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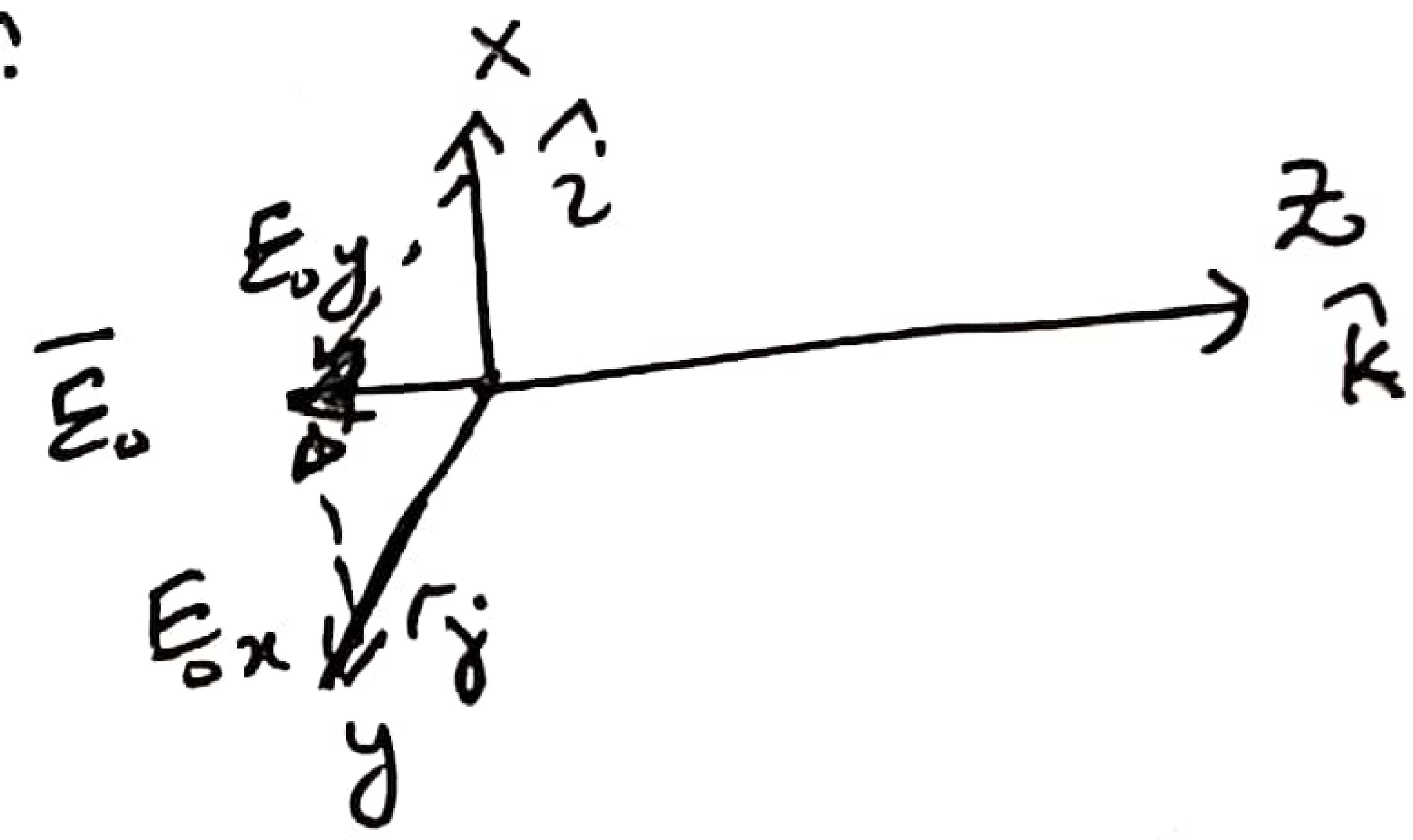
Solution of Plane monochromatic wave equation, travelling in a non-conducting medium:

We have a wave equation for space dependent travelling in a non-conducting medium with $\rho = 0$, $\bar{J} = 0$, $\sigma = 0$ given as

$$\nabla^2 \bar{E}_s + \omega^2 \epsilon\mu \bar{E}_s = 0 \quad \text{—— (1)}$$

Let us consider that the wave is moving in the +ve z-direction with a plane wave having its resultant amplitude

$$\bar{E}_0 = E_{0x}\hat{i} + E_{0y}\hat{j}$$



So we have

$$\bar{E}_s = \bar{E}_{s(z)}$$

putting in eq (1)

$$\frac{d^2 \bar{E}_{s(z)}}{dz^2} + \omega^2 \epsilon\mu \bar{E}_{s(z)} = 0 \quad \text{—— (2)}$$

The above equation is a second order differential equation & its general solution is of the form given as under

$$\bar{E}_{s(z)} = \bar{E}_0 \, e^{\pm(j\omega\sqrt{\epsilon\mu})z} \quad \text{—— (3)}$$

$$\bar{E}_{s(z)} = (E_{0x}\hat{i} + E_{0y}\hat{j}) \, e^{\pm j\omega\sqrt{\epsilon\mu}\, z} \quad \text{—— (4)}$$

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As we defined the $\bar{E}$ function $\left(\frac{2}{5}\right)$

$$\bar{E}_{(z,t)} = E_{s(z)} e^{-j\omega t} \quad (5)$$

putting (4) in eqn (5),

$$\bar{E}_{(z,t)} = (E_{ox}\hat{i} + E_{oy}\hat{j}) e^{\pm j\omega\sqrt{\epsilon\mu}\, z} \cdot e^{-j\omega t}$$

$$\bar{E}_{(z,t)} = \bar{E}_o\, e^{\pm j(\omega\sqrt{\epsilon\mu}\, z - \omega t)} \quad (6) \quad \text{Solution for } \bar{E}$$

Eqn (6) is the Solution for $\bar{E}$

Similarly defining $\bar{B}_{(r,t)} = \bar{B}_{s(z)} e^{-j\omega t}$

The Maxwell's 3rd equation given as

$$(\bar{\nabla} \times \bar{E}_{(z,t)} = -\frac{\partial \bar{B}_{(z,t)}}{\partial t} \quad (7)$$

$$\frac{\partial \bar{B}_{(r,t)}}{\partial t} = -(-j\omega\, B_{s(z)}\, e^{-j\omega t}) = j\omega\, B_{s(z)}\, e^{-j\omega t} \quad (8)$$

putting eqn (8) in eqn (7) we have

$$\bar{\nabla} \times \bar{E}_{s(z)} \cancel{e^{-j\omega t}} = j\omega\, \bar{B}_{s(z)} \cancel{e^{-j\omega t}}$$

$$\therefore \bar{\nabla} \times \bar{E}_{s(z)} = j\omega\, \bar{B}_{s(z)} \quad (9)$$

Now let us express L.H.S of eq (9) in its Component form.

$$\bar{\nabla} \times \bar{E}_{s(z)} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \bar{E}_{ox}\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z} & \bar{E}_{oy}\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z} & 0 \end{vmatrix}$$

Since

$$\bar{E}_{s(z)} = (E_{ox}\hat{i} + E_{oy}\hat{j})\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$

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$$\bar{\nabla}\times \bar{E}_{s(z)} = \hat{i}\left[\cancel{\frac{\partial}{\partial y}(0)} - \frac{\partial}{\partial z}\left(E_{oy}\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z}\right)\right]$$
$$+\hat{j}\left[\frac{\partial}{\partial z}\left(E_{ox}\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z}\right) - \cancel{\frac{\partial}{\partial x}(0)}\right]$$
$$+\hat{k}\left[0 - 0\right]$$

So we are left with

$$\bar{\nabla}\times \bar{E}_{s(z)} = \left(-\hat{i}\, E_{oy}(\pm j\omega\sqrt{\epsilon\mu}) + \hat{j}\, E_{ox}(\pm j\omega\sqrt{\epsilon\mu}\right) \rightarrow$$
$$\rightarrow e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$

$$\bar{\nabla}\times \bar{E}_{s(z)} = \underline{\left[-\hat{i}\, E_{oy} + \hat{j}\, E_{ox}\right]}(\pm j\omega\sqrt{\epsilon\mu}) e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$

We can put

$$\hat{k}\times \bar{E}_{s(z)} = \hat{k}\times(E_{ox}\hat{i} + E_{oy}\hat{j})\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$
$$= \underline{(+\hat{j}E_{ox} - \hat{i}E_{oy})}\, e^{\pm j\omega\sqrt{\epsilon\mu}\, z}$$

So replacing the term we get

$$\bar{\nabla}\times \bar{E}_{s(z)} = \hat{k}\times \bar{E}_{s(z)}(\pm j\sqrt{\epsilon\mu}\,\omega)$$

Now putting the above value of $\bar{\nabla}\times \bar{E}_{s(z)}$ in eq (9) we get

$$\cancel{j\omega}\, \bar{B}_{s(z)} = \pm \cancel{j\omega}\sqrt{\epsilon\mu}\; \hat{k}\times \bar{E}_{s(z)}$$

$$\bar{B}_{s(z)} = \pm\sqrt{\epsilon\mu}\; \hat{k}\times \bar{E}_{s(z)} \qquad (10)$$

$$\& \quad \bar{B}_{(z,t)} = \bar{B}_{s(z)}\, e^{-j\omega t}$$

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$$\overline{B}_{(z,t)} = \pm\sqrt{\epsilon\mu}\ \hat{k} \times e^{\pm j\omega\sqrt{\epsilon\mu}z - j\omega t}$$

$$\overline{B}_{(z,t)} = \pm\sqrt{\epsilon\mu}\ \hat{k} \times e^{\pm j\omega(\sqrt{\epsilon\mu}z - t)} \quad \text{—— (11)}$$

Equation (6) & (11) are the solutions for $\overline{E}_{(z,t)}$ & $\overline{B}_{(z,t)}$ fields

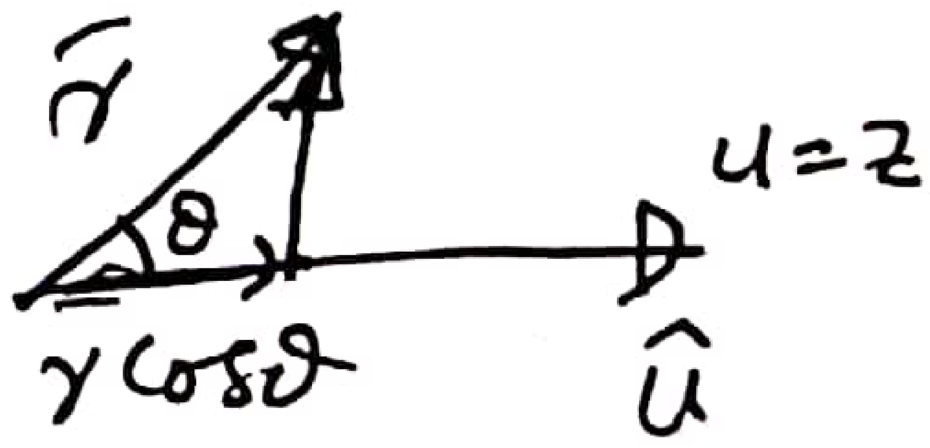
In both the equation (6) & (11) wave is propagating along the +ve z direction.

Let us assume that the wave is moving in an arbitrary direction $\hat{u}$ where $\hat{u}$ plays the role of $\hat{k}$.

So z is now replaced by "u" any arbitrary coordinate & $\overline{u} = u\hat{u}$

So we replace $z \Rightarrow r\cos\theta$

$u = \hat{u}\cdot\overline{r} = r\cos\theta$



$u = r\cos\theta$
$= \hat{u}\cdot\overline{r}$,
$u = r\cos\theta$

Now the field vectors $\overline{E}_{(r,t)}$ & $B_{(r,t)}$ are given as

$$\overline{E}_{(r,t)} = \overline{E}_0\, e^{\pm j\omega(\sqrt{\epsilon\mu}\,\hat{u}\cdot\overline{r} - t)} \quad \text{—— (12)}$$

& $$\overline{B}(r,t) = \pm\sqrt{\epsilon\mu}\ \hat{u} \times \overline{E}_0\, e^{\pm j\omega(\sqrt{\epsilon\mu}\,\hat{u}\cdot\overline{r} - t)} \quad \text{—— (13)}$$

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let us say $\bar{K} = \pm\omega\sqrt{\epsilon\mu}\,\hat{u}$ (5) $\overline{5}$

then

$$\bar{E}_{(r,t)} = \bar{E}_0 e^{j(\bar{K}\cdot\bar{r} - \omega t)} \quad \text{——— (14)}$$

$$\& \; \bar{B}_{(r,t)} = \frac{1}{\omega} \bar{K} \times e^{\pm j(\bar{K}\cdot\bar{r} - \omega t)} \quad \text{——— (15)}$$

Where $\bar{K}$ is called a wave vector or a wave number.

If the wave is moving with constant phase

$$\bar{K}\cdot\bar{r} - \omega t = \text{constant} \text{ — (16)}$$

Differentiating eq (16) w.r.t "t"

$$\frac{d}{dt}(\bar{K}\cdot\bar{r} - \omega t) = 0 \qquad \text{let } \bar{K} \parallel \bar{r}$$

$$K\frac{dr}{dt} - \omega = 0$$

$$\frac{dr}{dt} = v = \frac{\omega}{K}$$

Since $k = \omega\sqrt{\epsilon\mu}$

$$v = \frac{\omega}{\omega\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon\mu}}$$ gives the

Speed of the wave called phase velocity $v = \dfrac{1}{\sqrt{\epsilon\mu}}$