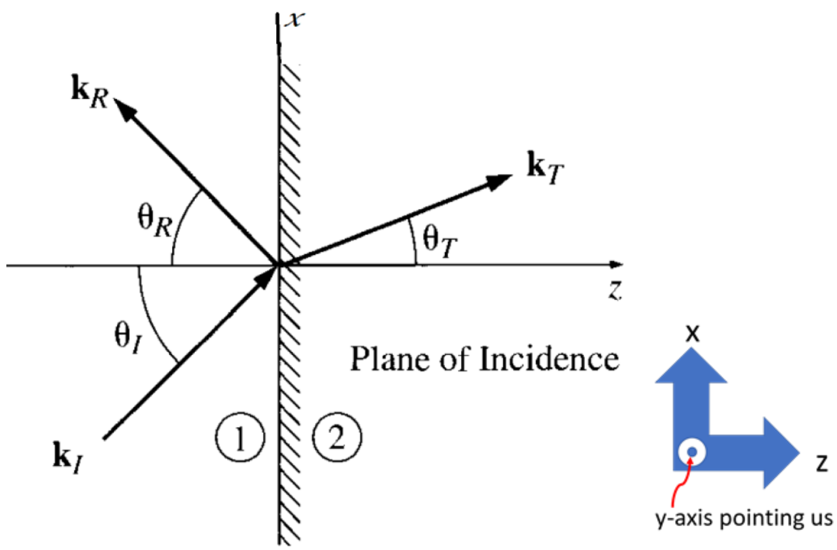

Figure 1.

To understand the phenomenon at the boundary at z=0, we should apply the appropriate boundary conditions as discussed in the earlier lectures.

Let us first write the equations of the waves in terms of electric and magnetic fields depending upon the wave vector $\boldsymbol{\kappa}$ and the frequency ω.

MEDIUM 1:

$$\tilde{\mathbf{E}}_I(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}$$

$$\tilde{\mathbf{B}}_I(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$$

Where $\mathbf{E}_I$ *and* $\mathbf{B}_I$ is the instantaneous magnitudes of the electric and magnetic vector, respectively, of the incident wave. Other symbols have their usual meanings.

For the reflected wave,

$$\tilde{\mathbf{E}}_R(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_R(\mathbf{r}, t) = \frac{1}{v_1}(\hat{\mathbf{k}}_R \times \tilde{\mathbf{E}}_R),$$

Similarly,

MEDIUM 2:

$$\tilde{\mathbf{E}}_T(\mathbf{r}, t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}, \quad \tilde{\mathbf{B}}_T(\mathbf{r}, t) = \frac{1}{v_2}(\hat{\mathbf{k}}_T \times \tilde{\mathbf{E}}_T).$$

Where $\mathbf{E}_T$ and $\mathbf{B}_T$ are the electric and magnetic instantaneous vectors of the transmitted part in medium 2.

BOUNDARY CONDITIONS (at z=0)

As the free charge on the surface is zero, the perpendicular component of the displacement vector is continuous across the surface.

$(\mathbf{D}_{I\perp} + \mathbf{D}_{R\perp})$ (In Medium 1) = $\mathbf{D}_{T\perp}$ (In Medium 2)

Where **D**s represent the perpendicular components of the displacement vector in both the media.

Converting **D** to **E**, we get,

$\varepsilon_1 E_{I\perp} + \varepsilon_1 E_{R\perp} = \varepsilon_2 E_{T\perp}$

$$\tilde{\mathbf{E}}_{0_I} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}$$

$$\varepsilon_1 \qquad\qquad \perp + \varepsilon_1 \tilde{\mathbf{E}}_{0_R} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} \perp = \varepsilon_2 \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \perp$$

Since the equation is valid for all x and y at z=0, and the coefficients of the exponentials are constants, only the exponentials will determine any change that is occurring. However, the exponentials are the phase factors and the waves must join smoothly at z=0 with the same phase. In other words, the exponentials must add up in such a way that the equation is valid for all points x and y on the plane z=0. (To understand it mathematically, problem 9.16 is very helpful.)

$$A e^{i(\mathbf{k}_I \cdot \mathbf{r}} + B e^{i(\mathbf{k}_R \cdot \mathbf{r}} = C \; e^{i(\mathbf{k}_T \cdot \mathbf{r}} \qquad (1)$$

Where A, B, and C are different constants replacing the fields. Since the frequency remains the same, the time part in the exponentials cancels out.

Equation 1 is satisfied for all x and y only when the powers of the exponents are separately equal. A crude way to understand this is to look at the method of algebraic summations in our school algebra: Only the coefficients of the variables having the same power can be added together.

So,

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{when } z = 0, \qquad (2)$$

Writing the above vector equations in component form, we get

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y, \qquad (3)$$

For all x and y. Remember z = 0 here.

Suppose the point of interest is x = 0 on the xy-plane.

Then equation 3 can be written as

$$y\,(k_I)_y = y\;(k_R)_y = y\;(k_T)_y$$

or

$$(k_I)_y = (k_R)_y = (k_T)_y, \tag{4}$$

Consider equation 4. If we suppose that $(k_I)_y = 0$, This means that the incident wave is moving in the xz plane. That is, the direction of the incident wave does not have a y component. Saying this automatically implies that the wave vector **k** of the reflected as well as the transmitted wave do not have a y component ($(k_I)_y = 0$ implies $(k_R)_y = 0$ and $(k_T)_y = 0$ by virtue of equation 4).

This leads us to the famous *First Law of Reflection and Refraction*: **All the incident, reflected, and refracted components of a wave lie in the same plane, the plane of incidence.** In our case, jus for example, the plane of incidence would be the xz-plane (because $k_y = 0$).

Now suppose $y = 0$ in equation 3, we get,

$$(k_I)_x = (k_R)_x = (k_T)_x \tag{5}$$

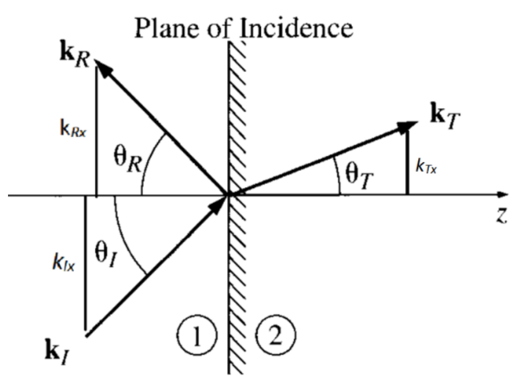

Figure 2

With the help of figure 2, equation 5 can be written as

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T \tag{6}$$

Or

$$k_I \sin\theta_I = k_R \sin\theta_R \tag{7}$$

As the incident and the reflected wave vectors are equal (the same medium), equation 7 can be simplified to

$$\sin\theta_I = \sin\theta_R$$

Leading us to the **Second Law or the Law of Reflection**,

$$\theta_I = \theta_R \tag{8}$$

The angle of incidence is equal to the angle of reflection.

Now, proceeding further, equation 6 can also be written as

$$k_I \sin\theta_I = k_T \sin\theta_T$$

Or

$$\frac{k_T}{k_I} = \frac{Sin\ \theta_I}{Sin\ \theta_T} \tag{9}$$

The frequency of the wave can be written as

$$\omega = kv$$

AS the frequency remains the same,

So,

$k_I v_1 = k_R v_1 = k_T v_2$ (v_1 and v_2 are the speeds in medium 1 and 2, respectively)

Or

$$\frac{k_T}{k_I} = \frac{v_1}{v_2} \tag{10}$$

Putting equation 10 in equation 9,

$$\frac{v_1}{v_2} = \frac{Sin\ \theta_I}{Sin\ \theta_T}$$

Or

$$\frac{\boldsymbol{Sin\ \theta_I}}{\boldsymbol{Sin\ \theta_T}} = \frac{\boldsymbol{n_2}}{\boldsymbol{n_1}} \tag{11}$$

Equation 11 is the **Third Law** or **Snell's Law of Refraction**.

Equation 4 , 8, and 11 give the three fundamental laws of geometrical optics.

Total Internal Reflection

The complete reflection of a wave back into the first medium is called total internal reflection. In this case, no part of the wave gets transmitted into the second medium. The phenomenon depends upon the indices of refraction of the two media and the angle of incidence. The total internal reflection always happens only above a certain angle of incidence for a set of two given media. This angle of incidence is called the critical angle. Figure 3 is a hypothetical situation in which light falls from water onto air.

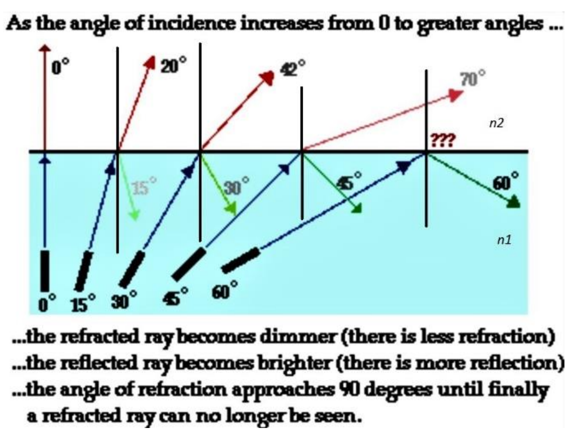

Figure 3 (Copied and modified from: https://www.physicsclassroom.com)

As you can see, when light starts falling oblique upon the interface, reflection starts. The reflection part increases in intensity and the transmitted part gets dimmer and dimmer upon increase in obliqueness. When the angle of incidence becomes 48.6° with the normal, the refracted or transmitted ray disappear and moves parallel to the interface grazing, with angle of refraction θ_T =90°. This angle is called the **Critical Angle**. Any angle of incidence greater than this will make light completely reflected into water (the first medium).

Mathematically

From Snell's law,

$$\frac{Sin\ \theta_I}{Sin\ \theta_T} = \frac{n_2}{n_1} \qquad (12)$$

At the critical angle θ_c,

$$\frac{Sin\ \theta_c}{Sin\ 90} = \frac{n_2}{n_1}$$

Or

$$\theta_c \equiv \sin^{-1}(n_2/n_1) \qquad (13)$$

Equation 13 help us to calculate the critical angle for a pair of two media.

From equation 12,

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I \qquad (14)$$

FOR $\theta_I > \theta_c$,

$$Sin\ \theta_I > Sin\ \theta_c$$

And with the help of 13,

$$Sin\ \theta_I > Sin\ (Sin^{-1}(\frac{n2}{n1}))$$

Or

$$Sin\ \theta_I > \frac{n2}{n1}$$

Or equation 14 implies

$$\boldsymbol{Sin\ \theta_T > \frac{n1}{n2}.\frac{n2}{n1}}$$

This means that

$$\boldsymbol{Sin\ \theta_T > 1} \tag{15}$$

Which renders the angle of transmission as meaningless. Hence no transmission at **θi** $>$ **θc.** (The total internal reflection).