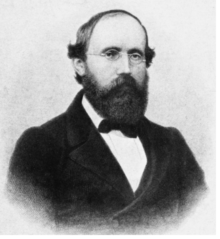
CHAPTER 7

THE RIEMANN INTEGRAL

We have already mentioned the developments, during the 1630s, by Fermat and Descartes leading to analytic geometry and the theory of the derivative. However, the subject we know as calculus did not begin to take shape until the late 1660s when Isaac Newton created his theory of "fluxions" and invented the method of "inverse tangents" to find areas under curves. The reversal of the process for finding tangent lines to find areas was also discovered in the 1680s by Gottfried Leibniz, who was unaware of Newton's unpublished work and who arrived at the discovery by a very different route. Leibniz introduced the terminology "calculus differentialis" and "calculus integralis," since finding tangent lines involved differences and finding areas involved summations. Thus, they had discovered that integration, being a process of summation, was inverse to the operation of differentiation.

During a century and a half of development and refinement of techniques, calculus consisted of these paired operations and their applications, primarily to physical problems. In the 1850s, Bernhard Riemann adopted a new and different viewpoint. He separated the concept of integration from its companion, differentiation, and examined the motivating summation and limit process of finding areas by itself. He broadened the scope by considering all functions on an interval for which this process of "integration" could be defined: the class of "integrable" functions. The Fundamental Theorem of Calculus became a result that held only for a restricted set of integrable functions. The viewpoint of Riemann led others to invent other integration theories, the most significant being Lebesgue's theory of integration. But there have been some advances made in more recent times that extend even the Lebesgue theory to a considerable extent. We will give a brief introduction to these results in Chapter 10.

Bernhard Riemann



(Georg Friedrich) Bernhard Riemann (1826–1866), the son of a poor Lutheran minister, was born near Hanover, Germany. To please his father, he enrolled (1846) at the University of Göttingen as a student of theology and philosophy, but soon switched to mathematics. He interrupted his studies at Göttingen to study at Berlin under C. G. J. Jacobi, P. G. J. Dirichlet, and F. G. Eisenstein, but returned to Göttingen in 1849 to complete his thesis under Gauss. His thesis dealt with what are now called "Riemann surfaces." Gauss was so enthusiastic about Riemann's work that he arranged for him to become a *privatdozent* at Göttingen in 1854. On admission as a *privatdozent*, Riemann was required to prove himself by delivering a probationary lecture before the entire faculty. As tradition dictated, he submitted three topics, the first two of which he was well prepared to discuss. To Riemann's surprise, Gauss chose that he should lecture on the third topic: "On the hypotheses that underlie the foundations of geometry." After its publication, this lecture had a profound effect on modern geometry.

Despite the fact that Riemann contracted tuberculosis and died at the age of 39, he made major contributions in many areas: the foundations of geometry, number theory, real and complex analysis, topology, and mathematical physics.

We begin by defining the concept of Riemann integrability of real-valued functions defined on a closed bounded interval of $\mathbb{R}$, using the Riemann sums familiar to the reader from calculus. This method has the advantage that it extends immediately to the case of functions whose values are complex numbers, or vectors in the space $\mathbb{R}^n$. In Section 7.2, we will establish the Riemann integrability of several important classes of functions: step functions, continuous functions, and monotone functions. However, we will also see that there are functions that are *not* Riemann integrable. The Fundamental Theorem of Calculus is the principal result in Section 7.3. We will present it in a form that is slightly more general than is customary and does not require the function to be a derivative at every point of the interval. A number of important consequences of the Fundamental Theorem are also given. In Section 7.3 we also give a statement of the definitive Lebesgue Criterion for Riemann integrability. This famous result is usually not given in books at this level, since its proof (given in Appendix C) is somewhat complicated. However, its statement is well within the reach of students, who will also comprehend the power of this result. In Section 7.4, we discuss an alternative approach to the Riemann integral due to Gaston Darboux that uses the concepts of upper integral and lower integral. The two approaches appear to be quite different, but in fact they are shown to be equivalent. The final section presents several methods of approximating integrals, a subject that has become increasingly important during this era of high-speed computers. While the proofs of these results are not particularly difficult, we defer them to Appendix D.

An interesting history of integration theory, including a chapter on the Riemann integral, is given in the book by Hawkins cited in the References.

Section 7.1 Riemann Integral

We will follow the procedure commonly used in calculus courses and define the Riemann integral as a kind of limit of the Riemann sums as the norm of the partitions tend to 0. Since we assume that the reader is familiar—at least informally—with the integral from a calculus course, we will not provide a motivation of the integral, or discuss its interpretation as the "area under the graph," or its many applications to physics, engineering, economics, etc. Instead, we will focus on the purely mathematical aspects of the integral.

However, we first define some basic terms that will be frequently used.

Partitions and Tagged Partitions

If $I := [a, b]$ is a closed bounded interval in $\mathbb{R}$, then a **partition** of I is a finite, ordered set $\mathcal{P} := (x_0, x_1, \ldots, x_{n-1}, x_n)$ of points in I such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

(See Figure 7.1.1.) The points of $\mathcal{P}$ are used to divide $I = [a, b]$ into non-overlapping subintervals

$$I_1 := [x_0, x_1], \quad I_2 := [x_1, x_2], \ldots, \quad I_n := [x_{n-1}, x_n].$$

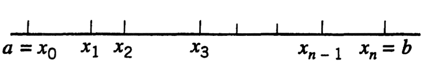


Figure 7.1.1 A partition of $[a, b]$

Often we will denote the partition $\mathcal{P}$ by the notation $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^n$. We define the **norm** (or **mesh**) of $\mathcal{P}$ to be the number

(1) $$||\mathcal{P}|| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Thus the norm of a partition is merely the length of the largest subinterval into which the partition divides $[a, b]$. Clearly, many partitions have the same norm, so the partition is *not* a function of the norm.

If a point t_i has been selected from each subinterval $I_i = [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$, then the points are called **tags** of the subintervals I_i. A set of ordered pairs

$$\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$$

of subintervals and corresponding tags is called a **tagged partition** of I; see Figure 7.1.2. (The dot over the $\mathcal{P}$ indicates that a tag has been chosen for each subinterval.) The tags can be chosen in a wholly arbitrary fashion; for example, we can choose the tags to be the left endpoints, or the right endpoints, or the midpoints of the subintervals, etc. Note that an endpoint of a subinterval can be used as a tag for two consecutive subintervals. Since each tag can be chosen in infinitely many ways, each partition can be tagged in infinitely many ways. The norm of a tagged partition is defined as for an ordinary partition and does not depend on the choice of tags.

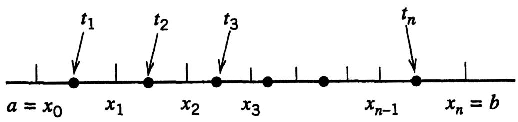


Figure 7.1.2 A tagged partition of $[a, b]$

If $\dot{\mathcal{P}}$ is the tagged partition given above, we define the **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ corresponding to $\dot{\mathcal{P}}$ to be the number

(2) $$S(f; \dot{\mathcal{P}}) := \sum_{i=1}^n f(t_i)(x_i - x_{i-1}).$$

We will also use this notation when $\dot{\mathcal{P}}$ denotes a *subset* of a partition, and not the entire partition.

The reader will perceive that if the function f is positive on $[a, b]$, then the Riemann sum (2) is the sum of the areas of n rectangles whose bases are the subintervals $I_i = [x_{i-1}, x_i]$ and whose heights are $f(t_i)$. (See Figure 7.1.3.)

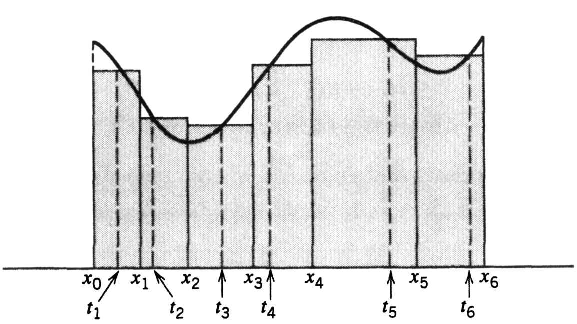


Figure 7.1.3 A Riemann sum

Definition of the Riemann Integral

We now define the Riemann integral of a function f on an interval $[a, b]$.

7.1.1 Definition A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$\left|S(f; \dot{\mathcal{P}}) - L\right| < \varepsilon.$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

Remark It is sometimes said that the integral L is "the limit" of the Riemann sums $S(f : \dot{\mathcal{P}})$ as the norm $||\dot{\mathcal{P}}|| \to 0$. However, since $S(f; \dot{\mathcal{P}})$ is not a function of $||\dot{\mathcal{P}}||$, this **limit** is not of the type that we have studied before.

First we will show that if $f \in \mathcal{R}[a, b]$, then the number L is uniquely determined. It will be called the **Riemann integral of** f over $[a, b]$. Instead of L, we will usually write

$$L = \int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

It should be understood that any letter other than x can be used in the latter expression, so long as it does not cause any ambiguity.

7.1.2 Theorem *If $f \in \mathcal{R}[a, b]$, then the value of the integral is uniquely determined.*

Proof. Assume that L' and L'' both satisfy the definition and let $\varepsilon > 0$. Then there exists $\delta'_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_1$ is any tagged partition with $||\dot{\mathcal{P}}_1|| < \delta'_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}_1) - L'\right| < \varepsilon/2.$$

Also there exists $\delta''_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_2$ is any tagged partition with $||\dot{\mathcal{P}}_2|| < \delta''_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}_2) - L''\right| < \varepsilon/2.$$

Now let $\delta_\varepsilon := \min\{\delta'_{\varepsilon/2}, \delta''_{\varepsilon/2}\} > 0$ and let $\dot{\mathcal{P}}$ be a tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$. Since both $||\dot{\mathcal{P}}|| < \delta'_{\varepsilon/2}$ and $||\dot{\mathcal{P}}|| < \delta''_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}) - L'\right| < \varepsilon/2 \qquad \text{and} \qquad \left|S(f; \dot{\mathcal{P}}) - L''\right| < \varepsilon/2,$$

whence it follows from the Triangle Inequality that

$$\begin{aligned} |L' - L''| &= \left|L' - S(f; \dot{\mathcal{P}}) + S(f; \dot{\mathcal{P}}) - L''\right| \\ &\le \left|L' - S(f; \dot{\mathcal{P}})\right| + \left|S(f; \dot{\mathcal{P}}) - L''\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $L' = L''$. Q.E.D.

The next result shows that changing a function at a finite number of points does not affect its integrability nor the value of its integral.

7.1.3 Theorem If g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, then f is Riemann integrable and $\int_a^b f = \int_a^b g$.

Proof. If we prove the assertion for the case of one exceptional point, then the extension to a finite number of points is done by a standard induction argument, which we leave to the reader.

Let c be a point in the interval and let $L = \int_a^b g$. Assume that $f(x) = g(x)$ for all $x \neq c$. For any tagged partition $\dot{\mathcal{P}}$, the terms in the two sums $S(f;\, \dot{\mathcal{P}})$ and $S(g;\, \dot{\mathcal{P}})$ are identical with the exception of at most two terms (in the case that $c = x_i = x_{i-1}$ is an endpoint). Therefore, we have

$$\left|S(f;\, \dot{\mathcal{P}}) - S(g;\, \dot{\mathcal{P}})\right| = |\Sigma(f(x_i) - g(x_i))(x_i - x_{i-1})| \leq 2(|g(c)| + |f(c)|)\, ||\dot{\mathcal{P}}||.$$

Now, given $\varepsilon > 0$, we let $\delta_1 > 0$ satisfy $\delta_1 < \varepsilon/(4(|f(c)| + |g(c)|)$, and let $\delta_2 > 0$ be such that $||\dot{\mathcal{P}}|| < \delta_2$ implies $\left|S(g;\, \dot{\mathcal{P}}) - L\right| < \varepsilon/2$. We now let $\delta := \min\{\delta_1,\, \delta_2\}$. Then, if $||\dot{\mathcal{P}}|| < \delta$, we obtain

$$\left|S(f;\, \dot{\mathcal{P}}) - L\right| \leq \left|S(f;\, \dot{\mathcal{P}}) - S(g;\, \dot{\mathcal{P}})\right| + \left|S(g;\, \dot{\mathcal{P}}) - L\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, the function f is integrable with integral L. Q.E.D.

Some Examples

If we use only the definition, in order to show that a function f is Riemann integrable we must (i) know (or guess correctly) the value L of the integral, and (ii) construct a δ_ε that will suffice for an arbitrary $\varepsilon > 0$. The determination of L is sometimes done by calculating Riemann sums and guessing what L must be. The determination of δ_ε is likely to be difficult.

In actual practice, we usually show that $f \in \mathcal{R}[a,\, b]$ by making use of some of the theorems that will be given later.

7.1.4 Examples **(a)** Every constant function on $[a,\, b]$ is in $\mathcal{R}[a,\, b]$.

Let $f(x) := k$ for all $x \in [a,\, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a,\, b]$, then it is clear that

$$S(f;\, \dot{\mathcal{P}}) = \sum_{i=1}^{n} k(x_i - x_{i-1}) = k(b - a).$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon := 1$ so that if $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$\left|S(f;\, \dot{\mathcal{P}}) - k(b - a)\right| = 0 < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = k(b - a)$.

(b) Let $g : [0,\, 3] \to \mathbb{R}$ be defined by $g(x) := 2$ for $0 \leq x \leq 1$, and $g(x) := 3$ for $1 < x \leq 3$. A preliminary investigation, based on the graph of g (see Figure 7.1.4), suggests that we might expect that $\int_0^3 g = 8$.

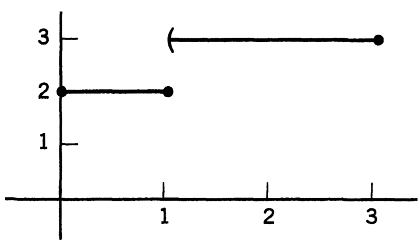


Figure 7.1.4 Graph of g

Let $\dot{\mathcal{P}}$ be a tagged partition of [0,3] with norm $< \delta$; we will show how to determine δ in order to ensure that $|S(g;\dot{\mathcal{P}}) - 8| < \varepsilon$. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having its tags in [0,1] where $g(x) = 2$, and let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ with its tags in (1, 3] where $g(x) = 3$. It is obvious that we have

$$(3) \qquad S(g;\dot{\mathcal{P}}) = S(g;\dot{\mathcal{P}}_1) + S(g;\dot{\mathcal{P}}_2).$$

If we let U_1 denote the union of the subintervals in $\dot{\mathcal{P}}_1$, then it is readily shown that

$$(4) \qquad [0,\, 1-\delta] \subset U_1 \subset [0,\, 1+\delta].$$

For example, to prove the first inclusion, we let $u \in [0,\, 1-\delta]$. Then u lies in an interval $I_k := [x_{k-1}, x_k]$ of $\dot{\mathcal{P}}_1$, and since $||\dot{\mathcal{P}}|| < \delta$, we have $x_k - x_{k-1} < \delta$. Then $x_{k-1} \le u \le 1-\delta$ implies that $x_k \le x_{k-1} + \delta \le (1-\delta) + \delta \le 1$. Thus the tag t_k in I_k satisfies $t_k \le 1$ and therefore u belongs to a subinterval whose tag is in [0,1], that is, $u \in U_1$. This proves the first inclusion in (4), and the second inclusion can be shown in the same manner. Since $g(t_k) = 2$ for the tags of $\dot{\mathcal{P}}_1$ and since the intervals in (4) have lengths $1-\delta$ and $1+\delta$, respectively, it follows that

$$2(1-\delta) \le S(g;\dot{\mathcal{P}}_1) \le 2(1+\delta).$$

A similar argument shows that the union of all subintervals with tags $t_i \in (1,3]$ contains the interval $[1+\delta, 3]$ of length $2-\delta$, and is contained in $[1-\delta,\, 3]$ of length $2+\delta$. Therefore,

$$3(2-\delta) \le S(g;\dot{\mathcal{P}}_2) \le 3(2+\delta).$$

Adding these inequalities and using equation (3), we have

$$8 - 5\delta \le S(g;\dot{\mathcal{P}}) = S(g;\dot{\mathcal{P}}_1) + S(g;\dot{\mathcal{P}}_2) \le 8 + 5\delta,$$

whence it follows that

$$|S(g;\dot{\mathcal{P}}) - 8| \le 5\delta.$$

To have this final term $< \varepsilon$, we are led to take $\delta_\varepsilon < \varepsilon/5$.

Making such a choice (for example, if we take $\delta_\varepsilon := \varepsilon/10$), we can retrace the argument and see that $|S(g;\dot{\mathcal{P}}) - 8| < \varepsilon$ when $||\dot{\mathcal{P}}|| < \delta_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have proved that $g \in \mathcal{R}[0,\, 3]$ and that $\int_0^3 g = 8$, as predicted,

(c) Let $h(x) := x$ for $x \in [0,\, 1]$; we will show that $h \in \mathcal{R}[0,\, 1]$.

We will employ a "trick" that enables us to guess the value of the integral by considering a particular choice of the tag points. Indeed, if $\{I_i\}_{i=1}^n$ is any partition of [0,1] and we choose the tag of the interval $I_i = [x_{i-1}, x_i]$ to be the midpoint $q_i := \frac{1}{2}(x_{i-1} + x_i)$, then the contribution of this term to the Riemann sum corresponding to the tagged partition $\dot{\mathcal{Q}} := \{(I_i, q_i)\}_{i=1}^n$ is

$$h(q_i)(x_i - x_{i-1}) = \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \frac{1}{2}\left(x_i^2 - x_{i-1}^2\right).$$

If we add these terms and note that the sum telescopes, we obtain

$$S(h;\dot{\mathcal{Q}}) = \sum_{i=1}^{n} \frac{1}{2}\left(x_i^2 - x_{i-1}^2\right) = \frac{1}{2}\left(1^2 - 0^2\right) = \frac{1}{2}.$$

Now let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ be an arbitrary tagged partition of [0,1] with $||\dot{\mathcal{P}}|| < \delta$ so that $x_i - x_{i-1} < \delta$ for $i = 1, \ldots, n$. Also let $\dot{\mathcal{Q}}$ have the same partition points, but where we choose the tag q_i to be the midpoint of the interval I_i. Since both t_i and q_i belong to this interval, we have $|t_i - q_i| < \delta$. Using the Triangle Inequality, we deduce

$$\begin{aligned}\left|S(h;\dot{\mathcal{P}}) - S(h;\ \dot{\mathcal{Q}})\right| &= \left|\sum_{i=1}^{n} t_i(x_i - x_{i-1}) - \sum_{i-1}^{n} q_i(x_i - x_{i-1})\right| \\ &\leq \sum_{i=1}^{n} |t_i - q_i|(x_i - x_{i-1}) < \delta \sum_{i=1}^{n}(x_i - x_{i-1}) = \delta(x_n - x_0) = \delta.\end{aligned}$$

Since $S(h;\ \dot{\mathcal{Q}}) = \frac{1}{2}$, we infer that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta$, then

$$\left|S(h;\ \dot{\mathcal{P}}) - \frac{1}{2}\right| < \delta.$$

Therefore we are led to take $\delta_\varepsilon \leq \varepsilon$. If we choose $\delta_\varepsilon := \varepsilon$, we can retrace the argument to conclude that $h \in \mathcal{R}[0,\ 1]$ and $\int_0^1 h = \int_0^1 x\,dx = \frac{1}{2}$.

(d) Let $G(x) := 1/n$ for $x = 1/n$ $(n \in \mathbb{N})$, and $G(x) := 0$ elsewhere on $[0,\ 1]$.

Given $\varepsilon > 0$, the set $E := \{x : G(x) \geq \varepsilon\}$ is a finite set. (For example, if $\varepsilon = 1/10$, then $E = \{1,\ 1/2,\ 1/3, \ldots, 1/10\}$.) If n is the number of points in E, we allow for the possibility that a tag may be counted twice if it is an endpoint and let $\delta := \varepsilon/2n$. For a given tagged partition $\dot{\mathcal{P}}$ such that $||\dot{\mathcal{P}}|| < \delta$, we let $\dot{\mathcal{P}}_0$ be the subset of $\dot{\mathcal{P}}$ with all tags outside of E and let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ with one or more tags in E. Since $G(x) < \varepsilon$ for each x outside of E and $G(x) \leq 1$ for all x in [0,1], we get

$$0 \leq S(G;\ \dot{\mathcal{P}}) = S(G;\ \dot{\mathcal{P}}_0) + S(G;\ \dot{\mathcal{P}}_1) < \varepsilon + (2n)\delta = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that G is Riemann integrable with integral equal to zero. □

Some Properties of the Integral

The difficulties involved in determining the value of the integral and of δ_ε suggest that it would be very useful to have some general theorems. The first result in this direction enables us to form certain algebraic combinations of integrable functions.

7.1.5 Theorem *Suppose that f and g are in $\mathcal{R}[a,\ b]$. Then:*

(a) If $k \in \mathbb{R}$, *the function kf is in $\mathcal{R}[a,\ b]$ and*

$$\int_a^b kf = k\int_a^b f.$$

(b) *The function $f + g$ is in $\mathcal{R}[a,\ b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

(c) *If $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a, b]$, then it is an easy exercise to show that

$$S(kf; \dot{\mathcal{P}}) = kS(f; \dot{\mathcal{P}}), \qquad S(f+g; \dot{\mathcal{P}}) = S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}),$$
$$S(f; \dot{\mathcal{P}}) \le S(g; \dot{\mathcal{P}}).$$

We leave it to the reader to show that the assertion (a) follows from the first equality. As an example, we will complete the proofs of (b) and (c).

Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem 7.1.2 to construct a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then both

$$(5) \qquad \left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon/2 \quad \text{and} \quad \left| S(g; \dot{\mathcal{P}}) - \int_a^b g \right| < \varepsilon/2.$$

To prove (b), we note that

$$\begin{aligned} \left| S(f+g; \dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g \right) \right| &= \left| S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}) - \int_a^b f - \int_a^b g \right| \\ &\le \left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| + \left| S(g; \dot{\mathcal{P}}) - \int_a^b g \right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f + g \in \mathcal{R}[a, b]$ and that its integral is the sum of the integrals of f and g.

To prove (c), we note that the Triangle Inequality applied to (5) implies

$$\int_a^b f - \varepsilon/2 < S(f; \dot{\mathcal{P}}) \quad \text{and} \quad S(g; \dot{\mathcal{P}}) < \int_a^b g + \varepsilon/2.$$

If we use the fact that $S(f; \dot{\mathcal{P}}) \le S(g; \dot{\mathcal{P}})$, we have

$$\int_a^b f \le \int_a^b g + \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that $\int_a^b f \le \int_a^b g$. Q.E.D.

Boundedness Theorem

We now show that an unbounded function cannot be Riemann integrable.

7.1.6 Theorem *If $f \in \mathcal{R}[a, b]$, then f is bounded on $[a, b]$.*

Proof. Assume that f is an unbounded function in $\mathcal{R}[a, b]$ with integral L. Then there exists $\delta > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta$, then we have $|S(f; \dot{\mathcal{P}}) - L| < 1$, which implies that

$$(6) \qquad |S(f; \dot{\mathcal{P}})| < |L| + 1.$$

Now let $\mathcal{Q} = \{[x_{i-1}, x_i]\}_{n=1}^n$ be a partition of $[a, b]$ with $\|\mathcal{Q}\| < \delta$. Since $|f|$ is not bounded on $[a, b]$, then there exists at least one subinterval in $\mathcal{Q}$, say $[x_{k-1}, x_k]$, on which $|f|$ is not bounded—for, if $|f|$ is bounded on each subinterval $[x_{i-1}, x_i]$ by M_i, then it is bounded on $[a, b]$ by $\max\{M_1, \ldots, M_n\}$.

We will now pick tags for $\mathcal{Q}$ that will provide a contradiction to (6). We tag $\mathcal{Q}$ by $t_t := x_i$ for $i \neq k$ and we pick $t_k \in [x_{k-1}, x_k]$ such that

$$|f(t_k)(x_k - x_{k-1})| > |L| + 1 + \Big|\sum_{i\neq k} f(t_i)(x_i - x_{i-1})\Big|.$$

From the Triangle Inequality (in the form $|A + B| \geq |A| - |B|$), we have

$$\big|S(f; \dot{\mathcal{Q}})\big| \geq |f(t_k)(x_k - x_{k-1})| - \Big|\sum_{i\neq k} f(t_i)(x_i - x_{i-1})\Big| > |L| + 1,$$

which contradicts (6). Q.E.D.

We will close this section with an example of a function that is discontinuous at every rational number and is not monotone, but is Riemann integrable nevertheless.

7.1.7 Example We consider Thomae's function $h : [0, 1] \to \mathbb{R}$ defined, as in Example 5.1.6(h), by $h(x) := 0$ if $x \in [0, 1]$ is irrational, $h(0) := 1$ and by $h(x) := 1/n$ if $x \in [0, 1]$ is the rational number $x = m/n$ where $m, n \in \mathbb{N}$ have no common integer factors except 1. It was seen in 5.1.6(h) that h is continuous at every irrational number and discontinuous at every rational number in [0, 1]. See Figure 5.1.2. We will now show that $h \in \mathcal{R}[0, 1]$.

For $\varepsilon > 0$, the set $E := \{x \in [0, 1] : h(x) \geq \varepsilon/2\}$ is a finite set. (For example, if $\varepsilon/2 = 1/5$, then there are eleven values of x such that $h(x) \geq 1/5$, namely, $E = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5\}$. (Sketch a graph.) We let n be the number of elements in E and take $\delta := \varepsilon/(4n)$. If $\dot{\mathcal{P}}$ is a given tagged partition such that $||\dot{\mathcal{P}}|| < \delta$, then we separate $\dot{\mathcal{P}}$ into two subsets. We let $\dot{\mathcal{P}}_1$ be the collection of tagged intervals in $\dot{\mathcal{P}}$ that have their tags in E, and we let $\dot{\mathcal{P}}_2$ be the subset of tagged intervals in $\dot{\mathcal{P}}$ that have their tags elsewhere in [0, 1]. Allowing for the possibility that a tag of $\dot{\mathcal{P}}_1$ may be an endpoint of adjacent intervals, we see that $\dot{\mathcal{P}}_1$ has at most $2n$ intervals and the total length of these intervals can be at most $2n\delta = \varepsilon/2$. Also, we have $0 < h(t_i) \leq 1$ for each tag t_i in $\dot{\mathcal{P}}_1$. Consequently, we have $S(h; \dot{\mathcal{P}}_1) \leq 1 \cdot 2n\delta \leq \varepsilon/2$. For tags t_i in $\dot{\mathcal{P}}_2$, we have $h(t_i) < \varepsilon/2$ and the total length of the subintervals in $\dot{\mathcal{P}}_2$ is clearly less than 1, so that $S(h; \dot{\mathcal{P}}_2) < (\varepsilon/2) \cdot 1 = \varepsilon/2$. Therefore, combining these results, we get

$$0 \leq S(h; \dot{\mathcal{P}}) = S(h; \dot{\mathcal{P}}_1) + S(h; \dot{\mathcal{P}}_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $h \in \mathcal{R}[0, 1]$ with integral 0. □

Exercises for Section 7.1

1. If $I := [0, 4]$, calculate the norms of the following partitions:
   (a) $\mathcal{P}_1 := (0, 1, 2, 4)$, (b) $\mathcal{P}_2 := (0, 2, 3, 4)$,
   (c) $\mathcal{P}_3 := (0, 1, 1.5, 2, 3.4, 4)$, (d) $\mathcal{P}_4 := (0, .5, 2.5, 3.5, 4)$.
2. If $f(x) := x^2$ for $x \in [0, 4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 1, and the tags are selected as indicated.
   (a) $\dot{\mathcal{P}}_1$ with the tags at the left endpoints of the subintervals.
   (b) $\dot{\mathcal{P}}_1$ with the tags at the right endpoints of the subintervals.
   (c) $\dot{\mathcal{P}}_2$ with the tags at the left endpoints of the subintervals.
   (d) $\dot{\mathcal{P}}_2$ with the tags at the right endpoints of the subintervals.
3. Show that $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with norm $||\dot{\mathcal{P}}|| \leq \delta_\varepsilon$, then $\big|S(f; \dot{\mathcal{P}}) - L\big| \leq \varepsilon$.

4. Let $\dot{\mathcal{P}}$ be a tagged partition of [0, 3].
   (a) Show that the union U_1 of all subintervals in $\dot{\mathcal{P}}$ with tags in [0, 1] satisfies $\left[0,\ 1-||\dot{\mathcal{P}}||\right] \subseteq U_1 \subseteq \left[0,\ 1+||\dot{\mathcal{P}}||\right]$.
   (b) Show that the union U_2 of all subintervals in $\dot{\mathcal{P}}$ with tags in [1,2] satisfies $\left[1+||\dot{\mathcal{P}}||,\ 2-||\dot{\mathcal{P}}||\right] \subseteq U_2 \subseteq \left[1-||\dot{\mathcal{P}}||, 2+||\dot{\mathcal{P}}||\right]$.
5. Let $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^{n}$ be a tagged partition of $[a,\ b]$ and let $c_1 < c_2$.
   (a) If u belongs to a subinterval I_i whose tag satisfies $c_1 \le t_i \le c_2$, show that $c_1 - ||\dot{\mathcal{P}}|| \le u \le c_2 + ||\dot{\mathcal{P}}||$.
   (b) If $v \in [a,\ b]$ and satisfies $c_1 + ||\dot{\mathcal{P}}|| \le v \le c_2 - ||\dot{\mathcal{P}}||$, then the tag t_i of any subinterval I_i that contains v satisfies $t_i \in [c_1,\ c_2]$.
6. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,\ 2]$ and evaluate its integral.
   (b) Let $h(x) := 2$ if $0 \le x < 1, h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,\ 2]$ and evaluate its integral.
7. Use Mathematical Induction and Theorem 7.1.5 to show that if $f_1, \ldots, f_n$ are in $\mathcal{R}[a,\ b]$ and if $k_1, \ldots, k_n \in \mathbb{R}$, then the linear combination $f = \sum_{i=1}^{n} k_i f_i$ belongs to $\mathcal{R}[a,\ b]$ and $\int_a^b f = \sum_{i=1}^{n} k_i \int_a^b f_i$.
8. If $f \in \mathcal{R}[a,\ b]$ and $|f(x)| \le M$ for all $x \in [a,\ b]$, show that $\left|\int_a^b f\right| \le M(b-a)$.
9. If $f \in \mathcal{R}[a,\ b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a,\ b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$, prove that $\int_a^b f = \lim_n S(f;\ \dot{\mathcal{P}}_n)$.
10. Let $g(x) := 0$ if $x \in [0,\ 1]$ is rational and $g(x) := 1/x$ if $x \in [0,\ 1]$ is irrational. Explain why $g \notin \mathcal{R}[0, 1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $\lim_n S(g;\ \dot{\mathcal{P}}_n)$ exists.
11. Suppose that f is bounded on $[a, b]$ and that there exists two sequences of tagged partitions of $[a, b]$ such that $||\dot{\mathcal{P}}_n|| \to 0$ and $||\dot{\mathcal{Q}}_n|| \to 0$, but such that $\lim_n S(f;\ \dot{\mathcal{P}}_n) \ne \lim_n S(f;\ \dot{\mathcal{Q}}_n)$. Show that f is not in $\mathcal{R}[a,\ b]$.
12. Consider the Dirichlet function, introduced in Example 5.1.6(g), defined by $f(x) := 1$ for $x \in [0,\ 1]$ rational and $f(x) := 0$ for $x \in [0, 1]$ irrational. Use the preceding exercise to show that f is *not* Riemann integrable on [0, 1].
13. Suppose that $c \le d$ are points in $[a, b]$. If $\varphi : [a,\ b] \to \mathbb{R}$ satisfies $\varphi(x) = \alpha > 0$ for $x \in [c,\ d]$ and $\varphi(x) = 0$ elsewhere in $[a, b]$, prove that $\varphi \in \mathcal{R}[a,\ b]$ and that $\int_a^b \varphi = \alpha(d-c)$. [*Hint*: Given $\varepsilon > 0$ let $\delta_\varepsilon := \varepsilon/4\alpha$ and show that if $||\dot{\mathcal{P}}|| < \delta_\varepsilon$ then we have $\alpha(d-c-2\delta_\varepsilon) \le S(\varphi;\ \dot{\mathcal{P}}) \le \alpha(d-c+2\delta_\varepsilon)$.]
14. Let $0 \le a < b$, let $Q(x) := x^2$ for $x \in [a,\ b]$ and let $\mathcal{P} := \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a partition of $[a, b]$. For each i, let q_i be the positive *square root* of

$$\tfrac{1}{3}\left(x_i^2 + x_i x_{i-1} + x_{i-1}^2\right).$$

   (a) Show that q_i satisfies $0 \le x_{i-1} \le q_i \le x_i$.
   (b) Show that $Q(q_i)(x_i - x_{i-1}) = \frac{1}{3}\left(x_i^3 - x_{i-1}^3\right)$.
   (c) If $\dot{\mathcal{Q}}$ is the tagged partition with the same subintervals as $\mathcal{P}$ and the tags q_i, show that $S(Q; \dot{\mathcal{Q}}) = \frac{1}{3}\left(b^3 - a^3\right)$.
   (d) Use the argument in Example 7.1.4(c) to show that $Q \in \mathcal{R}[a, b]$ and

$$\int_a^b Q = \int_a^b x^2\,dx = \tfrac{1}{3}\left(b^3 - a^3\right).$$

15. If $f \in \mathcal{R}[a,\ b]$ and $c \in \mathbb{R}$, we define g on $[a+c,\ b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c,\ b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

Section 7.2 Riemann Integrable Functions

We begin with a proof of the important Cauchy Criterion. We will then prove the Squeeze Theorem, which will be used to establish the Riemann integrability of several classes of functions (step functions, continuous functions, and monotone functions). Finally we will establish the Additivity Theorem.

We have already noted that direct use of the definition requires that we know the value of the integral. The Cauchy Criterion removes this need, but at the cost of considering two Riemann sums, instead of just one.

7.2.1 Cauchy Criterion *A function: $[a, b] \to \mathbb{R}$ belongs to $\mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then*

$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \varepsilon. \qquad \square$$

Proof. ($\Rightarrow$) If $f \in \mathcal{R}[a, b]$ with integral L, let $\eta_\varepsilon := \delta_{\varepsilon/2} > 0$ be such that if $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ are tagged partitions such that $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then

$$\left|S(f; \dot{\mathcal{P}}) - L\right| < \varepsilon/2 \quad \text{and} \quad \left|S(f; \dot{\mathcal{Q}}) - L\right| < \varepsilon/2.$$

Therefore we have

$$\begin{aligned} \left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| &\le \left|S(f; \dot{\mathcal{P}}) - L + L - S(f; \dot{\mathcal{Q}})\right| \\ &\le \left|S(f; \dot{\mathcal{P}}) - L\right| + \left|L - S(f; \dot{\mathcal{Q}})\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions with norms $< \delta_n$, then

$$\left|S(f; \dot{\mathcal{P}}) - f(f; \dot{\mathcal{Q}})\right| < 1/n.$$

Evidently we may assume that $\delta_n \ge \delta_{n+1}$ for $n \in \mathbb{N}$; otherwise, we replace δ_n by $\delta_n' := \min\{\delta_1, \ldots, \delta_n\}$.

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a tagged partition with $\|\dot{\mathcal{P}}_n\| < \delta_n$. Clearly, if $m > n$ then both $\dot{\mathcal{P}}_m$ and $\dot{\mathcal{P}}_n$ have norms $< \delta_n$, so that

$$(1) \qquad \left|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{P}}_m)\right| < 1/n \qquad \text{for} \qquad m > n.$$

Consequently, the sequence $\left(S(f; \dot{\mathcal{P}}_m)\right)_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. Therefore (by Theorem 3.5.5) this sequence converges in $\mathbb{R}$ and we let $A := \lim_m S(f; \dot{\mathcal{P}}_m)$.

Passing to the limit in (1) as $m \to \infty$, we have

$$\left|S(f; \dot{\mathcal{P}}_n) - A\right| \le 1/n \qquad \text{for all} \qquad n \in \mathbb{N}.$$

To see that A is the Riemann integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/\varepsilon$. If $\dot{\mathcal{Q}}$ is any tagged partition with $\|\dot{\mathcal{Q}}\| < \delta_K$, then

$$\begin{aligned} \left|S(f; \dot{\mathcal{Q}}) - A\right| &\le \left|S(f; \dot{\mathcal{Q}}) - S(f; \dot{\mathcal{P}}_K)\right| + \left|S(f; \dot{\mathcal{P}}_K) - A\right| \\ &\le 1/K + 1/K < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}[a, b]$ with integral A. Q.E.D.

We will now give two examples of the use of the Cauchy Criterion.

7.2.2 Examples **(a)** Let $g : [0, 3] \to \mathbb{R}$ be the function considered in Example 7.1.4(b). In that example we saw that if $\dot{\mathcal{P}}$ is a tagged partition of [0, 3] with norm $||\dot{\mathcal{P}}|| < \delta$, then

$$8 - 5\delta \leq S(g; \dot{\mathcal{P}}) \leq 8 + 5\delta.$$

Hence if $\dot{\mathcal{Q}}$ is another tagged partition with $||\dot{\mathcal{Q}}|| < \delta$, then

$$8 - 5\delta \leq S(g; \dot{\mathcal{Q}}) \leq 8 + 5\delta.$$

If we subtract these two inequalities, we obtain

$$\left|S(g; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{Q}})\right| \leq 10\delta.$$

In order to make this final term $< \varepsilon$, we are led to employ the Cauchy Criterion with $\eta_\varepsilon := \varepsilon/20$. (We leave the details to the reader.)

(b) The Cauchy Criterion can be used to show that a function $f : [a, b] \to \mathbb{R}$ is *not* Riemann integrable. To do this we need to show that: *There exists* $\varepsilon_0 > 0$ *such that for any* $\eta > 0$ *there exists tagged partitions* $\dot{\mathcal{P}}$ *and* $\dot{\mathcal{Q}}$ *with* $||\dot{\mathcal{P}}|| < \eta$ *and* $||\dot{\mathcal{Q}}|| < \eta$ *such that* $\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| \geq \varepsilon_0$.

We will apply these remarks to the Dirichlet function, considered in 5.1.6(g), defined by $f(x) := 1$ if $x \in [0, 1]$ is rational and $f(x) := 0$ if $x \in [0, 1]$ is irrational.

Here we take $\varepsilon_0 := \frac{1}{2}$. If $\dot{\mathcal{P}}$ is any partition all of whose tags are rational numbers then $S(f; \dot{\mathcal{P}}) = 1$, while if $\dot{\mathcal{Q}}$ is any tagged partition all of whose tags are irrational numbers then $S(f; \dot{\mathcal{Q}}) = 0$. Since we are able to take such tagged partitions with arbitrarily small norms, we conclude that the Dirichlet function is *not* Riemann integrable. □

The Squeeze Theorem

In working with the definition of Riemann integral, we have encountered two types of difficulties. First, for each partition, there are infinitely many choices of tags. And second, there are infinitely many partitions that have a norm less than a specified amount. We have experienced dealing with these difficulties in examples and proofs of theorems. We will now establish an important tool for proving integrability called the Squeeze Theorem that will provide some relief from those difficulties. It states that if a given function can be "squeezed" or bracketed between two functions that are known to be Riemann integrable with sufficient accuracy, then we may conclude that the given function is also Riemann integrable. The exact conditions are given in the statement of the theorem. (The idea of squeezing a function to establish integrability led the French mathematician Gaston Darboux to develop an approach to integration by means of upper and lower integrals, and this approach is presented in Section 7.4.)

7.2.3 Squeeze Theorem *Let* $f : [a, b] \to \mathbb{R}$. *Then* $f \in \mathcal{R}[a, b]$ *if and only if for every* $\varepsilon > 0$ *there exist functions* α_ε *and* ω_ε *in* $\mathcal{R}[a, b]$ *with*

(2) $$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \text{for all} \quad x \in [a, b],$$

and such that

(3) $$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon.$$

Proof. ($\Rightarrow$) Take $\alpha_\varepsilon = \omega_\varepsilon = f$ for all $\varepsilon > 0$.

($\Leftarrow$) Let $\varepsilon > 0$. Since α_ε and ω_ε belong to $\mathcal{R}[a, b]$, there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$ then

$$\left| S(\alpha_\varepsilon; \dot{\mathcal{P}}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \qquad \text{and} \qquad \left| S(\omega_\varepsilon; \dot{\mathcal{P}}) - \int_a^b \omega_\varepsilon \right| < \varepsilon.$$

It follows from these inequalities that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon; \dot{\mathcal{P}}) \qquad \text{and} \qquad S(\omega_\varepsilon; \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

In view of inequality (2), we have $S(\alpha_\varepsilon; \dot{\mathcal{P}}) \le S(f; \dot{\mathcal{P}}) \le S(\omega_\varepsilon; \dot{\mathcal{P}})$, whence

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If $\dot{\mathcal{Q}}$ is another tagged partition with $||\dot{\mathcal{Q}}|| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{\mathcal{Q}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If we subtract these two inequalities and use (3), we conclude that

$$\begin{aligned} \left| S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}}) \right| &< \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon \\ &= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f \in \mathcal{R}[a, b]$. Q.E.D.

Classes of Riemann Integrable Functions

The Squeeze Theorem is often used in connection with the class of step functions. It will be recalled from Definition 5.4.9 that a function $\varphi : [a, b] \to \mathbb{R}$ is a **step function** if it has only a finite number of distinct values, each value being assumed on one or more subintervals of $[a, b]$. For illustrations of step functions, see Figures 5.4.3 or 7.1.4.

7.2.4 Lemma *If J is a subinterval of $[a, b]$ having endpoints $c < d$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere in $[a, b]$, then $\varphi_J \in \mathcal{R}[a, b]$ and $\int_a^b \varphi_J = d - c$.*

Proof. If $J = [c, d]$ with $c \le d$, this is Exercise 7.1.13 and we can choose $\delta_\varepsilon := \varepsilon/4$.

There are three other subintervals J having the same endpoints c and d, namely, $[c, d)$, $(c, d]$, and (c, d). Since, by Theorem 7.1.3, we can change the value of a function at finitely many points without changing the integral, we have the same result for these other three subintervals. Therefore, we conclude that all four functions φ_J are integrable with integral equal to $d - c$. Q.E.D.

It is an important fact that any step function is Riemann integrable.

7.2.5 Theorem *If $\varphi : [a, b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a, b]$.*

Proof. Step functions of the type appearing in 7.2.4 are called “elementary step functions.” In Exercise 5 it is shown that an arbitrary step function φ can be expressed as a linear combination of such elementary step functions:

$$(4)\qquad \varphi = \sum_{j=1}^{m} k_j \varphi_{J_j},$$

where J_j has endpoints $c_j < d_j$. The lemma and Theorem 7.1.5(a, b) imply that $\varphi \in \mathcal{R}[a,b]$ and that

$$(5)\qquad \int_a^b \varphi = \sum_{j=1}^{m} k_j(d_j - c_j). \qquad \text{Q.E.D.}$$

We illustrate the use of step functions and the Squeeze Theorem in the next two examples. The first reconsiders a function that originally required a complicated calculation.

7.2.6 Examples **(a)** The function g in Example 7.1.4(b) is defined by $g(x) = 2$ for $0 \le x \le 1$ and $g(x) = 3$ for $1 < x \le 3$. We now see that g is a step function and therefore we calculate its integral to be $\int_0^3 g = 2\cdot(1-0) + 3\cdot(3-1) = 2 + 6 = 8$.

(b) Let $h(x) := x$ on [0,1] and let $P_n := (0,\, 1/n,\, 2/n, \ldots, (n-1)/n,\, n/n = 1)$. We define the step functions α_n and ω_n on the disjoint subintervals $[0,\, 1/n)$, $[1/n,\, 2/n)$, $\ldots, [(n-2)/n),\, (n-1)/n)$, $[(n-1)/n,\, 1]$ as follows: $\alpha_n(x) := h((k-1)/n) = (k-1)/n$ for x in $[(k-1)/n,\, k/n)$ for $k = 1, 2, \ldots, n-1$, and $\alpha_n(x) := h((n-1)/n) = (n-1)/n$ for x in $[(n-1)/n,\, 1]$. That is, α_n has the minimum value of h on each subinterval. Similarly, we define ω_n to be the maximum value of h on each subinterval, that is, $\omega_n(x) := k/n$ for x in $[(k-1)/n,\, k/n)$ for $k = 1, 2, \ldots, n-1$, and $\omega_n(x) := 1$ for x in $[(n-1)/n,\, 1]$. (The reader should draw a sketch for the case $n = 4$.)

Then we get

$$\begin{aligned}\int_0^1 \alpha_n &= \frac{1}{n}(0 + 1/n + 2/n + \cdots + (n-1)/n) \\ &= \frac{1}{n^2}(1 + 2 + \cdots + (n-1)) \\ &= \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{1}{2}(1 - 1/n).\end{aligned}$$

In a similar manner, we also get $\int_0^1 \omega_n = \frac{1}{2}(1 + 1/n)$. Thus we have

$$\alpha_n(x) \le h(x) \le \omega_n(x)$$

for $x \in [0,1]$ and

$$\int_0^1 (\omega_n - \alpha_n) = \frac{1}{n}.$$

Since for a given $\varepsilon > 0$, we can choose n so that $\frac{1}{n} < \varepsilon$, it follows from the Squeeze Theorem that h is integrable. We also see that the value of the integral of h lies between the integrals of α_n and ω_n for all n and therefore has value $\frac{1}{2}$. □

We will now use the Squeeze Theorem to show that an arbitrary continuous function is Riemann integrable.

7.2.7 Theorem *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$, *then* $f \in \mathcal{R}[a, b]$.

Proof. It follows from Theorem 5.4.3 that f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta_\varepsilon$, then we have $|f(u) - f(v)| < \varepsilon/(b - a)$.

Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition such that $||\mathcal{P}|| < \delta_\varepsilon$. Applying Theorem 5.3.4 we let $u_i \in I_i$ be a point where f attains its minimum value on I_i, and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.

Let α_ε be the step function defined by $\alpha_\varepsilon(x) := f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, \ldots, n - 1)$ and $\alpha_\varepsilon(x) := f(u_n)$ for $x \in [x_{n-1}, x_n]$. Let ω_ε be defined similarly using the points v_i instead of the u_i. Then one has

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \qquad \text{for all} \qquad x \in [a, b].$$

Moreover, it is clear that

$$\begin{aligned} 0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) &= \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1}) \\ &< \sum_{i=1}^n \left(\frac{\varepsilon}{b - a}\right)(x_i - x_{i-1}) = \varepsilon. \end{aligned}$$

Therefore it follows from the Squeeze Theorem that $f \in \mathcal{R}[a, b]$. Q.E.D.

Monotone functions are not necessarily continuous at every point, but they are also Riemann integrable.

7.2.8 Theorem *If* $f : [a, b] \to \mathbb{R}$ *is monotone on* $[a, b]$, *then* $f \in \mathcal{R}[a, b]$.

Proof. Assume that f is increasing on $I = [a, b]$. Partitioning the interval into n equal subintervals $I_k = [x_{k-1}, x_k]$ gives us $x_k - x_{k-1} = (b - a)/n$, $k = 1, 2, \ldots, n$. Since f is increasing on I_k, its minimum value is attained at the left endpoint x_{k-1} and its maximum value is attained at the right endpoint x_k. Therefore, we define the step functions $\alpha(x) := f(x_{k-1})$ and $\omega(x) := f(x_k)$ for $x \in [x_{k-1}, x_k)$, $k = 1, 2, \ldots, n - 1$, and $\alpha(x) := f(x_{n-1})$ and $\omega(x) := f(x_n)$ for $x \in [x_{n-1}, x_n]$. Then we have $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in I$, and

$$\int_a^b \alpha = \frac{b - a}{n}(f(x_0) + f(x_1) + \cdots + f(x_{n-1}))$$

$$\int_a^b \omega = \frac{b - a}{n}(f(x_1) + \cdots + f(x_{n-1}) + f(x_n)).$$

Subtracting, and noting the many cancellations, we obtain

$$\int_a^b (\omega - \alpha) = \frac{b - a}{n}(f(x_n) - f(x_0)) = \frac{b - a}{n}(f(b) - f(a)).$$

Thus for a given $\varepsilon > 0$, we choose n such that $n > (b - a)(f(b) - f(a))/\varepsilon$. Then we have $\int_a^b (\omega - \alpha) < \varepsilon$ and the Squeeze Theorem implies that f is integrable on I. Q.E.D.

The Additivity Theorem

We now return to arbitrary Riemann integrable functions. Our next result shows that the integral is an "additive function" of the interval over which the function is integrated. This property is no surprise, but its proof is a bit delicate and may be omitted on a first reading.

7.2.9 Additivity Theorem *Let* $f := [a,b] \to \mathbb{R}$ *and let* $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$ *if and only if its restrictions to* $[a, c]$ *and* $[c, b]$ *are both Riemann integrable. In this case*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{6}$$

Proof. ($\Leftarrow$) Suppose that the restriction f_1 of f to $[a, c]$, and the restriction f_2 of f to $[c, b]$ are Riemann integrable to L_1 and L_2 respectively. Then, given $\varepsilon > 0$ there exists $\delta' > 0$ such that if $\dot{\mathcal{P}}_1$ is a tagged partition of $[a, c]$ with $||\dot{\mathcal{P}}_1|| < \delta'$, then $|S(f_1; \dot{\mathcal{P}}_1) - L_1| < \varepsilon/3$. Also there exists $\delta'' > 0$ such that if $\dot{\mathcal{P}}_2$ is a tagged partition of $[c, b]$ wilh $||\dot{\mathcal{P}}_2|| < \delta''$ then $|S(f_2; \dot{\mathcal{P}}_2) - L_2| < \varepsilon/3$. If M is a bound for $|f|$, we define $\delta_\varepsilon := \min\{\delta', \delta'', \varepsilon/6M\}$ and let $\dot{\mathcal{P}}$ be a tagged partition of $[a, b]$ with $||\dot{\mathcal{Q}}|| < \delta$. We will prove that

$$\left|S(f; \dot{\mathcal{Q}}) - (L_1 + L_2)\right| < \varepsilon. \tag{7}$$

(i) If c is a partition point of $\dot{\mathcal{Q}}$, we split $\dot{\mathcal{Q}}$ into a partition $\dot{\mathcal{Q}}_1$ of $[a, c]$ and a partition $\dot{\mathcal{Q}}_2$ of $[c, b]$. Since $S(f; \dot{\mathcal{Q}}) = S(f; \dot{\mathcal{Q}}_1) + S(f; \dot{\mathcal{Q}}_2)$, and since $\dot{\mathcal{Q}}_1$ has norm $< \delta'$ and $\dot{\mathcal{Q}}_2$ has norm $< \delta''$, the inequality (7) is clear.

(ii) If c is not a partition point in $\dot{\mathcal{Q}} = \{(I_k, t_k)\}_{k=1}^m$, there exists $k \le m$ such that $c \in (x_{k-1}, x_k)$. We let $\dot{\mathcal{Q}}_1$ be the tagged partition of $[a, c]$ defined by

$$\dot{\mathcal{Q}}_1 := \{(I_1, t_1), \ldots, (I_{k-1}, t_{k-1}), ([x_{k-1}, c], c)\},$$

and $\dot{\mathcal{Q}}_2$ be the tagged partition of $[c, b]$ defined by

$$\dot{\mathcal{Q}}_2 := \{([c, x_k], c), (I_{k+1}, t_{k+1}), \ldots, (I_m, t_m)\}.$$

A straightforward calculation shows that

$$\begin{aligned} S(f; \dot{\mathcal{Q}}) - S(f; \dot{\mathcal{Q}}_1) - S(f; \dot{\mathcal{Q}}_2) &= f(t_k)(x_k - x_{k-1}) - f(c)(x_k - x_{k-1}) \\ &= (f(t_k) - f(c)) \cdot (x_k - x_{k-1}), \end{aligned}$$

whence it follows that

$$\left|S(f; \dot{\mathcal{Q}}) - S(f; \dot{\mathcal{Q}}_1) - S(f; \dot{\mathcal{Q}}_2)\right| \le 2M(x_k - x_{k-1}) < \varepsilon/3.$$

But since $||\dot{\mathcal{Q}}_1|| < \delta \le \delta'$ and $||\dot{\mathcal{Q}}_2|| < \delta \le \delta''$, it follows that

$$\left|S(f; \dot{\mathcal{Q}}_1) - L_1\right| < \varepsilon/3 \qquad \text{and} \qquad \left|S(f; \dot{\mathcal{Q}}_2) - L_2\right| < \varepsilon/3,$$

from which we obtain (7). Since $\varepsilon > 0$ is arbitrary, we infer that $f \in \mathcal{R}[a,b]$ and that (6) holds.

($\Rightarrow$) We suppose that $f \in \mathcal{R}[a,b]$ and, given $\varepsilon > 0$, we let $\eta_\varepsilon > 0$ satisfy the Cauchy Criterion 7.2.1. Let f_1 be the restriction of f to $[a, c]$ and let $\dot{\mathcal{P}}_1, \dot{\mathcal{Q}}_1$ be tagged partitions of

$[a, c]$ with $||\dot{\mathcal{P}}_1|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}_1|| < \eta_\varepsilon$. By adding additional partition points and tags from $[c, b]$, we can extend $\dot{\mathcal{P}}_1$ and $\dot{\mathcal{Q}}_1$ to tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ of $[a, b]$ that satisfy $||\dot{\mathcal{P}}|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}|| < \eta_\varepsilon$. If we use the *same* additional points and tags in $[c, b]$ for both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$, then

$$S(f_1; \dot{\mathcal{P}}_1) - S(f_1; \dot{\mathcal{Q}}_1) = S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}}).$$

Since both $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ have norm η_ε, then $\left|S(f_1; \dot{\mathcal{P}}_1) - S(f_1; \dot{\mathcal{Q}}_1)\right| < \varepsilon$. Therefore the Cauchy Condition shows that the restriction f_1 of f to $[a, c]$ is in $\mathcal{R}[a, c]$. In the same way, we see that the restriction f_2 of f to $[c, b]$ is in $\mathcal{R}[c, d]$.

The equality (6) now follows from the first part of the theorem. Q.E.D.

7.2.10 Corollary *If $f \in \mathcal{R}[a, b]$, and if $[c, d] \subseteq [a, b]$, then the restriction of f to $[c, d]$ is in $\mathcal{R}[c, d]$.*

Proof. Since $f \in \mathcal{R}[a, b]$ and $c \in [a, b]$, it follows from the theorem that its restriction to $[c, b]$ is in $\mathcal{R}[c, b]$. But if $d \in [c, b]$, then another application of the theorem shows that the restriction of f to $[c, d]$ is in $\mathcal{R}[c, d]$. Q.E.D.

7.2.11 Corollary *If $f \in \mathcal{R}[a, b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and*

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f.$$

Until now, we have considered the Riemann integral over an interval $[a, b]$ where $a < b$. It is convenient to have the integral defined more generally.

7.2.12 Definition If $f \in \mathcal{R}[a, b]$ and if $\alpha, \beta \in [a, b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \qquad \text{and} \qquad \int_\alpha^\alpha f := 0.$$

7.2.13 Theorem *If $f \in \mathcal{R}[a, b]$ and if α, β, γ are any numbers in $[a, b]$, then*

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f, \tag{8}$$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (8).

Proof. If any two of the numbers α, β, γ are equal, then (8) holds. Thus we may suppose that all three of these numbers are distinct.

For the sake of symmetry, we introduce the expression

$$L(\alpha, \beta, \gamma) := \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f.$$

It is clear that (8) holds if and only if $L(\alpha, \beta, \gamma) = 0$. Therefore, to establish the assertion, we need to show that $L = 0$ for all six permutations of the arguments α, β, and γ.

We note that the Additivity Theorem 7.2.9 implies that $L(\alpha, \beta, \gamma) = 0$ when $\alpha < \gamma < \beta$. But it is easily seen that both $L(\beta, \gamma, \alpha)$ and $L(\gamma, \alpha, \beta)$ equal $L(\alpha, \beta, \gamma)$.

Moreover, the numbers

$$L(\beta, \alpha, \gamma), \qquad L(\alpha, \gamma, \beta), \qquad \text{and} \qquad L(\gamma, \beta, \alpha)$$

are all equal to $-L(\alpha, \beta, \gamma)$. Therefore, L vanishes for all possible configurations of these three points. Q.E.D.

Exercises for Section 7.2

1. Let $f : [a, b] \to \mathbb{R}$. Show that $f \notin \mathcal{R}[a, b]$ if and only if there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exist tagged partitions $\dot{\mathcal{P}}_n$ and $\dot{\mathcal{Q}}_n$ with $||\dot{\mathcal{P}}_n|| < 1/n$ and $||\dot{\mathcal{Q}}_n|| < 1/n$ such that $\left|S(f; \dot{\mathcal{P}}_n) - S(f; \dot{\mathcal{Q}}_n)\right| \geq \varepsilon_0$.
2. Consider the function h defined by $h(x) := x + 1$ for $x \in [0, 1]$ rational, and $h(x) := 0$ for $x \in [0, 1]$ irrational. Show that h is not Riemann integrable.
3. Let $H(x) := k$ for $x = 1/k$ ($k \in \mathbb{N}$) and $H(x) := 0$ elsewhere on [0, 1]. Use Exercise 1, or the argument in 7.2.2(b), to show that H is not Riemann integrable.
4. If $\alpha(x) := -x$ and $\omega(x) := x$ and if $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in [0, 1]$, does it follow from the Squeeze Theorem 7.2.3 that $f \in \mathcal{R}[0, 1]$?
5. If J is any subinterval of $[a, b]$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere on $[a, b]$, we say that φ_J is an *elementary step function* on $[a, b]$. Show that every step function is a linear combination of elementary step functions.
6. If $\psi : [a, b] \to \mathbb{R}$ takes on only a finite number of distinct values, is ψ a step function?
7. If $S(f; \dot{\mathcal{P}})$ is any Riemann sum of $f : [a, b] \to \mathbb{R}$, show that there exists a step function $\varphi : [a, b] \to \mathbb{R}$ such that $\int_a^b \varphi = S(f; \dot{\mathcal{P}})$.
8. Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
9. Show that the continuity hypothesis in the preceding exercise cannot be dropped.
10. If f and g are continuous on $[a, b]$ and if $\int_a^b f = \int_a^b g$, prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.
11. If f is bounded by M on $[a, b]$ and if the restriction of f to every interval $[c, b]$ where $c \in (a, b)$ is Riemann integrable, show that $f \in \mathcal{R}[a, b]$ and that $\int_c^b f \to \int_a^b f$ as $c \to a+$. [*Hint:* Let $\alpha_c(x) := -M$ and $\omega_c(x) := M$ for $x \in [a, c)$ and $\alpha_c(x) := \omega_c(x) := f(x)$ for $x \in [c, b]$. Apply the Squeeze Theorem 7.2.3 for c sufficiently near a.]
12. Show that $g(x) := \sin(1/x)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0, 1]$.
13. Give an example of a function $f : [a, b] \to \mathbb{R}$ that is in $\mathcal{R}[c, b]$ for every $c \in (a, b)$ but which is not in $\mathcal{R}[a, b]$.
14. Suppose that $f : [a, b] \to \mathbb{R}$, that $a = c_0 < c_1 < \cdots < c_m = b$ and that the restrictions of f to $[c_{i-1}, c_i]$ belong to $\mathcal{R}[c_{i-1}, c_i]$ for $i = 1, \ldots, m$. Prove that $f \in \mathcal{R}[a, b]$ and that the formula in Corollary 7.2.11 holds.
15. If f is bounded and there is a finite set E such that f is continuous at every point of $[a, b]\backslash E$, show that $f \in \mathcal{R}[a, b]$.
16. If f is continuous on $[a, b]$, $a < b$, show that there exists $c \in [a, b]$ such that we have $\int_a^b f = f(c)(b - a)$. This result is sometimes called the *Mean Value Theorem for Integrals.*
17. If f and g are continuous on $[a, b]$ and $g(x) > 0$ for all $x \in [a, b]$, show that there exists $c \in [a, b]$ such that $\int_a^b fg = f(c) \int_a^b g$. Show that this conclusion fails if we do not have $g(x) > 0$. (Note that this result is an extension of the preceding exercise.)

18. Let f be continuous on $[a, b]$, let $f(x) \geq 0$ for $x \in [a, b]$, and let $M_n := (\int_a^b f^n)^{1/n}$. Show that $\lim(M_n) = \sup\{f(x) : x \in [a, b]\}$.
19. Suppose that $a > 0$ and that $f \in \mathcal{R}[-a, a]$.
    (a) If f is *even* (that is, if $f(-x) = f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 2\int_0^a f$.
    (b) If f is *odd* (that is, if $f(-x) = -f(x)$ for all $x \in [0, a]$), show that $\int_{-a}^{a} f = 0$.
20. If f is continuous on $[-a, a]$, show that $\int_{-a}^{a} f(x^2)dx = 2\int_0^a f(x^2)dx$.

Section 7.3 The Fundamental Theorem

We will now explore the connection between the notions of the derivative and the integral. In fact, there are *two* theorems relating to this problem: one has to do with integrating a derivative, and the other with differentiating an integral. These theorems, taken together, are called the Fundamental Theorem of Calculus. Roughly stated, they imply that the operations of differentiation and integration are inverse to each other. However, there are some subtleties that should not be overlooked.

The Fundamental Theorem (First Form)

The First Form of the Fundamental Theorem provides a theoretical basis for the method of calculating an integral that the reader learned in calculus. It asserts that if a function f is the derivative of a function F, *and if f belongs to* $\mathcal{R}[a, b]$, then the integral $\int_a^b f$ can be calculated by means of the evaluation $F|_a^b := F(b) - F(a)$. A function F such that $F'(x) = f(x)$ for all $x \in [a, b]$ is called an **antiderivative** or a **primitive of** f on $[a, b]$. Thus, when f has an antiderivative, it is a very simple matter to calculate its integral.

In practice, it is convenient to allow some exceptional points c where $F'(c)$ does not exist in $\mathbb{R}$, or where it does not equal $f(c)$. It turns out that we can permit a *finite* number of such exceptional points.

7.3.1 Fundamental Theorem of Calculus (First Form) *Suppose there is a* ***finite*** *set* $\boldsymbol{E}$ *in* $[a, b]$ *and functions* $f, F := [a, b] \to \mathbb{R}$ *such that*:

(a) *F is continuous on* $[a, b]$,

(b) $F'(x) = f(x)$ *for all* $x \in [a, b]\backslash E$,

(c) *f belongs to* $\mathcal{R}[a, b]$.

Then we have

$$\int_a^b f = F(b) - F(a). \tag{1}$$

Proof. We will prove the theorem in the case where $E := \{a, b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals (see Exercise 1).

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}[a, b]$ by assumption (c), there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is *any* tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then

$$\left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon. \tag{2}$$

If the subintervals in $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$, then the Mean Value Theorem 6.2.4 applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_{i-1}) \qquad \text{for} \qquad i = 1, \ldots, n.$$

If we add these terms, note the telescoping of the sum, and use the fact that $F'(u_i) = f(u_i)$, we obtain

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}).$$

Now let $\dot{\mathcal{P}}_u := \{([x_{i-1}, x_i], u_i)\}_{i=1}^{n}$, so the sum on the right equals $S(f; \dot{\mathcal{P}}_u)$. If we substitute $F(b) - F(a) = S(f; \dot{\mathcal{P}}_u)$ into (2), we conclude that

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we infer that equation (1) holds. Q.E.D.

Remark If the function F is differentiable at every point of $[a, b]$, then (by Theorem 6.1.2) hypothesis (a) is automatically satisfied. If f is not defined for some point $c \in E$, we take $f(c) := 0$. Even if F is differentiable at every point of $[a, b]$, condition (c) is *not automatically satisfied*, since there exist functions F such that F' is not Riemann integrable. (See Example 7.3.2(e).)

7.3.2 Examples **(a)** If $F(x) := \frac{1}{2}x^2$ for all $x \in [a, b]$, then $F'(x) = x$ for all $x \in [a, b]$. Further, $f = F'$ is continuous so it is in $\mathcal{R}[a, b]$. Therefore the Fundamental Theorem (with $E = \emptyset$) implies that

$$\int_a^b x\, dx = F(b) - F(a) = \tfrac{1}{2}\left(b^2 - a^2\right).$$

(b) If $G(x) := \operatorname{Arctan} x$ for $x \in [a, b]$, then $G'(x) = 1/(x^2 + 1)$ for all $x \in [a, b]$; also G' is continuous, so it is in $\mathcal{R}[a, b]$. Therefore the Fundamental Theorem (with $E = \emptyset$) implies that

$$\int_a^b \frac{1}{x^2 + 1}\, dx = \operatorname{Arctan} b - \operatorname{Arctan} a.$$

(c) If $A(x) := |x|$ for $x \in [-10, 10]$, then $A'(x) = -1$ if $x \in [-10, 0)$ and $A'(x) = +1$ for $x \in (0, 10]$. Recalling the definition of the signum function (in 4.1.10(b)), we have $A'(x) = \operatorname{sgn}(x)$ for all $x \in [-10, 10] \backslash \{0\}$. Since the signum function is a step function, it belongs to $\mathcal{R}[-10, 10]$. Therefore the Fundamental Theorem (with $E = \{0\}$) implies that

$$\int_{-10}^{10} \operatorname{sgn}(x)\, dx = A(10) - A(-10) = 10 - 10 = 0.$$

(d) If $H(x) := 2\sqrt{x}$ for $x \in [0, b]$, then H is continuous on $[0, b]$ and $H'(x) = 1/\sqrt{x}$ for $x \in (0, b]$. Since $h := H'$ is not bounded on $(0, b]$, it does not belong to $\mathcal{R}[0, b]$ no matter how we define $h(0)$. Therefore, the Fundamental Theorem 7.3.1 does not apply. (However, we will see in Example 10.1.10(a) that h is *generalized* Riemann integrable on $[0, b]$.)

(e) Let $K(x) := x^2 \cos(1/x^2)$ for $x \in (0,\ 1]$ and let $K(0) := 0$. It follows from the Product Rule 6.1.3(c) and the Chain Rule 6.1.6 that

$$K'(x) = 2x\cos(1/x^2) + (2/x)\sin(1/x^2) \qquad \text{for} \qquad x \in (0,\ 1].$$

Further, as in Example 6.1.7(e), it can be shown that $K'(0) = 0$. Thus K is continuous and differentiable at *every point* of [0, 1]. Since it can be seen that the function K' is not bounded on [0, 1], it does not belong to $\mathcal{R}[0,\ 1]$ and the Fundamental Theorem 7.3.1 does not apply to K'. (However, we will see from Theorem 10.1.9 that K' is *generalized* Riemann integrable on [0, 1].) □

The Fundamental Theorem (Second Form)

We now turn to the Fundamental Theorem (Second Form) in which we wish to differentiate an integral involving a variable upper limit.

7.3.3 Definition If $f \in \mathcal{R}[a, b]$, then the function defined by

(3)
$$F(z) := \int_a^z f \qquad \text{for} \qquad z \in [a,\ b],$$

is called the **indefinite integral** of f with **basepoint** a. (Sometimes a point other than a is used as a basepoint; see Exercise 6.)

We will first show that if $f \in \mathcal{R}[a,\ b]$, then its indefinite integral F satisfies a Lipschitz condition; hence F is continuous on $[a,\ b]$.

7.3.4 Theorem *The indefinite integral F defined by* (3) *is continuous on* $[a,\ b]$. *In fact, if* $|f(x)| \le M$ *for all* $x \in [a, b]$, *then* $|F(z) - F(w)| \le M|z - w|$ *for all* $z,\ w \in [a,\ b]$.

Proof. The Additivity Theorem 7.2.9 implies that if $z,\ w \in [a, b]$ and $w \le z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f,$$

whence we have

$$F(z) - F(w) = \int_w^z f.$$

Now if $-M \le f(x) \le M$ for all $x \in [a, b]$, then Theorem 7.1.5(c) implies that

$$-M(z - w) \le \int_w^z f \le M(z - w),$$

whence it follows that

$$|F(z) - F(w)| \le \left| \int_w^z f \right| \le M|z - w|,$$

as asserted. Q.E.D.

We will now show that the indefinite integral F is differentiable at any point where f is continuous.

7.3.5 Fundamental Theorem of Calculus (Second Form) *Let $f \in \mathcal{R}[a,b]$ and let f be continuous at a point $c \in [a,b]$. Then the indefinite integral, defined by* (3), *is differentiable at c and $F'(c) = f(c)$.*

Proof. We will suppose that $c \in [a,b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \tag{4}$$

Let h satisfy $0 < h < \eta_\varepsilon$. The Additivity Theorem 7.2.9 implies that f is integrable on the intervals $[a,c]$, $[a,\ c+h]$ and $[c,\ c+h]$ and that

$$F(c+h) - F(c) = \int_c^{c+h} f.$$

Now on the interval $[c,\ c+h]$ the function f satisfies inequality (4), so that we have

$$(f(c) - \varepsilon) \cdot h \le F(c+h) - F(c) = \int_c^{c+h} f \le (f(c) + \varepsilon) \cdot h.$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| \le \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{x \to 0+} \frac{F(c+h) - F(c)}{h} = f(c).$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a,\ b]$, whence the assertion follows. Q.E.D.

If f is continuous on all of $[a,\ b]$, we obtain the following result.

7.3.6 Theorem *If f is continuous on* $[a,\ b]$, *then the indefinite integral F, defined by* (3), *is differentiable on* $[a,\ b]$ *and $F'(x) = f(x)$* for all $x \in [a,b]$.

Theorem 7.3.6 can be summarized: *If f is continuous on* $[a,\ b]$, *then its indefinite integral is an antiderivative of f.* We will now see that, in general, the indefinite integral need not be an antiderivative (either because the derivative of the indefinite integral does not exist or does not equal $f(x)$).

7.3.7 Examples **(a)** If $f(x) := \operatorname{sgn} x$ on $[-1,\ 1]$, then $f \in \mathcal{R}[-1,\ 1]$ and has the indefinite integral $F(x) := |x| - 1$ with the basepoint -1. However, since $F'(0)$ does not exist, F is not an antiderivative of f on $[-1,\ 1]$.

(b) If h denotes Thomae's function, considered in 7.1.7, then its indefinite integral $H(x) := \int_0^x h$ is identically 0 on [0, 1]. Here, the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \ne h(x)$ whenever $x \in \mathbb{Q} \cap [0,\ 1]$, so that H is not an antiderivative of h on [0, 1]. □

Substitution Theorem

The next theorem provides the justification for the "change of variable" method that is often used to evaluate integrals. This theorem is employed (usually implicitly) in the evaluation by means of procedures that involve the manipulation of "differentials," common in elementary courses.

7.3.8 Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx. \tag{5}$$

The proof of this theorem is based on the Chain Rule 6.1.6, and will be outlined in Exercise 17. The hypotheses that f and φ' are continuous are restrictive, but are used to ensure the existence of the Riemann integral on the left side of (5).

7.3.9 Examples **(a)** Consider the integral $\displaystyle\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}dt$.

Here we substitute $\varphi(t) := \sqrt{t}$ for $t \in [1,4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on [1, 4]. If we let $f(x) := 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem 7.3.8 implies that the integral equals $\int_1^2 2\sin x\, dx = -2\cos x|_1^2 = 2(\cos 1 - \cos 2)$.

(b) Consider the integral $\displaystyle\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}dt$.

Since $\varphi(t) := \sqrt{t}$ does not have a continuous derivative on [0, 4], the Substitution Theorem 7.3.8 is not applicable, at least with this substitution. (In fact, it is not obvious that this integral exists; however, we can apply Exercise 7.2.11 to obtain this conclusion. We could then apply the Fundamental Theorem 7.3.1 to $F(t) := -2\cos\sqrt{t}$ with $E := \{0\}$ to evaluate this integral.) □

We will give a more powerful Substitution Theorem for the *generalized* Riemann integral in Section 10.1.

Lebesgue's Integrability Criterion

We will now present a statement of the definitive theorem due to Henri Lebesgue (1875–1941) giving a necessary and sufficient condition for a function to be Riemann integrable, and will give some applications of this theorem. In order to state this result, we need to introduce the important notion of a null set.

Warning Some people use the term "null set" as a synonym for the terms "empty set" or "void set" referring to $\emptyset$ (= the set that has no elements). However, we will always use the term "null set" in conformity with our next definition, as is customary in the theory of integration.

7.3.10 Definition **(a)** A set $Z \subset \mathbb{R}$ is said to be a **null set** if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that

$$Z \subseteq \bigcup_{k=1}^{\infty}(a_k,\, b_k) \qquad \text{and} \qquad \sum_{k=1}^{\infty}(b_k - a_k) \le \varepsilon. \tag{6}$$

(b) If $Q(x)$ is a statement about the point $x \in I$, we say that $Q(x)$ holds **almost everywhere** on I (or for **almost every** $x \in I$), if there exists a null set $Z \subset I$ such that $Q(x)$ holds for all $x \in I \backslash Z$. In this case we may write

$$Q(x) \quad \text{for a.e.} \quad x \in I.$$

It is trivial that any subset of a null set is also a null set, and it is easy to see that the union of two null sets is a null set. We will now give an example that may be very surprising.

7.3.11 Example The $\mathbb{Q}_1$ of rational numbers in [0, 1] is a null set.

We enumerate $\mathbb{Q}_1 = \{r_1, r_2, \ldots\}$. Given $\varepsilon > 0$, note that the open interval $J_1 := (r_1 - \varepsilon/4,\ r_1 + \varepsilon/4)$ contains r_1 and has length $\varepsilon/2$; also the open interval $J_2 := (r_2 - \varepsilon/8,\ r_2 + \varepsilon/8)$ contains r_2 and has length $\varepsilon/4$. In general, the open interval

$$J_k := \left(r_k - \frac{\varepsilon}{2^{k+1}},\ r_k + \frac{\varepsilon}{2^{k+1}} \right)$$

contains the point r_k and has length $\varepsilon/2^k$. Therefore, the union $\bigcup_{k=1}^{\infty} J_k$ of these open intervals contains every point of $\mathbb{Q}_1$; moreover, the sum of the lengths is $\sum_{k=1}^{\infty} \left(\varepsilon/2^k\right) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mathbb{Q}_1$ is a null set. □

The argument just given can be modified to show that: *Every countable set is a null set*. However, it can be shown that there exist uncountable null sets in $\mathbb{R}$; for example, the Cantor set that will be introduced in Definition 11.1.10.

We now state Lebesgue's Integrability Criterion. It asserts that a bounded function on an interval is Riemann integrable if and only if its points of discontinuity form a null set.

7.3.12 Lebesgue's Integrability Criterion *A bounded function* $f : [a, b] \to \mathbb{R}$ *is Riemann integrable if and only if it is continuous almost everywhere on* $[a, b]$.

A proof of this result will be given in Appendix C. However, we will apply Lebesgue's Theorem here to some specific functions, and show that some of our previous results follow immediately from it. We shall also use this theorem to obtain the important Composition and Product Theorems.

7.3.13 Examples **(a)** The step function g in Example 7.1.4(b) is continuous at every point except the point $x = 1$. Therefore it follows from the Lebesgue Integrability Criterion that g is Riemann integrable.

In fact, since every step function has at most a finite set of points of discontinuity, then: *Every step function on* $[a, b]$ *is Riemann integrable.*

(b) Since it was seen in Theorem 5.6.4 that the set of points of discontinuity of a monotone function is countable, we conclude that: *Every monotone function on* $[a, b]$ *is Riemann integrable.*

(c) The function G in Example 7.1.4(d) is discontinuous precisely at the points $D := \{1, 1/2, \ldots, 1/n, \ldots\}$. Since this is a countable set, it is a null set and Lebesgue's Criterion implies that G is Riemann integrable.

(d) The Dirichlet function was shown in Example 7.2.2(b) not to be Riemann integrable. Note that it is discontinuous at *every* point of [0, 1]. Since it can be shown that the interval [0, 1] is not a null set, Lebesgue's Criterion yields the same conclusion.

(e) Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function, defined in Examples 5.1.6(h) and 7.1.7.

In Example 5.1.6(h), we saw that h is continuous at every irrational number and is discontinuous at every rational number in [0, 1]. By Example 7.3.11, it is discontinuous on a null set, so Lebesgue's Criterion implies that Thomae's function is Riemann integrable on [0, 1], as we saw in Example 7.1.7. □

We now obtain a result that will enable us to take other combinations of Riemann integrable functions.

7.3.14 Composition Theorem *Let $f \in \mathcal{R}[a, b]$ with $f([a, b]) \subseteq [c, d]$ and let $\varphi : [c, d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ belongs to $\mathcal{R}[a, b]$.*

Proof. If f is continuous at a point $u \in [a, b]$, then $\varphi \circ f$ is also continuous at u. Since the set D of points of discontinuity of f is a null set, it follows that the set $D_1 \subseteq D$ of points of discontinuity of $\varphi \circ f$ is also a null set. Therefore the composition $\varphi \circ f$ also belongs to $\mathcal{R}[a, b]$. Q.E.D.

It will be seen in Exercise 22 that the hypothesis that φ is continuous cannot be dropped. The next result is a corollary of the Composition Theorem.

7.3.15 Corollary *Suppose that $f \in \mathcal{R}[a, b]$. Then its absolute value $|f|$ is in $\mathcal{R}[a, b]$, and*

$$\left| \int_a^b f \right| \le \int_a^b |f| \le M(b - a),$$

where $|f(x)| \le M$ for all $x \in [a, b]$.

Proof. We have seen in Theorem 7.1.6 that if f is integrable, then there exists M such that $|f(x)| \le M$ for all $x \in [a, b]$. Let $\varphi(t) := |t|$ for $t \in [-M, M]$; then the Composition Theorem implies that $|f| = \varphi \circ f \in \mathcal{R}[a, b]$. The first inequality follows from the fact that $-|f| \le f \le |f|$ and 7.1.5(c), and the second from the fact that $|f(x)| \le M$. Q.E.D.

7.3.16 The Product Theorem *If f and g belong to $\mathcal{R}[a, b]$, then the product fg belongs to $\mathcal{R}[a, b]$.*

Proof. If $\varphi(t) := t^2$ for $t \in [-M, M]$, it follows from the Composition Theorem that $f^2 = \varphi \circ f$ belongs to $\mathcal{R}[a, b]$. Similarly, $(f + g)^2$ and g^2 belong to $\mathcal{R}[a, b]$. But since we can write the product as

$$fg = \frac{1}{2}\left[(f + g)^2 - f^2 - g^2\right],$$

it follows that $fg \in \mathcal{R}[a, b]$. Q.E.D.

Integration by Parts

We will conclude this section with a rather general form of Integration by Parts for the Riemann integral, and Taylor's Theorem with the Remainder.

7.3.17 Integration by Parts *Let F, G be differentiable on $[a, b]$ and let $f := F'$ and $g := G'$ belong to $\mathcal{R}[a, b]$. Then*

$$\int_a^b fG = FG \Big|_a^b - \int_a^b Fg. \tag{7}$$

Proof. By Theorem 6.1.3(c), the derivative $(FG)'$ exists on $[a, b]$ and

$$(FG)' = F'G + FG' = fG + Fg.$$

Since F, G are continuous and f, g belong to $\mathcal{R}[a, b]$, the Product Theorem 7.3.16 implies that fG and Fg are integrable. Therefore the Fundamental Theorem 7.3.1 implies that

$$FG\Big|_a^b = \int_a^b (FG)' = \int_a^b fG + \int_a^b Fg,$$

from which (7) follows. Q.E.D.

A special, but useful, case of this theorem is when f and g are continuous on $[a, b]$ and F, G are their indefinite integrals $F(x) := \int_a^x f$ and $G(x) := \int_a^x g$.

We close this section with a version of Taylor's Theorem for the Riemann Integral.

7.3.18 Taylor's Theorem with the Remainder *Suppose that* $f', \ldots, f^{(n)}, f^{(n+1)}$ *exist on* $[a, b]$ *and that* $f^{(n+1)} \in \mathcal{R}[a, b]$. *Then we have*

(8) $$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where the remainder is given by

(9) $$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t)\cdot(b-t)^n dt.$$

Proof. Apply Integration by Parts to equation (9), with $F(t) := f^{(n)}(t)$ and $G(t) := (b-t)^n/n!$, so that $g(t) = -(b-t)^{n-1}/(n-1)!$, to get

$$\begin{aligned} R_n &= \frac{1}{n!}f^{(n)}(t)\cdot(b-t)^n\Big|_{t=a}^{t=b} + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)\cdot(b-a)^{n-1}dt \\ &= -\frac{f^{(n)}(a)}{n!}\cdot(b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)\cdot(b-t)^{n-1}dt. \end{aligned}$$

If we continue to integrate by parts in this way, we obtain (8). Q.E.D.

Exercises for Section 7.3

1. Extend the proof of the Fundamental Theorem 7.3.1 to the case of an arbitrary finite set E.
2. If $n \in \mathbb{N}$ and $H_n(x) := x^{n+1}/(n+1)$ for $x \in [a, b]$, show that the Fundamental Theorem 7.3.1 implies that $\int_a^b x^n dx = (b^{n+1} - a^{n+1})/(n+1)$. What is the set E here?
3. If $g(x) := x$ for $|x| \ge 1$ and $g(x) := -x$ for $|x| < 1$ and if $G(x) := \frac{1}{2}|x^2 - 1|$, show that $\int_{-2}^3 g(x)dx = G(3) - G(-2) = 5/2$.
4. Let $B(x) := -\frac{1}{2}x^2$ for $x < 0$ and $B(x) := \frac{1}{2}x^2$ for $x \ge 0$. Show that $\int_a^b |x|dx = B(b) - B(a)$.
5. Let $f : [a, b] \to \mathbb{R}$ and let $C \in \mathbb{R}$.
   (a) If $\Phi : [a, b] \to \mathbb{R}$ is an antiderivative of f on $[a, b]$, show that $\Phi_C(x) := \Phi(x) + C$ is also an antiderivative of f on $[a, b]$.
   (b) If Φ_1 and Φ_2 are antiderivatives of f on $[a, b]$, show that $\Phi_1 - \Phi_2$ is a constant function on $[a, b]$.

6. If $f \in \mathcal{R}[a, b]$ and if $c \in [a, b]$, the function defined by $F_c(z) := \int_c^z f$ for $z \in [a, b]$ is called the **indefinite integral** of f with **basepoint** c. Find a relation between F_a and F_c.

7. We have seen in Example 7.1.7 that Thomae's function is in $\mathcal{R}[0, 1]$ with integral equal to 0. Can the Fundamental Theorem 7.3.1 be used to obtain this conclusion? Explain your answer.

8. Let $F(x)$ be defined for $x \geq 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n)$, $n \in \mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [\![x]\!]dx$ for $0 \leq a < b$, where $[\![x]\!]$ denotes the greatest integer in x, as defined in Exercise 5.1.4.

9. Let $f \in \mathcal{R}[a, b]$ and define $F(x) := \int_a^x f$ for $x \in [a, b]$.
   (a) Evaluate $G(x) := \int_c^x f$ in terms of F, where $c \in [a, b]$.
   (b) Evaluate $H(x) := \int_x^b f$ in terms of F.
   (c) Evaluate $S(x) := \int_x^{\sin x} f$ in terms of F.

10. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $v : [c, d] \to \mathbb{R}$ be differentiable on $[c, d]$ with $v([c, d]) \subseteq [a, b]$. If we define $G(x) := \int_a^{v(x)} f$, show that $G'(x) = f(v(x)) \cdot v'(x)$ for all $x \in [c, d]$.

11. Find $F'(x)$ when F is defined on $[0, 1]$ by:
   (a) $F(x) := \int_0^{x^2} (1 + t^3)^{-1} dt$.
   (b) $F(x) := \int_{x^2}^{x} \sqrt{1 + t^2}\, dt$.

12. Let $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) := x$ for $0 \leq x < 1$, $f(x) := 1$ for $1 \leq x < 2$ and $f(x) := x$ for $2 \leq x \leq 3$. Obtain formulas for $F(x) := \int_0^x f$ and sketch the graphs of f and F. Where is F differentiable? Evaluate $F'(x)$ at all such points.

13. The function g is defined on $[0, 3]$ by $g(x) := -1$ if $0 \leq x < 2$ and $g(x) := 1$ if $2 \leq x \leq 3$. Find the indefinite integral $G(x) = \int_0^x g$ for $0 \leq x \leq 3$, and sketch the graphs of g and G. Does $G'(x) = g(x)$ for all x in $[0, 3]$?

14. Show there does not exist a continuously differentiable function f on $[0, 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for $0 \leq x \leq 2$. (Apply the Fundamental Theorem.)

15. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := \int_{x-c}^{x+c} f(t)dt$. Show that g is differentiable on $\mathbb{R}$ and find $g'(x)$.

16. If $f : [0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$, show that $f(x) = 0$ for all $x \in [0, 1]$.

17. Use the following argument to prove the Substitution Theorem 7.3.8. Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = F(\varphi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

18. Use the Substitution Theorem 7.3.8 to evaluate the following integrals.
   (a) $\displaystyle\int_0^1 t\sqrt{1 + t^2}\, dt$,
   (b) $\displaystyle\int_0^2 t^2 \left(1 + t^3\right)^{-1/2} dt = 4/3$,
   (c) $\displaystyle\int_1^4 \frac{\sqrt{1 + \sqrt{t}}}{\sqrt{t}} dt$,
   (d) $\displaystyle\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}} dt = 2(\sin 2 - \sin 1)$.

19. Explain why Theorem 7.3.8 and/or Exercise 7.3.17 cannot be applied to evaluate the following integrals, using the indicated substitution.
   (a) $\displaystyle\int_0^4 \frac{\sqrt{t}dt}{1 + \sqrt{t}}$ $\quad \varphi(t) = \sqrt{t}$,
   (b) $\displaystyle\int_0^4 \frac{\cos\sqrt{t}\, dt}{\sqrt{t}}$ $\quad \varphi(t) = \sqrt{t}$,
   (c) $\displaystyle\int_{-1}^1 \sqrt{1 + 2|t|}\, dt$ $\quad \varphi(t) = |t|$,
   (d) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1 - t^2}}$ $\quad \varphi(t) = \operatorname{Arcsin} t$.

20. (a) If Z_1 and Z_2 are null sets, show that $Z_1 \cup Z_2$ is a null set.
(b) More generally, if Z_n is a null set for each $n \in \mathbb{N}$, show that $\bigcup_{n=1}^{\infty} Z_n$ is a null set. [*Hint:* Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\{J_k^n : k \in \mathbb{N}\}$ be a countable collection of open intervals whose union contains Z_n and the sum of whose lengths is $\le \varepsilon/2^n$. Now consider the countable collection $\{J_k^n : n, k \in \mathbb{N}\}$.]

21. Let $f, g \in \mathcal{R}[a, b]$.
(a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \ge 0$.
(b) Use (a) to show that $2\left|\int_a^b fg\right| \le t\int_a^b f^2 + (1/t)\int_a^b g^2$ for $t > 0$.
(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$.
(d) Now prove that $\left|\int_a^b fg\right|^2 \le \left(\int_a^b |fg|\right)^2 \le \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right)$. This inequality is called the **Cauchy-Bunyakovsky-Schwarz Inequality** (or simply the **Schwarz Inequality**).

22. Let $h : [0, 1] \to \mathbb{R}$ be Thomae's function and let sgn be the signum function. Show that the composite function sgn $\circ\, h$ is not Riemann integrable on [0, 1].

Section 7.4 The Darboux Integral

An alternative approach to the integral is due to the French mathematician Gaston Darboux (1842–1917). Darboux had translated Riemann's work on integration into French for publication in a French journal and inspired by a remark of Riemann, he developed a treatment of the integral in terms of upper and lower integrals that was published in 1875. Approximating sums in this approach are obtained from partitions using the infima and suprema of function values on subintervals, which need not be attained as function values and thus the sums need not be Riemann sums.

This approach is technically simpler in the sense that it avoids the complications of working with infinitely many possible choices of tags. But working with infima and suprema also has its complications, such as lack of additivity of these quantities. Moreover, the reliance on the order properties of the real numbers causes difficulties in extending the Darboux integral to higher dimensions, and, more importantly, impedes generalization to more abstract surfaces such as manifolds. Also, the powerful Henstock-Kurzweil approach to integration presented in Chapter 10, which includes the Lebesgue integral, is based on the Riemann definition as given in Section 7.1.

In this section we introduce the upper and lower integrals of a bounded function on an interval, and define a function to be Darboux integrable if these two quantities are equal. We then look at examples and establish a Cauchy-like integrability criterion for the Darboux integral. We conclude the section by proving that the Riemann and Darboux approaches to the integral are in fact equivalent, that is, a function on a closed, bounded interval is Riemann integrable if and only if it is Darboux integrable. Later topics in the book do not depend on the Darboux definition of integral so that this section can be regarded as optional.

Upper and Lower Sums

Let $f : I \to \mathbb{R}$ be a bounded function on $I = [a, b]$ and let $\mathcal{P} = (x_0, x_1, \ldots, x_n)$ be a partition of I. For $k = 1, 2, \ldots, n$ we let

$$m_k := \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \qquad M_k := \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

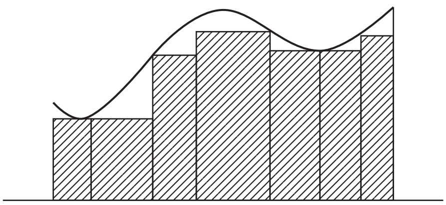


Figure 7.4.1 $L(f;\mathcal{P})$ a lower sum

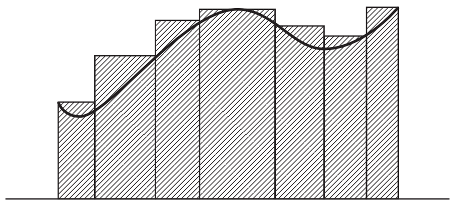


Figure 7.4.2 $U(f;\mathcal{P})$ an upper sum

The **lower sum** of f corresponding to the partition $\mathcal{P}$ is defined to be

$$L(f;\mathcal{P}) := \sum_{k=1}^{n} m_k(x_k - x_{k-1}),$$

and the **upper sum** of f corresponding to $\mathcal{P}$ is defined to be

$$U(f;\mathcal{P}) := \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

If f is a positive function, then the lower sum $L(f;\mathcal{P})$ can be interpreted as the area of the union of rectangles with base $[x_{k-1},\ x_k]$ and height m_k. (See Figure 7.4.1.) Similarly, the upper sum $U(f;\mathcal{P})$ can be interpreted as the area of the union of rectangles with base $[x_{k-1}, x_k]$ and height M_k. (See Figure 7.4.2.) The geometric interpretation suggests that, for a given partition, the lower sum is less than or equal to the upper sum. We now show this to be the case.

7.4.1 Lemma *If $f := I \to \mathbb{R}$ is bounded and $\mathcal{P}$ is any partition of I, then $L(f;\mathcal{P}) \le U(f;\mathcal{P})$.*

Proof. Let $\mathcal{P} := (x_0, x_1, \ldots, x_n)$. Since $m_k \le M_k$ for $k = 1, 2, \ldots, n$ and since $x_k - x_{k-1} > 0$ for $k = 1, 2, \ldots, n$, it follows that

$$L(f;\mathcal{P}) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \le \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = U(f;\mathcal{P}). \qquad \text{Q.E.D.}$$

If $\mathcal{P} := (x_0, x_1, \ldots, x_n)$ and $Q := (y_0, y_1, \ldots, y_m)$ are partitions of I, we say that Q is a **refinement of** $\mathcal{P}$ if each partition point $x_k \in \mathcal{P}$ also belongs to Q (that is, if $\mathcal{P} \subseteq Q$). A refinement Q of a partition $\mathcal{P}$ can be obtained by adjoining a finite number of points to $\mathcal{P}$. In this case, each one of the intervals $[x_{k-1},\ x_k]$ into which $\mathcal{P}$ divides I can be written as the union of intervals whose end points belong to Q; that is,

$$[x_{k-1}, x_k] = [y_{j-1}, y_j] \cup [y_j, y_{j+1}] \cup \cdots \cup [y_{h-1},\ y_h].$$

We now show that refining a partition increases lower sums and decreases upper sums.

7.4.2 Lemma *If $f : I \to \mathbb{R}$ is bounded, if $\mathcal{P}$ is a partition of I, and if Q is a refinement of $\mathcal{P}$, then*

$$L(f;\mathcal{P}) \le L(f;Q) \quad \text{and} \quad U(f;Q) \le U(f;\mathcal{P})$$

Proof. Let $\mathcal{P} = (x_0, x_1, \ldots, x_n)$. We first examine the effect of adjoining *one* point to $\mathcal{P}$. Let $z \in I$ satisfy $x_{k-1} < z < x_k$ and let $\mathcal{P}'$ be the partition

$$\mathcal{P}' := (x_0, x_1, \ldots, x_{k-1},\ z,\ x_k, \ldots, x_n),$$

obtained from $\mathcal{P}$ by adjoining z to $\mathcal{P}$. Let m'_k and m''_k be the numbers

$$m'_k := \inf\{f(x) : x \in [x_{k-1}, z]\}, \quad m''_k := \inf\{f(x) : x \in [z, x_k]\}.$$

Then $m_k \le m'_k$ and $m_k \le m''_k$ (why?) and therefore

$$m_k(x_k - x_{k-1}) = m_k(z - x_{k-1}) + m_k(x_k - z) \le m'_k(z - x_{k-1}) + m''_k(x_k - z).$$

If we add the terms $m_j(x_j - x_{j-1})$ for $j \neq k$ to the above inequality, we obtain $L(f;\mathcal{P}) \le L(f;\mathcal{P}')$.

Now if Q is any refinement of $\mathcal{P}$ (i.e., if $\mathcal{P} \subseteq Q$), then Q can be obtained from $\mathcal{P}$ by adjoining a finite number of points to $\mathcal{P}$ one at a time. Hence, repeating the preceding argument, we infer that $L(f;\mathcal{P}) \le L(f;Q)$.

Upper sums are handled similarly; we leave the details as an exercise. Q.E.D.

These two results are now combined to conclude that a lower sum is *always* smaller than an upper sum even if they correspond to different partitions.

7.4.3 Lemma *Let $f : I \to \mathbb{R}$ be bounded. If $\mathcal{P}_1, \mathcal{P}_2$ are any two partitions of I, then $L(f;\mathcal{P}_1) \le U(f;\mathcal{P}_2)$.*

Proof. Let $Q := \mathcal{P}_1 \cup \mathcal{P}_2$ be the partition obtained by combining the points of $\mathcal{P}_1$ and $\mathcal{P}_2$. Then Q is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Hence, by Lemmas 7.4.1 and 7.4.2, we conclude that

$$L(f;\mathcal{P}_1) \le L(f;Q) \le U(f;Q) \le U(f;\mathcal{P}_2).$$ Q.E.D.

Upper and Lower Integrals

We shall denote the collection of all partitions of the interval I by $\mathscr{P}(I)$. If $f : I \to \mathbb{R}$ is bounded, then each $\mathcal{P}$ in $\mathscr{P}(I)$ determines two numbers: $L(f;\mathcal{P})$ and $U(f;\mathcal{P})$. Thus, the collection $\mathscr{P}(I)$ determines two sets of numbers: the set of lower sums $L(f;\mathcal{P})$ for $\mathcal{P} \in \mathscr{P}(I)$, and the set of upper sums $U(f;\mathcal{P})$ for $\mathcal{P} \in \mathscr{P}(I)$. Hence, we are led to the following definitions.

7.4.4 Definition Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. The **lower integral of f on I** is the number

$$L(f) := \sup\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\},$$

and the **upper integral of f on I** is the number

$$U(f) := \inf\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}.$$

Since f is a bounded function, we are assured of the existence of the numbers

$$m_I := \inf\{f(x) : x \in I\} \quad \text{and} \quad M_I := \sup\{f(x) : x \in I\}.$$

It is readily seen that for any $\mathcal{P} \in \mathscr{P}(I)$, we have

$$m_I(b - a) \le L(f;\mathcal{P}) \le U(f;\mathcal{P}) \le M_I(b - a).$$

Hence it follows that

(1) $$m_I(b - a) \le L(f) \quad \text{and} \quad U(f) \le M_I(b - a).$$

The next inequality is also anticipated.

7.4.5 Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. Then the lower integral $L(f)$ and the upper integral $U(f)$ of f on I exist. Moreover,*

$$L(f) \le U(f). \tag{2}$$

Proof. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are any partitions of I, then it follows from Lemma 7.4.3 that $L(f;\mathcal{P}_1) \le U(f;\mathcal{P}_2)$. Therefore the number $U(f;\mathcal{P}_2)$ is an upper bound for the set $\{L(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$. Consequently, $L(f)$, being the supremum of this set, satisfies $L(f) \le U(f;\mathcal{P}_2)$. Since $\mathcal{P}_2$ is an arbitrary partition of I, then $L(f)$ is a lower bound for the set $\{U(f;\mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\}$. Consequently, the infimum $U(f)$ of this set satisfies the inequality (2). Q.E.D.

The Darboux Integral

If I is a closed bounded interval and $f : I \to \mathbb{R}$ is a bounded function, we have proved in Theorem 7.4.5 that the lower integral $L(f)$ and the upper integral $U(f)$ always exist. Moreover, *we always have* $L(f) \le U(f)$. However, it is possible that we might have $L(f) < U(f)$, as we will see in Example 7.4.7(d). On the other hand, there is a large class of functions for which $L(f) = U(f)$.

7.4.6 Definition Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. Then f is said to be **Darboux integrable on** I if $L(f) = U(f)$. In this case the **Darboux integral of** f **over** I is defined to be the value $L(f) = U(f)$.

Thus we see that if the Darboux integral of a function on an interval exists, then the integral is the *unique* real number that lies between the lower sums and the upper sums.

Since we will soon establish the equivalence of the Darboux and Riemann integrals, we will use the standard notation $\int_a^b f$ or $\int_a^b f(x)\,dx$ for the Darboux integral of a function f on $[a, b]$. The context should prevent any confusion from arising.

7.4.7 Examples **(a)** A constant function is Darboux integrable.

Let $f(x) := c$ for $x \in I := [a, b]$. If $\mathcal{P}$ is any partition of I, it is easy to see that $L(f;\mathcal{P}) = c(b-a) = U(f;\mathcal{P})$ (See Exercise 7.4.2). Therefore the lower and upper integrals are given by $L(f) = c(b-a) = U(f)$. Consequently, f is integrable on I and $\int_a^b f = \int_a^b c\,dx = c(b-a)$.

(b) Let g be defined on [0, 3] as follows: $g(x) := 2$ if $0 \le x \le 1$ and $g(x) := 3$ if $2 < x \le 3$. (See Example 7.1.4(b).) For $\varepsilon > 0$, if we define the partition $\mathcal{P}_\varepsilon := (0,\ 1,\ 1+\varepsilon,\ 3)$, then we get the upper sum

$$U(g;\mathcal{P}_\varepsilon) = 2\cdot(1-0) + 3(1+\varepsilon-1) + 3(2-\varepsilon) = 2 + 3\varepsilon + 6 - 3\varepsilon = 8.$$

Therefore, the upper integral satisfies $U(g) \le 8$. (Note that we cannot yet claim equality because $U(g)$ is the infimum over *all* partitions of [0, 3].) Similarly, we get the lower sum

$$L(g;\mathcal{P}_\varepsilon) = 2 + 2\varepsilon + 3(2-\varepsilon) = 8 - \varepsilon,$$

so that the lower integral satisfies $L(g) \ge 8$. Then we have $8 \le L(g) \le U(g) \le 8$, and hence $L(g) = U(g) = 8$. Thus the Darboux integral of g is $\int_0^3 g = 8$.

(c) The function $h(x) := x$ is integrable on [0, 1].
Let $\mathcal{P}_n$ be the partition of $I := [0, 1]$ into n subintervals given by

$$\mathcal{P}_n := \left(0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\right).$$

Since h is an increasing function, its infimum and supremum on the subinterval $[(k-1)/n, k/n]$ are attained at the left and right end points, respectively, and are thus given by $m_k = (k-1)/n$ and $M_k = k/n$. Moreover, since $x_k - x_{k-1} = 1/n$ for all $k = 1, 2, \ldots, n$, we have

$$L(h; \mathcal{P}_n) = (0 + 1 + \cdots + (n-1))/n^2, \quad U(h; \mathcal{P}_n) = (1 + 2 + \cdots + n)/n^2.$$

If we use the formula $1 + 2 + \cdots + m = m(m+1)/2$, for $m \in N$, we obtain

$$L(h; \mathcal{P}_n) = \frac{(n-1)n}{2n^2} = \frac{1}{2}\left(1 - \frac{1}{n}\right), \quad U(h; \mathcal{P}_n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

Since the set of partitions $\{\mathcal{P}_n : n \in N\}$ is a subset of the set of all partitions of $\mathcal{P}(I)$ of I, it follows that

$$\frac{1}{2} = \sup\{L(h; \mathcal{P}_n) : n \in N\} \leq \sup\{L(h; \mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\} = L(h),$$

and also that

$$U(h) = \inf\{U(h; \mathcal{P}) : \mathcal{P} \in \mathscr{P}(I)\} \leq \inf\{U(h; \mathcal{P}_n) : n \in N\} = \frac{1}{2}.$$

Since $\frac{1}{2} \leq L(h) \leq U(h) \leq \frac{1}{2}$, we conclude that $L(h) = U(h) = \frac{1}{2}$. Therefore h is Darboux integrable on $I = [0, 1]$ and $\int_0^1 h = \int_0^1 x\,dx = \frac{1}{2}$.

(d) A nonintegrable function.
Let $I := [0, 1]$ and let $f : I \to \mathbb{R}$ be the Dirichlet function defined by

$$\begin{aligned} f(x) &:= 1 \quad \text{for } x \text{ rational}, \\ &:= 0 \quad \text{for } x \text{ irrational}. \end{aligned}$$

If $\mathcal{P} := (x_0, x_1, \ldots, x_n)$ is any partition of [0, 1], then since every nontrivial interval contains both rational numbers and irrational numbers (see the Density Theorem 2.4.8 and its corollary), we have $m_k = 0$ and $M_k = 1$. Therefore, we have $L(f; \mathcal{P}) = 0$, $U(f; \mathcal{P}) = 1$, for all $\mathcal{P} \in \mathscr{P}(I)$, so that $L(f) = 0$, $U(f) = 1$. Since $L(f) \neq U(f)$, the function f is not Darboux integrable on [0, 1].

We now establish some conditions for the existence of the integral.

7.4.8 Integrability Criterion *Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function on I. Then f is Darboux integrable on I if and only if for each $\varepsilon > 0$ there is a partition $\mathcal{P}_\varepsilon$ of I such that*

(3) $$U(f; \mathcal{P}_\varepsilon) - L(f; \mathcal{P}_\varepsilon) < \varepsilon.$$

Proof. If f is integrable, then we have $L(f) = U(f)$. If $\varepsilon > 0$ is given, then from the definition of the lower integral as a supremum, there is a partition $\mathcal{P}_1$ of I such that $L(f) - \varepsilon/2 < L(f; \mathcal{P}_1)$. Similarly, there is a partition $\mathcal{P}_2$ of I such that $U(f; \mathcal{P}_2) < U(f) + \varepsilon/2$. If we let $\mathcal{P}_\varepsilon := \mathcal{P}_1 \cup \mathcal{P}_2$, then $\mathcal{P}_\varepsilon$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Consequently, by

Lemmas 7.4.1 and 7.4.2, we have

$$L(f) - \varepsilon/2 < L(f;\mathcal{P}_1) \le L(f;\mathcal{P}_\varepsilon) \\ \le U(f;\mathcal{P}_\varepsilon) \le U(f;\mathcal{P}_2) < U(f) + \varepsilon/2.$$

Since $L(f) = U(f)$, we conclude that (3) holds.

To establish the converse, we first observe that for any partition $\mathcal{P}$ we have $L(f;\mathcal{P}) \le L(f)$ and $U(f) \le U(f;\mathcal{P})$. Therefore,

$$U(f) - L(f) \le U(f;\mathcal{P}) - L(f;\mathcal{P}).$$

Now suppose that for each $\varepsilon > 0$ there exists a partition $\mathcal{P}_\varepsilon$ such that (3) holds.

Then we have

$$U(f) - L(f) \le U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $U(f) \le L(f)$. Since the inequality $L(f) \le U(f)$ is always valid, we have $L(f) = U(f)$. Hence f is Darboux integrable. Q.E.D.

7.4.9 Corollary *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a bounded function. If $\{P_n : n \in N\}$ is a sequence of partitions of I such that*

$$\lim_n (U(f;P_n) - L(f;P_n)) = 0,$$

then f is integrable and $\lim_n L(f;\mathcal{P}_n) = \int_a^b f = \lim_n U(f;\mathcal{P}_n)$.

Proof. If $\varepsilon > 0$ is given, it follows from the hypothesis that there exists K such that if $n \ge K$ then $U(f;\mathcal{P}_n) - L(f;\mathcal{P}_n) < \varepsilon$, whence the integrability of f follows from the Integrability Criterion. We leave the remainder of the proof as an exercise. Q.E.D.

The significance of the corollary is the fact that although the definition of the Darboux integral involves the set of all possible partitions of an interval, for a given function, the existence of the integral and its value can often be determined by a special sequence of partitions.

For example, if $h(x) = x$ on [0, 1] and $\mathcal{P}_n$ is the partition as in Example 7.4.7(c), then

$$\lim(U(h;\mathcal{P}_n) - L(h;\mathcal{P}_n)) = \lim 1/n = 0$$

and therefore $\int_0^1 x\,dx = \lim U(h;\mathcal{P}_n) = \lim \frac{1}{2}(1 + 1/n) = \frac{1}{2}$.

Continuous and Monotone Functions

It was shown in Section 7.2 that functions that are continuous or monotone on a closed bounded interval are Riemann integrable. (See Theorems 7.2.7 and 7.2.8.) The proofs employed approximation by step functions and the Squeeze Theorem 7.2.3 as the main tools. Both proofs made essential use of the fact that both continuous functions and monotone functions attain a maximum value and a minimum value on a closed bounded interval. That is, if f is a continuous or monotone function on $[a, b]$, then for a partition $\mathcal{P} = (x_0, x_1, \ldots, x_n)$, the numbers $M_k = \sup\{f(x) : x \in I_k\}$ and $m_k = \inf\{f(x) : x \in I_k\}$, $k = 1, 2, \ldots, n$, are attained as function values. For continuous functions, this is Theorem 5.3.4, and for monotone functions, these values are attained at the right and left endpoints of the interval.

If we define the step function ω on $[a, b]$ by $\omega(x) := M_k$ for $x \in [x_{k-1}, x_k)$ for $k = 1, 2, \ldots, n-1$, and $\omega(x) := M_n$ for $x \in [x_{n-1}, x_n]$, then we observe that the Riemann integral of ω is given by $\int_a^b \omega = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$. (See Theorem 7.2.5.) Now we recognize the sum on the right as the upper Darboux sum $U(f;\mathcal{P})$, so that we have

$$\int_a^b \omega = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = U(f;\mathcal{P}).$$

Similarly, if the step function α is defined by $\alpha(x) := m_k$ for $x \in [x_{k-1}, x_k)$, $k = 1, 2, \ldots, n-1$, and $\alpha(x) := m_n$ for $x \in [x_{n-1}, x_n]$, then we have the Riemann integral

$$\int_a^b \alpha = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = L(f;\mathcal{P}).$$

Subtraction then gives us

$$\int_a^b (\omega - \alpha) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = U(f;\mathcal{P}) - L(f;\mathcal{P}).$$

We thus see that the Integrability Criterion 7.4.8 is the Darboux integral counterpart to the Squeeze Theorem 7.2.3 for the Riemann integral.

Therefore, if we examine the proofs of Theorems 7.2.7 and 7.2.8 that establish the Riemann integrability of continuous and monotone functions, respectively, and replace the integrals of step functions by the corresponding lower and upper sums, then we obtain proofs of the theorems for the Darboux integral. (For example, in Theorem 7.2.7 for continuous functions, we would have $\alpha_\varepsilon(x) = f(u_i) = m_i$ and $\omega_\varepsilon(x) = f(v_i) = M_i$ and replace the integral of $\omega_\varepsilon - \alpha_\varepsilon$ with $U(f;\mathcal{P}) - L(f;\mathcal{P})$.)

Thus we have the following theorem. We leave it as an exercise for the reader to write out the proof.

7.4.10 Theorem If the function f on the interval $I = [a, b]$ is either continuous or monotone on I, then f is Darboux integrable on I.

The preceding observation that connects the Riemann and Darboux integrals plays a role in the proof of the equivalence of the two approaches to integration, which we now discuss. Of course, once equivalence has been established, then the preceding theorem would be an immediate consequence.

Equivalence

We conclude this section with a proof that the Riemann and Darboux definitions of the integral are equivalent in the sense that a function on a closed, bounded interval is Riemann integrable if and only if it is Darboux integrable, and their integrals are equal. This is not immediately apparent. The Riemann integral is defined in terms of sums that use function values (tags) together with a limiting process based on the length of subintervals in a partition. On the other hand, the Darboux integral is defined in terms of sums that use infima and suprema of function values, which need *not* be function values, and a limiting process based on refinement of partitions, not the size of subintervals in a partition. Yet the two are equivalent.

The background needed to prove equivalence is at hand. For example, if a function is Darboux integrable, we recognize that upper and lower Darboux sums are Riemann integrals of step functions. Thus the Integrability Criterion 7.4.8 for the Darboux integral corresponds to the Squeeze Theorem 7.2.3 for the Riemann integral in its application. In the other direction, if a function is Riemann integrable, the definitions of supremum and infimum enable us to choose tags to obtain Riemann sums that are as close to upper and lower Darboux sums as we wish. In this way, we connect the Riemann integral to the upper and lower Darboux integrals. The details are given in the proof.

7.4.11 Equivalence Theorem A function f on $I = [a, b]$ is Darboux integrable if and only if it is Riemann integrable.

Proof. Assume that f is Darboux integrable. For $\varepsilon > 0$, let $\mathcal{P}_\varepsilon$ be a partition of $[a, b]$ such that $U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon$. For this partition, as in the preceding discussion, we define the step functions α_ε and ω_ε on $[a, b]$ by $\alpha_\varepsilon(x) := m_k$ and $\omega_\varepsilon(x) := M_k$ for $x \in [x_{k-1}, x_k)$, $k = 1, 2, \ldots, n-1$, and $\alpha_\varepsilon(x) := m_n$, $\omega_\varepsilon(x) := M_n$ for $x \in [x_{n-1}, x_n]$, where, as usual, M_k is the supremum and m_k the infimum of f on $I_k = [x_{k-1}, x_k]$. Clearly we have

$$(4) \qquad \alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \text{for all } x \text{ in } [a, b].$$

Moreover, by Theorem 7.2.5, these functions are Riemann integrable and their integrals are equal to

$$(5) \quad \int_a^b \omega_\varepsilon = \sum_{k=1}^n M_k(x_k - x_{k-1}) = U(f;\mathcal{P}_\varepsilon), \quad \int_a^b \alpha_\varepsilon = \sum_{k=1}^n m_k(x_k - x_{k-1}) = L(f;\mathcal{P}_\varepsilon).$$

Therefore, we have

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = U(f;\mathcal{P}_\varepsilon) - L(f;\mathcal{P}_\varepsilon) < \varepsilon.$$

By the Squeeze Theorem 7.2.3, it follows that f is Riemann integrable. Moreover, we note that (4) and (5) are valid for any partition $\mathcal{P}$ and therefore the Riemann integral of f lies between $L(f;\mathcal{P})$ and $U(f;\mathcal{P})$ for any partition $\mathcal{P}$. Therefore the Riemann integral of f is equal to the Darboux integral of f.

Now assume that f is Riemann integrable and let $A = \int_a^b f$ denote the value of the integral. Then, f is bounded by Theorem 7.1.6, and given $\varepsilon > 0$, there exists $\delta > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ with $||\dot{\mathcal{P}}|| < \delta$, we have $\left|S(f;\dot{\mathcal{P}}) - A\right| < \varepsilon$, which can be written

$$(6) \qquad A - \varepsilon < S(f;\dot{\mathcal{P}}) < A + \varepsilon.$$

If $\mathcal{P} = (x_0, x_1, \ldots, x_n)$, then because $M_k = \sup\{f(x) : x \in I_k\}$ is a supremum, we can choose tags t_k in I_k such that $f(t_k) > M_k - \varepsilon/(b-a)$. Summing, and noting that $\sum_{k=1}^n (x_k - x_{k-1}) = b - a$, we obtain

$$(7) \quad S(f;\dot{\mathcal{P}}) = \sum_{k=1}^n f(t_k)(x_k - x_{k-1}) > \sum_{k=1}^n M_k(x_k - x_{k-1}) - \varepsilon = U(f;\mathcal{P}) - \varepsilon \ge U(f) - \varepsilon.$$

Combining inequalities (6) and (7), we get

$$A + \varepsilon > S(f;\dot{\mathcal{P}}) \ge U(f) - \varepsilon,$$

and hence we have $U(f) < A + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that $U(f) \le A$.

In the same manner, we can approximate lower sums by Riemann sums and show that $L(f) > A - 2\varepsilon$ for arbitrary $\varepsilon > 0$, which implies $L(f) \ge A$. Thus we have obtained the inequality $A \le L(f) \le U(f) \le A$, which gives us $L(f) = U(f) = A = \int_a^b f$. Hence, the function f is Darboux integrable with value equal to the Riemann integral. Q.E.D.

Exercises for Section 7.4

1. Let $f(x) := |x|$ for $-1 \leq x \leq 2$. Calculate $L(f;\mathcal{P})$ and $U(f;\mathcal{P})$ for the following partitions: (a) $\mathcal{P}_1 := (-1, 0, 1, 2)$, (b) $\mathcal{P}_2 := (-1, -1/2, 0, 1/2, 1, 3/2, 2)$.
2. Prove if $f(x) := c$ for $x \in [a, b]$, then its Darboux integral is equal to $c(b - a)$.
3. Let f and g be bounded functions on $I := [a, b]$. If $f(x) \leq g(x)$ for all $x \in I$, show that $L(f) \leq L(g)$ and $U(f) \leq U(g)$.
4. Let f be bounded on $[a, b]$ and let $k > 0$. Show that $L(kf) = kL(f)$ and $U(kf) = kU(f)$.
5. Let f, g, h be bounded functions on $I := [a, b]$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in I$. Show that if f and h are Darboux integrable and if $\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\int_a^b g = \int_a^b f$.
6. Let f be defined on [0, 2] by $f(x) := 1$ if $x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.
7. (a) Prove that if $g(x) := 0$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) := 1$ for $\frac{1}{2} < x \leq 1$, then the Darboux integral of g on [0, 1] is equal to $\frac{1}{2}$.
   (b) Does the conclusion hold if we change the value of g at the point $\frac{1}{2}$ to 13?
8. Let f be continuous on $I := [a, b]$ and assume $f(x) \geq 0$ for all $x \in I$. Prove if $L(f) = 0$, then $f(x) = 0$ for all $x \in I$.
9. Let f_1 and f_2 be bounded functions on $[a, b]$. Show that $L(f_1) + L(f_2) \leq L(f_1 + f_2)$.
10. Give an example to show that strict inequality can hold in the preceding exercise.
11. If f is a bounded function on $[a, b]$ such that $f(x) = 0$ except for x in $\{c_1, c_2, \ldots, c_n\}$ in $[a, b]$, show that $U(f) = L(f) = 0$.
12. Let $f(x) = x^2$ for $0 \leq x \leq 1$. For the partition $\mathcal{P}_n := (0, 1/n, 2/n, \ldots, (n-1)/n, 1)$, calculate $L(f, \mathcal{P}_n)$ and $U(f, \mathcal{P}_n)$, and show that $L(f) = U(f) = \frac{1}{3}$. (Use the formula $1^2 + 2^2 + \cdots + m^2 = \frac{1}{6}m(m+1)(2m+1)$.)
13. Let $\mathcal{P}_\varepsilon$ be the partition whose existence is asserted in the Integrability Criterion 7.4.8. Show that if $\mathcal{P}$ is any refinement of $\mathcal{P}_\varepsilon$, then $U(f;\mathcal{P}) - L(f;\mathcal{P}) < \varepsilon$.
14. Write out the proofs that a function f on $[a, b]$ is Darboux integrable if it is either (a) continuous, or (b) monotone.
15. Let f be defined on $I := [a, b]$ and assume that f satisfies the Lipschitz condition $|f(x) - f(y)| \leq K|x - y|$ for all x, y in I. If $\mathcal{P}_n$ is the partition of I into n equal parts, show that $0 \leq U(f;\mathcal{P}_n) - \int_a^b f \leq K(b-a)^2/n$.

Section 7.5 Approximate Integration

The Fundamental Theorem of Calculus 7.3.1 yields an effective method of evaluating the integral $\int_a^b f$ *provided* we can find an antiderivative F such that $F'(x) = f(x)$ when $x \in [a, b]$. However, when we cannot find such an F, we may not be able to use the Fundamental Theorem. Nevertheless, *when f is continuous*, there are a number of techniques for approximating the Riemann integral $\int_a^b f$ by using sums that resemble the Riemann sums.

One very elementary procedure to obtain quick estimates of $\int_a^b f$, based on Theorem 7.1.5(c), is to note that if $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b]$, then

$$\int_a^b g \leq \int_a^b f \leq \int_a^b h.$$