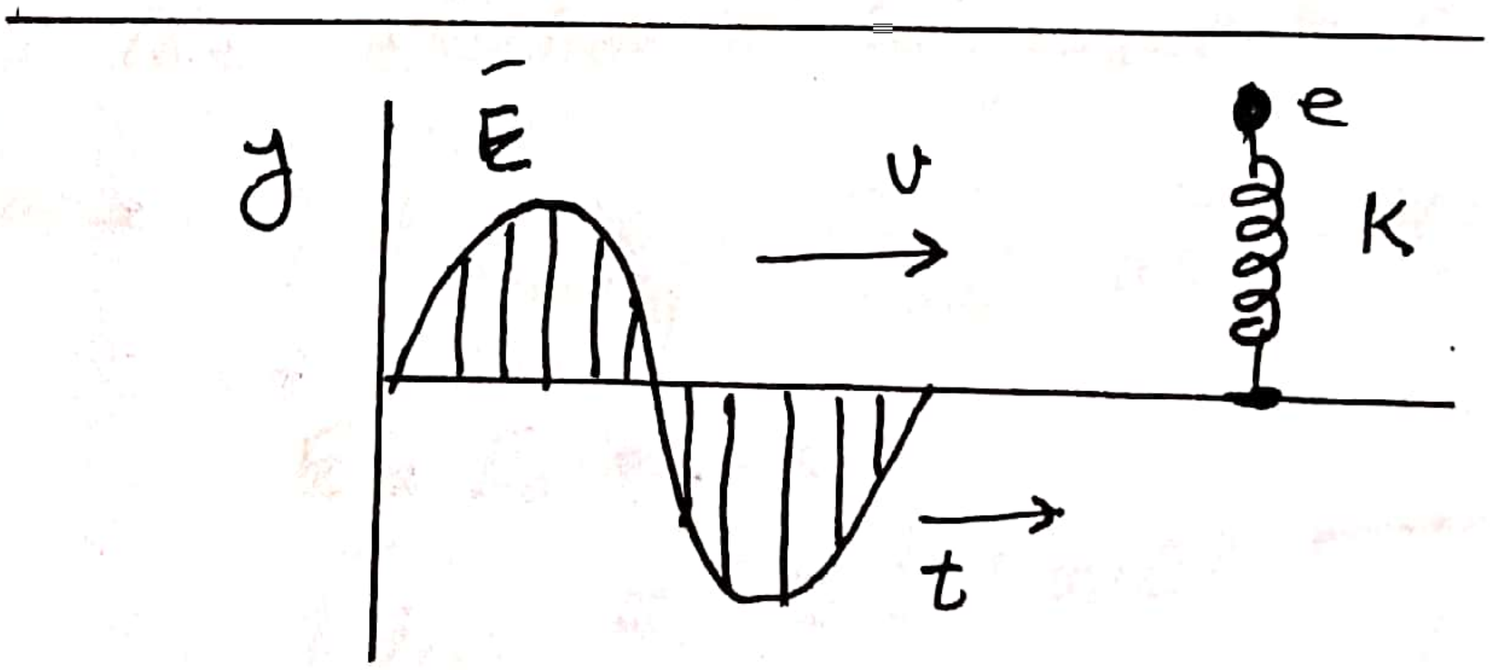
Dispersion in non-conductors:

The electrons in a non-conducter are attached to specific molecules by binding forces whose detailed structure may be extremly complicated (specifically for large molecules) and would at any point require quantum mechanics for proper treatment. But here let us we consider the electron attached to the end of a spring with a force constant K as shown in fig-1



The binding force F_b is given as

$$F_b = -Ky = -m\omega_0^2 y \quad \text{——(1)}$$

where "m" is the mass of the electron

& ω_0 is the natural oscillation frequency given as $\omega_0 = \sqrt{\frac{k}{m}}$

& K is the force consth; $K = \omega_0^2 m$

When the electron oscillates there will be presumebly some damping force which we assume ; ∴ ; negative times proportional to velocity

$$F_{damping} = f_{damp} = -m\gamma \frac{dy}{dt} \quad \text{—(2)}$$

Where " γ " is a damping factor. The damping must be opposite to velocity

In the presence of EM-wave of frequency "ω" the electron is subjected to a driving force $\bar{F}_{driving} = \bar{F}_{drive} = q\bar{E}$ —(3)

$$E = E_0 \cos\omega t$$

$$\bar{F}_{drive} = q\bar{E}_0 \cos\omega t \quad \text{—(4)}$$

Where "q" is the charge of electron & $\bar{E}_0$ is the amplitude of oscillating electron.

According to the Newton's 2nd law of motion putting all above equations

Contd.

$$m\frac{d^2y}{dt^2} = F_{bind} + F_{damp} + F_{drive}$$

$$m\frac{d^2y}{dt^2} + m\gamma\frac{dy}{dt} + Km\omega_0^2 y = qE_0\cos\omega t \quad -(5)$$

Where $F_{bind} = -.\omega_0^2 my$, $F_{damp} = -m\gamma\frac{dy}{dt}$, $F_{drive} = qE_0\cos\omega t$

The equation(5) describes a Model of electron as a damped harmonic oscillator driven at frequency "ω". We assume that massive nuclei remain at rest.

Equation(5) is easier to handle if we consider it a real part of a complex equation.

$$\frac{d^2\tilde{y}}{dt^2} + \gamma\frac{d\tilde{y}}{dt} + \omega_0^2\tilde{y} = \frac{q}{m}E_0 e^{-i\omega t} \quad -(6)$$

The electron oscillating at driving force

let $\tilde{y}_{(t)} = \tilde{y}_0 e^{-i\omega t}$ ——— (7)

Where $\tilde{y}_t$ is the instantanous displacement & $\tilde{y}_0$ is the amplitude.

*Note:

The "~" tilde sign represents the ——— a Complex function.

Contd.

$$\frac{d\tilde{y}}{dt} = -j y_0 \omega e^{-i\omega t}$$

$$\frac{d^2\tilde{y}}{dt^2} = -y_0 \omega^2 e^{-i\omega t}$$

Putting in eq (6) We get

$$\tilde{y}_0 = \frac{e/m}{(\omega_0^2 - \omega^2) - j\gamma\omega} \tilde{E}_0 \quad \text{---(8)}$$

The dipole moment associated with the motion of electron is therefore the part of a complex dipole moment

$$\tilde{p}_{ind} = q\,\tilde{y}_{ind} \quad \text{---(9)}$$

$$\tilde{p}_{ind} = \frac{q^2/m}{(\omega_0^2 - \omega^2) - i\gamma\omega} \tilde{E}_0 e^{-i\omega t} \quad \text{---(10)}$$

The imaginary term in the denominator indicates that $\tilde{p}$ is out of phase with $\tilde{E}$ Lagging behind by an angle $\tan^{-1}\left(\frac{-\gamma\omega}{\omega_0^2 - \omega^2}\right)$ which is very small when $\omega << \omega_0$ & rises to π when $\omega >> \omega_0$ $\phi = (0 - \pi)$

Continued:

In general differently situated electrons in a given molecule exhibits different natural frequencies & damp coefficients.

Let us say there are f_j electrons with frequency ω_j & damping factor γ_j in each molecule. When $j = 1, 2, \ldots$

If N is the total number of molecules per unit volume

The polarisation $\tilde{P}$ is given a

$$\tilde{P} = N\tilde{p}$$

$$\tilde{P} = N\frac{q^2}{m}\sum \frac{f_j}{(\omega_j^2 - \omega^2) - 2i\gamma_j\omega}\tilde{E} \quad \text{— (11)}$$

The Electric susceptibility is given by the relation ($\tilde{P} = \epsilon_0 \chi_e \tilde{E}$)

$$\tilde{P} = \epsilon_0 \chi_e \tilde{E} \quad \text{— (12)}$$

In the present case $\tilde{P}$ is not proportional to $\tilde{E}$ which is not a linear medium. However the complex polarization $\tilde{P}$ is proportional to complex $\tilde{E}$ & this suggests

Contd.

That we introduce a complex Susceptibility $\tilde{\chi}_e$

$$\tilde{P} = \epsilon_0 \tilde{\chi}_e \tilde{E} \quad \text{---(13)}$$

The complex permitivity $\tilde{\epsilon} = \epsilon_0 (1 + \tilde{\chi}_e)$ ---(13a)

i.e $\tilde{\epsilon} = \epsilon_0 \tilde{\epsilon}_r$ Which in this model takes the form

$$\chi_e = \left(\frac{\bar{P}}{\bar{E}\epsilon_0}\right) \quad \text{putting } \bar{P} \text{ from equation (11)}$$

$$\tilde{\chi}_e = \left(\frac{N q^2}{m \epsilon_0} \sum_j \frac{f_j}{(\omega_j - \omega_j^2 - j\gamma_j \omega} \right)$$

Since

$$\tilde{\epsilon} = \epsilon_0 (1 + \tilde{\chi}_e) \quad \text{putting } \tilde{\chi}_e \text{ we get}$$

$$\tilde{\epsilon} = \epsilon_0 \left[1 + \frac{N q^2}{m \epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - 2\gamma_j \omega} \right] \quad \text{---14}$$

Contd: —

Generally the imaginary term is insignificant (for lenear medium) However when "ω" is very Close to of the resonant frequencies ω_j. it plays an important role.

In dispersive medium the wave equation for a given frequency is given as

$$\nabla^2 \tilde{E} = \tilde{\epsilon} \mu_0 \frac{\partial^2 \tilde{E}}{\partial t^2} \quad \text{——(15)}$$

Where $\tilde{\epsilon}$ is now a complex function of ω_j. It gives a plane wave Solution

$$\tilde{E}_{(x,t)} = \tilde{E}_0 e^{i(\tilde{k}x - \omega t)} \quad \text{——(16)}$$

where $\tilde{k}$ wave vector is complex as $\tilde{\epsilon}$ is complex related as

$$\tilde{k} = \omega \sqrt{\tilde{\epsilon} \mu_0} \quad \text{—(17)}$$

$$\text{So let } \tilde{k} = k_+ + i k_- \quad \text{——(18)}$$

$$\tilde{E}_{(x,t)} = \tilde{E}_0 e^{-k_- x} \cdot e^{i(k_+ x - \omega t)} \quad \text{——(19)}$$

Eventually k_- measures the attenuation

contd:

& the wave as the intensity is proportional to $(\tilde{E})^2 = (e^{-\tilde{k}x})^2$

Let $\alpha_A = 2k$ ——(20)

When α_A is called the absorption coefficient

The velocity of the wave

$$v = \frac{\omega}{k_+}$$

& the index of refraction

$$n = \frac{c}{\omega} k_+ \text{ ——— (21)} \quad \text{As } n = \left(\frac{c}{v}\right)$$

For gases the second term in equation (14) is typically small & we can approximate Square root by using binomial expansion $(1+x)^{1/2} = (1 + \frac{1}{2}x + \cdots)$

Since $\tilde{K} = \frac{\omega}{c}\sqrt{\frac{\tilde{\epsilon}}{\epsilon_0}}$ ——— (22)

When $\tilde{\epsilon} = \epsilon_0 \left[1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j\omega}\right]$

putting in eqn (22),

$$\tilde{K} = \frac{\omega}{c}\sqrt{\frac{\epsilon_0}{\epsilon_0}}\left(1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j\omega}\right)$$

$$\tilde{K} = \frac{\omega}{c}\left[1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2) - i\gamma_j\omega}\right]^{1/2}$$

Applying binomial expansion (contd.)

$$\tilde{K} = \frac{\omega}{c}\left[1 + \frac{N q^2 \sum f_j}{2m\epsilon_0 (\omega_j^2 - \omega^2) - i\gamma_j \omega}\right] \quad -(23)$$

Since $$n = \frac{c}{\omega} K_+ \quad -(24)$$

& $$\bar{K} = K_+ + i K_- \quad -(25)$$

First rationalizing equation (23) we get

$$\tilde{K} = \left[\frac{\omega}{c} + \frac{\omega N q^2 \sum f_j (\omega_j^2 - \omega^2)}{2m\epsilon_0 c \left[(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2\right]}\right] + i\left[\frac{\omega N q^2 \sum f_j \omega \gamma_j}{2cm\epsilon_0 \left[(\omega_j^2 - \omega^2)^2 + (\gamma_j \omega)^2\right]}\right] \quad -(26)$$

Comparing the equn (25) & (26), for real & imaginary parts

$$K_+ = \frac{\omega}{c}\left[1 + \frac{N q^2}{2m\epsilon_0} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}\right] \quad -(27)$$

&

$$\alpha_A = 2K_- = \left(\frac{N q^2 \omega^2}{m\epsilon_0 c} \sum_j \frac{f \gamma_j}{(\omega_j^2 - \omega^2)^2 - \gamma_j^2 \omega^2}\right) \quad -28$$