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Electromagnetic Wave Equation in 1/4
a linear homogeneous (free space) medium
Non-conducting! ($\rho = \bar{J} = 0$, $g = 0$)

From Maxwell's Equations
 we have

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad (1)$$

Taking the curl of the both sides of
 equation (1)

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = \bar{\nabla} \times \bar{J} + \bar{\nabla} \times \frac{\partial \bar{D}}{\partial t}$$

$$\text{Since } \bar{D} = \epsilon \bar{E}$$

$$\& \quad \bar{J} = g \bar{E} \text{ when "g" is the conductivity}$$

in field; form of ohms law
 Putting in eq (1)

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = g (\bar{\nabla} \times \bar{E}) + \epsilon \frac{\partial}{\partial t} (\bar{\nabla} \times \bar{E}) \quad (2)$$

Since time & space differentials are
 interchangeable as they are independent
 of each other for a well behaved " \bar{E} ".

From Maxwell's 3rd equation we have

$$\bar{\nabla} \times \bar{E} = - \frac{\partial \bar{B}}{\partial t} \quad \text{putting it in eq (2)}$$

We have

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = -g\mu \frac{\partial \bar{H}}{\partial t} - \epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2} \quad (3)$$

Using the vector identity

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = \bar{\nabla} (\bar{\nabla} \cdot \bar{H}) - \nabla^2 \bar{H}$$

$$\text{Since } \bar{\nabla} \cdot \bar{H} = 0$$

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 \bar{H} \quad \text{putting back in eqn. (3)}$$

$$\nabla^2 \bar{H} = \nabla \left(g\mu \frac{\partial \bar{H}}{\partial t} + \epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2} \right)$$

$$\nabla^2 \bar{H} - g\mu \frac{\partial \bar{H}}{\partial t} - \epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2} = 0 \quad \text{--- (4)}$$

Now for free space $\bar{J}=0$ $\bar{J}=0$ & $g=0$
 putting in eq (4) we get

$$\nabla^2 \bar{H} - \epsilon\mu \frac{\partial^2 \bar{H}}{\partial t^2} = 0 \quad \text{--- (4a)}$$

Now Solving for the \bar{E} field

Let us consider Maxwell's 3rd eqn:

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{--- (5)}$$

Taking the curl of the both sides of eq (5),

$$\nabla \times \nabla \times \bar{E} = -\frac{\partial}{\partial t} (\nabla \times \bar{B}) \quad \text{since } \bar{B} = \mu \bar{H}$$

$$\nabla \times \nabla \times \bar{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \bar{H})$$

Also from Maxwell's 4th eqn we have

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{when } \bar{D} = \epsilon \bar{E} \text{ and } \bar{J} = g \bar{E}$$

$$\nabla \times \bar{H} = g \bar{E} + \epsilon \frac{\partial \bar{E}}{\partial t}$$

We have

$$\nabla \times \nabla \times \bar{E} = -\mu g \frac{\partial \bar{E}}{\partial t} - \epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2} \quad \text{--- (6)}$$

Using the vector ID to the L.H.S
of the above eqn:

$$\nabla \times \nabla \times \bar{E} = \nabla (\nabla \cdot \bar{E}) - \nabla^2 \bar{E}$$

Since $\nabla \cdot \bar{E} = \rho/\epsilon$ & $\rho = 0$ (free space)
 $\nabla \cdot \bar{E} = 0$

We have

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E}$$

$$+\nabla^2 \bar{E} = \nabla \left(\mu g \frac{\partial \bar{E}}{\partial t} + \epsilon \mu \frac{\partial^2 \bar{E}}{\partial t^2} \right)$$

$$\nabla^2 \bar{E} - \mu g \frac{\partial \bar{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \quad \text{--- (7)}$$

For free space $g = 0$

Equation 7 reduces to

$$\nabla^2 \bar{E} - \epsilon \mu \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \quad \text{--- (8)}$$

Now to get a relation the wave equation is position dependent or time independent

Let us define $\bar{E}(x, t) = \bar{E}_s e^{-j\omega t}$
 \bar{E} as space time function.

Where \bar{E}_s is the spatial (space) part.

Putting in (7)

$$\nabla^2 (\bar{E}_s e^{-j\omega t}) - \mu g \frac{\partial}{\partial t} (\bar{E}_s e^{-j\omega t}) - \epsilon \mu \frac{\partial^2}{\partial t^2} (\bar{E}_s e^{-j\omega t})$$

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Let us apply the Space and time Operators one by one to each respective part of the above equation

$$\nabla^2 (\bar{E}_s e^{-j\omega t}) = e^{-j\omega t} \nabla^2 \bar{E}_s$$

$$\frac{\partial}{\partial t} (\bar{E}_s e^{-j\omega t}) = -j\omega \bar{E}_s e^{-j\omega t} = -j\omega \bar{E}_{r,t}$$

$$\frac{\partial^2}{\partial t^2} (\bar{E}_s e^{-j\omega t}) = -\omega^2 \bar{E}_s e^{-j\omega t}$$

Now putting all above terms the respective slots, we have

$$e^{-j\omega t} (\nabla^2 \bar{E}_s + j\sigma\mu\omega \bar{E}_s + \omega^2\mu\epsilon \bar{E}_s) = 0$$

So we have the space dependent wave equation given as

$$\nabla^2 \bar{E}_s + j\sigma\mu\omega \bar{E}_s + \omega^2\mu\epsilon \bar{E}_s = 0$$

For free space $\sigma=0$ the equation will reduce to

$$\nabla^2 \bar{E}_s + \omega^2\mu\epsilon \bar{E}_s = 0$$

Similarly for \bar{H} field we will have

$$\nabla^2 \bar{H}_s + \omega^2\mu\epsilon \bar{H}_s = 0$$

$$\text{As } \bar{H}_{r,t} = \bar{H}_s e^{-j\omega t}$$