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Electromagnetic Wave Equation in a linear homogeneous (free Space) medium Non-conducting $(\rho = \bar{J}) = 0$, $g = 0$

From Maxwell's Equations we have

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{---} (1)$$

Taking the curl of the both sides of equation (1)

$$\bar{\nabla} \times \bar{\nabla} \times \vec{H} = \bar{\nabla} \times \bar{J} + \bar{\nabla} \times \frac{\partial \bar{D}}{\partial t}$$

Since $\bar{D} = \epsilon \bar{E}$

& $\bar{J} = g\bar{E}$ where "g" is the conductivity in field's form of ohms Law

putting in eq (1)

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = g(\bar{\nabla} \times \bar{E}) + \epsilon \frac{\partial}{\partial t}(\bar{\nabla} \times \bar{E}) \quad \text{---} (2)$$

Since time & space differentials are interchangable as they are independent of each other for a well behaved "$\bar{E}$".

From Maxwell's 3rd equation we have

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$ putting it in eq (2)

As $\bar{B} = \mu \bar{H}$

We have

$$\bar{\nabla} \times \bar{\nabla} \times \vec{H} = -g\mu \frac{\partial H}{\partial t} - \epsilon\mu \frac{\partial^2 H}{\partial t^2} \quad \text{---} (3)$$

Using the Vector Identity

$$\bar{\nabla} \times \bar{\nabla} \times \bar{H} = \bar{\nabla}(\bar{\nabla} \cdot \vec{H} - \nabla^2 \vec{H}$$

Since $\bar{\nabla} \cdot \bar{H} = 0$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{H} = -\nabla^2 H$$ putting back in eqn, (3)

$$\cancel{-}\nabla^2 H = \cancel{-}\left(g\mu\frac{\partial H}{\partial t} + \epsilon\mu\frac{\partial^2 H}{\partial t^2}\right)$$

$$\nabla^2 H - g\mu\frac{\partial H}{\partial t} - \epsilon\mu\frac{\partial^2 H}{\partial t^2} = 0 \quad (4)$$

Now for free space $\bar{J}=0$ $\rho=0$ & $g=0$

putting in eq (4) we get

$$\nabla^2 H - \epsilon\mu\frac{\partial^2 H}{\partial t^2} = 0 \quad (4a)$$

Now Solving for the $\bar{E}$ field

Let us consider Maxwell's 3rd eqn:

$$\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \quad (5)$$

Taking the curl of the both sides of eq (5)

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{B})$$ since, $\bar{B}=\mu\bar{H}$

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\mu\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{H})$$

Also from Maxwell's 4th eqn we have

$$\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t}$$ when $\bar{D}=\epsilon\bar{E}$ & $\bar{J}=g\bar{E}$

$$\bar{\nabla}\times\bar{H} = g\bar{E} + \epsilon\frac{\partial\bar{E}}{\partial t}$$

We have

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\mu g\frac{\partial\bar{E}}{\partial t} - \epsilon\mu\frac{\partial^2\bar{E}}{\partial t^2} \quad (6)$$

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Using the vector ID to the L.H.S of the above eqn:

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = \bar{\nabla}(\bar{\nabla}\cdot\bar{E}) - \nabla^2\bar{E}$$

Since $\bar{\nabla}\cdot\bar{E} = \rho/\epsilon$ & $\rho = 0$ (free space)

$\bar{\nabla}\cdot\bar{E} = 0$

we have

$$\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\bar{\nabla}^2 E$$

$$\cancel{+}\bar{\nabla}^2 E = \cancel{+}\left(\mu g \frac{\partial \bar{E}}{\partial t} + \epsilon\mu \frac{\partial^2 E}{\partial t^2}\right)$$

$$\nabla^2 E - g\mu \frac{\partial \bar{E}}{\partial t} - \epsilon\mu \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{---} (7)$$

For free space $g = 0$

Equation 7 reduces to

$$\bar{\nabla}^2 E - \epsilon\mu \frac{\partial^2 \bar{E}}{\partial t^2} = 0 \quad \text{---} (8)$$

Now to get a relation the wave equation is position dependent or time independent

Let us define $\bar{E}(r,t) = \bar{E}_s e^{-j\omega t}$

$\bar{E}$ as space time function.

Where $\bar{E}_s$ is the spatial (space) part.

putting in (7)

$$\nabla^2(\bar{E}_s e^{-j\omega t}) - \mu g \frac{\partial}{\partial t}(\bar{E}_s e^{-j\omega t}) - \epsilon\mu \frac{\partial^2}{\partial t^2}(E_s e^{-j\omega t})$$

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Let us apply the space and time operators one by one to each respective part of the above equation

$$\nabla^2(\bar{E}_s e^{-j\omega t}) = e^{-j\omega t}\nabla^2\bar{E}_s$$

$$\frac{\partial}{\partial t}(\bar{E}_s e^{-j\omega t}) = -j\omega \bar{E}_s e^{-j\omega t}$$
$$= -j\omega \bar{E}_{(r,t)}$$

$$\frac{\partial^2}{\partial t^2}(\bar{E}_s e^{-j\omega t}) = -\omega^2 \bar{E}_s e^{-j\omega t}$$

Now putting all above terms the respective slots, we have

$$\cancel{e^{-j\omega t}}(\nabla^2\bar{E}_s + jg\mu\omega E_s + \omega^2\mu\epsilon \bar{E}_s) = 0$$

So we have the space dependent wave equation given as

$$\nabla^2\bar{E}_s + jg\omega\mu \bar{E}_s + \omega^2\epsilon\mu \bar{E}_s = 0$$

for free space $g=0$ the equation will reduce to

$$\nabla^2\bar{E}_s + \omega^2\epsilon\mu \bar{E}_s = 0$$

Similarly for $\bar{H}$ field we will have $\nabla^2\bar{H}_s + \omega^2\epsilon\mu \bar{H}_s = 0$

As $\bar{H}_{(r,t)} = \bar{H} e^{-j\omega t}$