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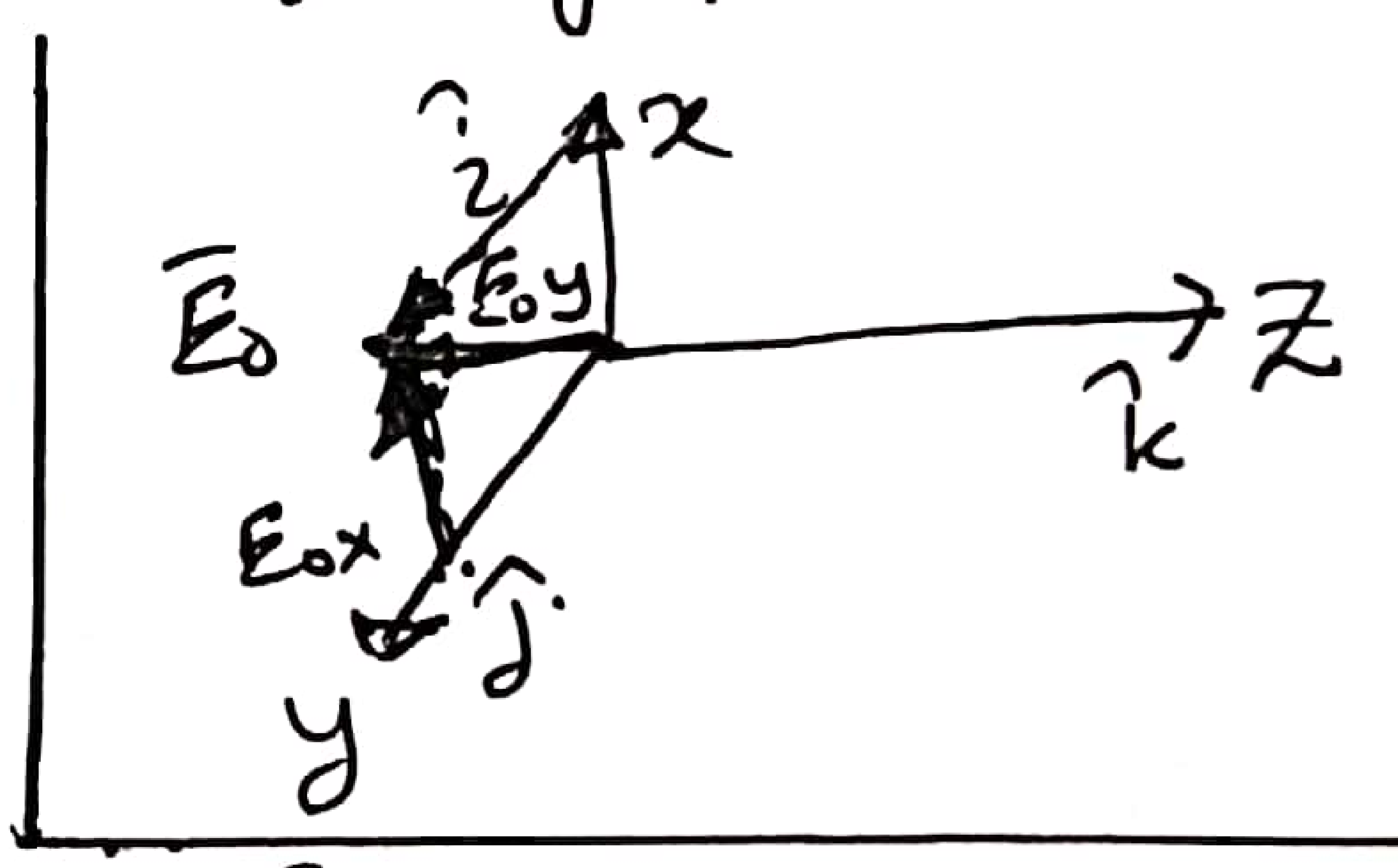
Solution of Plane monochromatic Wave equation travelling in a Conducting medium:

We have the plane monochromatic wave with $\bar{E}_{(s)}$ field, space dependent wave equation given as

$$\nabla^2 \bar{E}_{(s)} + \omega^2 \epsilon \mu \bar{E}_{(s)} + j g \omega \mu \bar{E}_{(s)} = 0 \quad \text{—(1)}$$

When g is the conductivity of the medium.

Let us consider that Wave is moving along the +ve z-direction & its amplitude $\bar{E}_0$ is having xy-plane



The equation (1) is written as

$$\frac{d^2 \bar{E}_{s(z)}}{dz^2} + \omega^2 \epsilon \mu \bar{E}_{s(z)} + j \omega g \mu E_{s(z)} = 0 \quad \text{—(2)}$$

Let us consider that

$$\left.\begin{aligned} E_{s(z)} &= \bar{E}_0 e^{j\gamma z} \\ \frac{d^2 E_{s(z)}}{dz^2} &= -\gamma^2 \bar{E}_0 e^{j\gamma z} \end{aligned}\right] \text{putting in eqn (2)}$$

$$-\gamma^2 \bar{E}_0 \cancel{e^{j\gamma z}} + \omega^2 \epsilon \mu \bar{E}_0 \cancel{e^{j\gamma z}} + j\omega g \bar{E}_0 \cancel{e^{j\gamma z}} = 0$$

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$$-\gamma^2 + \omega^2\epsilon\mu + j\omega g\mu = 0$$

or
$$\gamma^2 = \omega^2\epsilon\mu + j\omega g\mu \quad \text{--- (3)}$$

Let $\gamma = \alpha + j\beta$ putting in eq (3)

$$\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + 2j\alpha\beta \quad \text{--- (4)}$$

$$(\alpha^2 - \beta^2) + 2j\alpha\beta = \omega^2\epsilon\mu + j\omega g\mu \quad \text{--- (5)}$$

Comparing the Real & Imaginary parts of eq (5)

$$\alpha^2 - \beta^2 = \omega^2\epsilon\mu \quad \text{--- (6)}$$

$$2\alpha\beta = \omega g\mu \quad \text{--- (7)}$$

From eq (7) $\beta = \dfrac{\omega g\mu}{2\alpha}$ putting in eq (6)

$$\alpha^2 - \left(\frac{\omega g\mu}{2\alpha}\right)^2 = \omega^2\epsilon\mu$$

after Simplifying we get

$$4\alpha^4 - (4\omega^2\epsilon\mu)\alpha^2 - \omega^2 g^2\mu^2 = 0 \quad \text{--- (8)}$$

Equation (8) is a quadratic equation

$$(\alpha^2)^2 - 4\omega^2\epsilon\mu(\alpha^2)^1 - \omega^2 g^2\mu^2 = 0$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $a = 4$, $b = -4\omega^2\epsilon\mu$ & $c = -\omega^2 g^2\mu^2$

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$$\alpha^2 = \frac{4\omega^2\epsilon\mu \pm \sqrt{(4\omega^2\epsilon\mu)^2 - 4(-4\omega^2 g^2}}{2\times 4}$$

$$= \frac{\cancel{4}\omega^2\epsilon\mu \pm \cancel{4}\sqrt{\omega^4\epsilon^2\mu^2 + \omega^2 g^2\mu^2}}{2(\cancel{4})}$$

$$= \omega^2\epsilon\mu \pm \omega^2(\epsilon\mu)\sqrt{1 + \frac{g^2}{\omega^2\epsilon^2}} \Big/ 2$$

$$\alpha^2 = \omega^2\epsilon\mu\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + g^2/\omega^2 c^2}\right]$$

$$\alpha = \pm\,\omega\sqrt{\epsilon\mu}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + g^2/\omega^2\epsilon^2}\right]^{1/2}$$

Since $\beta = \frac{\omega g\mu}{2\alpha}$ (putting the value of (α))

$$\beta = \frac{\omega g\mu}{\pm 2\omega\sqrt{\epsilon\mu}\left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + g^2/\omega^2\epsilon^2}\right]^{1/2}}$$

The plane wave travelling in the +ve z direction with time given by $\bar{E}_{(z,t)} = E_{s,z}\, e^{-j\omega t} = \bar{E}_0 . e^{j\gamma z} . e^{-j\omega t}$

Solution

$$\bar{E}_{z,t} = \bar{E}_0\, e^{j(\alpha z - \omega t)} . e^{-\beta z} \quad (9)$$

putting ($\gamma = \alpha + j\beta$) in the above eqn (9)

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Equation (9) represents an exponentially damped wave travelling in the +ve-Z axis ($e^{-\beta z}$) factor is the damping part as $\beta = \frac{\omega g \mu}{2\alpha}$ containing "g" conductivity part which as denominator $\left(\frac{1}{e^{\beta z}}\right)$ offering the resistive part causing the damping.

Now let us solve for $\bar{B}_{(z,t)}$

We have the Maxwell's 3rd equation

$$\bar{\nabla} \times \bar{E}_{(z,t)} = -\frac{\partial \bar{B}_{(z,t)}}{\partial t} \quad \text{--- (10)}$$

Let $\bar{B}_{(z,t)} = \bar{B}_{s_z} e^{-j\omega t}$

R.H.S $\rightarrow$ $-\frac{\partial \bar{B}_{(z,t)}}{\partial t} = +j\omega \bar{B}_{(z,t)}$ --- (11)

Taking the L.H.S of the eqn (10)

$$\bar{\nabla} \times \bar{E}_{(z,t)} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ \{E_{0x} e^{j(\alpha z - \omega t)} \cdot e^{-\beta z}\} & \{E_{0y} e^{j(\alpha z - \omega t)} e^{-\beta z}\} & \{0\} \end{vmatrix}$$

after simplifying we get

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$$\left(\bar{\nabla}\times\bar{E}_{z,t}\right) = j(\alpha+j\beta)\left[-i E_{oy} + j E_{ox}\right] e^{j(\alpha+\beta j)z - j\omega t}$$

$$\left(\bar{\nabla}\times\bar{E}_{z,t}\right) = j(\alpha+j\beta)\ \hat{k}\times\bar{E}_o\, e^{j\gamma z}\cdot e^{j\omega t} \quad \text{—(12)}$$

Comparing equations (11) & (12),

$$-\cancel{j}\omega\,\bar{B}_{s(z)}\,\cancel{e^{-j\omega t}} = \cancel{j}(\gamma)\ \hat{k}\times \bar{E}_{s(z)}\,\cancel{e^{-j\omega t}}$$

$$\bar{B}_{s(z)} = \frac{\gamma}{\omega}\,\hat{k}\times\bar{E}_{s(z)}$$

or

$$\bar{B}_{(z,t)} = \frac{\gamma}{\omega}\,\hat{k}\times\bar{E}_o\, e^{j(\gamma z-\omega t)} \quad \text{—(13)}$$

The equations (9) & (13) are the Solutions for the given Wave equation, in terms of $\bar{E}$ & $\bar{B}$