Chapter 7

Improper integrals

7.1 Introduction

The goal of this chapter is to meaningfully extend our theory of integrals to *improper* integrals. There are two types of so-called improper integrals: the first involves integrating a function over an infinite domain and the second involves integrands that are undefined at points within the domain of integration. In order to integrate over an infinite domain, we consider limits of the form $\lim_{b\to\infty}\int_a^b f(x)dx$. If the integrand is not defined at c ($a < c < b$) then we split the integral and consider the limits $\int_a^b f(x)dx = \lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0}\int_{c+\epsilon}^b f(x)dx$. The latter is sometimes also referred to as improper integrals of the second kind. Such situations occur, for example, for rational functions $f(x) = p(x)/q(x)$ whenever $q(x)$ has zeroes in the domain of integration.

The notions of *convergence* and *divergence* as discussed in Chapter 10 & 11 in the context of sequences and series will be very important to determine these limits. Improper integrals (of both types) arise frequently in applications and in probability. By relating improper integrals to infinite series we derive the last convergence test: the *Integral Comparison test*. As an application we finally prove that the p-series $\sum_{k=1}^{\infty} k^{-p}$ converges for $p > 1$ and diverges otherwise.

7.2 Integration over an infinite domain

We will see that there is a close connection between certain infinite series and improper integrals, which involve integrals over an infinite domain. We have already encountered examples of improper integrals in Section 3.8 and in the context of radioactive decay in Section 8.4. Recall the following definition:

Definition 1 (Improper integral (first kind)). *An improper integral of the first kind is an integral performed over an infinite domain, e.g.*

$$\int_a^{\infty} f(x)\, dx.$$

The value of such an integral is understood to be a limit, as given in the following definition:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

We evaluate an improper integral by first computing a definite integral over a finite domain $a \le x \le b$, and then taking a limit as the endpoint b moves off to larger and larger values. The definite integral can be interpreted as an area under the graph of the function. The essential question being addressed here is whether that area remains bounded when we include the "infinite tail" of the function (i.e. as the endpoint b moves to larger values.) For some functions (whose values get small enough, fast enough) the answer is "yes".

Definition 2 (Convergence). *If the limit*

$$\lim_{b\to\infty}\int_a^b f(x)\,dx$$

exists, we say that the improper integral converges. *Otherwise we say that the improper integral* diverges.

With these definitions in mind, we can compute a number of classic integrals.

7.2.1 Example: Decaying exponential

Recall that the improper integral of a decaying exponential converges – we have seen this earlier, in Section 3.8.5, and again in applications in Sections 7.3 and 8.4.1. Here we recapitulate this important result in the context of improper integrals. Suppose that $r > 0$ and let

$$I = \int_0^\infty e^{-rt}\,dt \equiv \lim_{b\to\infty}\int_0^b e^{-rt}\,dt.$$

Then

$$I = \lim_{b\to\infty} -\frac{1}{r}e^{-rt}\bigg|_0^b = -\frac{1}{r}\lim_{b\to\infty}(e^{-rb} - e^0) = -\frac{1}{r}(\underbrace{\lim_{b\to\infty} e^{-rb}}_{0} - 1) = \frac{1}{r},$$

where we have used the fact that $\lim_{b\to\infty} e^{-rb} = 0$ for $r > 0$. Thus the limit exists (is finite) and hence the integral *converges*. More precisely, it converges to the value $I = 1/r$.

7.2.2 Example: The improper integral of $1/x$ diverges

We now consider a classic and counter-intuitive result, and *one of the most important results in this chapter*. Consider the function

$$y = f(x) = \frac{1}{x}.$$

Examining the graph of this function for positive x, e.g. for the interval $(1, \infty)$, we know that values decrease to zero as x increases[26]. The function is not only bounded, but also falls to arbitrarily small values as x increases (see Figure 7.1). Nevertheless, this is *insufficient*

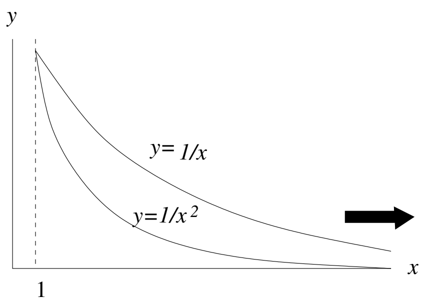

Figure 7.1. *In Sections 7.2.2 and 7.2.3, we consider two functions whose values decrease along the x axis, $f(x) = 1/x$ and $f(x) = 1/x^2$. We show that one, but not the other encloses a finite (bounded) area over the interval $(1, \infty)$. To do so, we compute an improper integral for each one. The heavy arrow is meant to remind us that we are considering areas over an unbounded domain.*

to guarantee that the enclosed area remains finite! We made a similar observation in the context of series in Section 11.3.1. We show this in the following calculation.

$$I = \int_1^\infty \frac{1}{x}\, dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\, dx = \lim_{b\to\infty} \ln(x)\Big|_1^b = \lim_{b\to\infty} (\ln(b) - \ln(1))$$
$$= \lim_{b\to\infty} \ln(b) = \infty$$

The fact that we get an infinite value for this integral follows from the observation that $\ln(b)$ increases without bound as b increases, that is *the limit does not exist (is not finite)*. Thus, the area under the curve $f(x) = 1/x$ over the interval $1 \leq x \leq \infty$ is infinite. We say that the improper integral of $1/x$ *diverges* (or does not converge). We will use this result again in Section 11.4.2.

7.2.3 Example: The improper integral of $1/x^2$ converges

Now consider the related function

$$y = f(x) = \frac{1}{x^2}$$

and the corresponding integral

$$I = \int_1^\infty \frac{1}{x^2}\, dx.$$

[26]We do not chose the interval $(0, \infty)$ because this function is undefined at $x = 0$. Here we want to emphasize the behaviour at infinity, not the blow up that occurs close to $x = 0$.

Then

$$I = \lim_{b\to\infty} \int_1^b x^{-2}\, dx = \lim_{b\to\infty} (-x^{-1})\Big|_1^b = -\lim_{b\to\infty} \left(\frac{1}{b} - 1\right) = 1.$$

Thus, the limit *exists*, and is $I = 1$. In contrast to the example in Section 7.2.2, this integral *converges*.

We observe that the behaviours of the improper integrals of the functions $1/x$ and $1/x^2$ are very different. The former diverges, while the latter converges. The only difference between these functions is the power of x. As shown in Figure 7.1, that power affects how rapidly the graph "falls off" to zero as x increases. The function $1/x^2$ decreases much faster than $1/x$. Consequently $1/x^2$ has a sufficiently "slim" infinite "tail", such that the area under its graph does not become infinite - not an easy concept to digest! This observations leads us to wonder what power p is needed to make the improper integral of a function $1/x^p$ converge. We answer this question below.

7.2.4 When does the integral of $1/x^p$ converge?

Here we consider an arbitrary power, p, that can be any real number. We ask when the corresponding improper integral converges or diverges. Let

$$I = \int_1^\infty \frac{1}{x^p}\, dx.$$

For $p = 1$ we have already established that this integral diverges (see Section 7.2.2), and for $p = 2$ we have seen that it is convergent (see Section 7.2.3). By a similar calculation, we find that

$$I = \lim_{b\to\infty} \frac{x^{1-p}}{(1-p)}\Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(b^{1-p} - 1\right).$$

Thus, this integral converges provided that the term b^{1-p} does not "blow up" as b increases. For this to be true, we require that the exponent $(1-p)$ should be negative, i.e. $1 - p < 0$ or $p > 1$. In this case, we have

$$I = \frac{1}{p-1}.$$

To summarize our result,

$\int_1^\infty \frac{1}{x^p}\, dx$ converges if $p > 1$, and diverges if $p \leq 1$.

Examples:

(i) The integral

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx, \quad \text{diverges.}$$

We see this from the following argument: $\sqrt{x} = x^{\frac{1}{2}}$, so $p = \frac{1}{2} < 1$. Thus, by the general result, this integral diverges.

(ii) The integral

$$\int_1^\infty x^{-1.01}\,dx, \quad \text{converges.}$$

Here $p = 1.01 > 1$, so the result implies convergence of the integral.

7.3 Application: Present value of a continuous income stream

Here we discuss the value of an annuity, which is a kind of savings account that guarantees a continuous stream of income. You would like to pay P dollars to purchase an annuity that will pay you an income $f(t)$ every year from now on, for $t > 0$. In some cases, we might want a constant income every year, in which case $f(t)$ would be constant. More generally, we can consider the case that at each future year t, we ask for income $f(t)$ that could vary from year to year. If the bank interest rate is r, how much should you pay now?

Solution

If we invest P dollars (the "principal" i.e., the amount deposited) in the bank with interest r then the amount $A(t)$ in the account at time t (in years), will grow as follows:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where r is the annual interest rate (e.g. 5%) and n is the number of times per year that interest is compound (e.g. $n = 2$ means interest compounded twice per year, $n = 12$ means monthly compounded interest, etc.). Define $h = \frac{r}{n}$. Then at time t, we have that

$$\begin{aligned} A(t) &= P(1+h)^{\frac{1}{h}rt} \\ &= P\left[(1+h)^{\frac{1}{h}}\right]^{rt} \\ &\approx Pe^{rt} \quad \text{for large } n \text{ or small } h. \end{aligned}$$

Here we have used the fact that when h is small (i.e. frequent intervals of compounding) the expression in square brackets above can be approximated by e, the base of the natural logarithms. Recall that

$$e = \lim_{h\to 0}\left[(1+h)^{\frac{1}{h}}\right].$$

This result was obtained in a first semester calculus course by selecting the base of exponentials such that the derivative of e^x is just e^x itself. Thus, we have found that the amount in the bank at time t will grow as

$$A(t) = Pe^{rt}, \quad \text{continually compounded interest.} \tag{7.1}$$

Having established the exponential growth of an investment, we return to the question of how to set up an annuity for a continuous stream of income in the future. Rewriting

Eqn. (7.1), the principle amount that we should invest in order to have $A(t)$ to spend at time t is

$$P = A(t)e^{-rt}.$$

Suppose we want to have $f(t)$ spending money for each year t. We refer to the *present value* of year t as the quantity

$$P = f(t)e^{-rt},$$

i.e. we must pay P now, in the present, to get $f(t)$ in a future year t. Summing over all the years, we find that the present value of the continuous income stream is

$$P = \sum_{t=1}^{L} f(t)e^{-rt} \cdot \underbrace{1}_{\text{“}\Delta t\text{”}} \approx \int_0^L f(t)e^{-rt}\, dt,$$

where L is the expected number of years left in the lifespan of the individual to whom this annuity will be paid, and where we have approximated a sum of payments by an integral (of a continuous income stream). One problem is that we do not know in advance how long the lifespan L will be. As a crude approximation, we could assume that this income stream continues forever, i.e. that $L \approx \infty$. In such an approximation, we have to compute the integral:

$$P = \int_0^\infty f(t)e^{-rt}\, dt. \tag{7.2}$$

The integral in Eqn. (7.2) is an **improper integral** (i.e. integral over an unbounded domain), as we have already encountered in Section 3.8.5. We shall have more to say about the properties of such integrals, and about their technical definition, existence, and properties in Chapter 7. We refer to the quantity

$$P = \int_0^\infty f(t)e^{-rt}\, dt, \tag{7.3}$$

as the *present value of a continuous income stream* $f(t)$.

Example: Setting up an annuity

Suppose we want an annuity that provides us with an annual payment of $10,000$ from the bank, i.e. in this case $f(t) = \$10,000$ is a function that has a constant value for every year. Then from Eqn (7.3),

$$P = \int_0^\infty 10000e^{-rt}\, dt = 10000 \int_0^\infty e^{-rt}\, dt.$$

By a previous calculation in Section 3.8.5, we find that

$$P = 10000 \cdot \frac{1}{r},$$

e.g. if interest rate is 5% (and assumed constant over future years), then

$$P = \frac{10000}{0.05} = \$200,000.$$

Therefore, we need to pay \$200,000 today to get $10,000$ annually for every future year.

7.4 Integral comparison test

The integrals discussed above can be used to make comparisons that help us to identify when other improper integrals converge or diverge[27]. The following important result establishes how these comparisons work:

> Suppose we are given two functions, $f(x)$ and $g(x)$, both continuous on some infinite interval $[a, \infty)$. Suppose, moreover, that at all points on this interval the first function is smaller than the second, i.e.
>
> $$0 \leq f(x) \leq g(x).$$
>
> Then the following conclusions can be made:[a]
>
> (i) $\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$
> The area under $f(x)$ is smaller than the area under $g(x)$.
>
> (ii) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
> If the larger area is finite, so is the smaller one.
>
> (iii) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.
> If the smaller area is infinite, so is the larger one.
>
> [a]These statements have to be carefully noted. What is assumed and what is concluded works "one way". That is the order "if ... then" is important. Reversing that order leads to a common error.

Example: Determine whether the following integral converges:

$$\int_1^\infty \frac{x}{1+x^3}\,dx.$$

Solution: by noting that for all $x > 0$

$$0 \leq \frac{x}{1+x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}.$$

Note that for $x > 0$

$$0 \leq \frac{x}{1+x^3} \leq \frac{x}{x^3} = \frac{1}{x^2}.$$

Thus,

$$\int_1^\infty \frac{x}{1+x^3}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx.$$

Since the larger integral on the right is known to converge, so does the smaller integral on the left. ♢

[27]Similar ideas will be employed for the comparison of infinite series in Chapter 11. A recurring theme in this course is the close connection between series and integrals, for example, recall the Riemann sums in Chapter 2.

7.5 Integration of an unbounded integrand

The second kind of improper integrals refers to integrands that are undefined at one (or more) points of the domain of integration $[a, b]$. Suppose $f(x)$ is continuous on the open interval (a, b) but becomes infinite at the lower bound, $x = a$. Then the integral of $f(x)$ over the domain $[a+\epsilon, b]$ for $\epsilon > 0$ has a definite value regardless of how small ϵ is chosen. Therefore, we can consider the limit

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} f(x)dx,$$

where $\epsilon \to 0^+$ means that ϵ approaches 0 from 'above', i.e. $\epsilon > 0$ always holds. If this limit exists and is equal to L, then we define

$$\int_{a}^{b} f(x)dx = L.$$

If an anti-derivative of $f(x)$, say $F(x)$ is known, then the fundamental theorem of calculus permits us to compute

$$\int_{a+\epsilon}^{b} f(x)dx = F(b) - F(a+\epsilon).$$

We are thus led to determine the existence (or nonexistence) of the limit

$$\lim_{\epsilon \to 0^+} F(a+\epsilon).$$

Example 1: Calculate the following integral for $p \neq 1$:

$$I = \int_{a}^{b} \frac{dx}{(x-a)^p}.$$

Solution: We interpret the integral as the following limit:

$$I = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b} \frac{dx}{(x-a)^p} = -\lim_{\epsilon \to 0^+} \frac{1}{p-1} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}} \right].$$

Thus, for $p > 1$ the term ϵ^{1-p} becomes arbitrarily large as $\epsilon \to 0^+$ and hence the area diverges and the integral does not exist. Conversely, for $p < 1$ the term ϵ^{1-p} converges to 0 as $\epsilon \to 0^+$ and hence the improper integral exists and is

$$I = \frac{(b-a)^{1-p}}{1-p}.$$

Finally, note that for $p = 1$ the anti-derivative is undefined and the integral does not exist. Alternatively, for $p = 1$ we directly see that

$$\int_{a+\epsilon}^{b} \frac{dx}{x-a} = \ln\left(\frac{b-a}{\epsilon}\right)$$

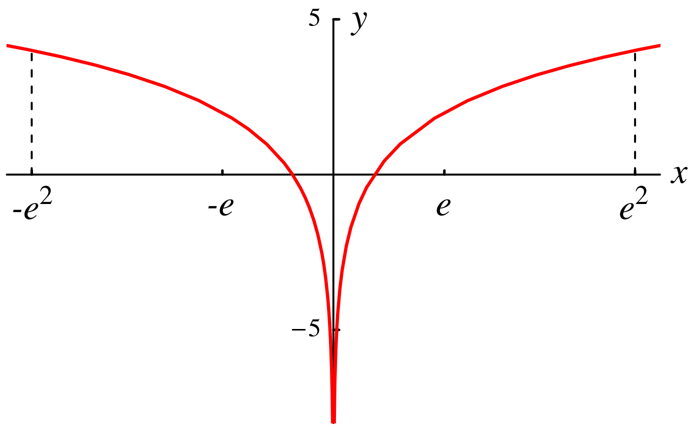

Figure 7.2. *Consider the improper integral* $\int_{-e^2}^{e^2} \ln\left(x^2\right) dx$. *The integrand* $\ln\left(x^2\right)$ *does not exist for* $x = 0$. *Nevertheless, the definite integral exists and equals* $4e^2$.

diverges as $\epsilon \to 0^+$. ♢

Note that for $a = 0$ the above example recovers the integrand $1/x^p$ that was discussed in Section 7.2.4. In particular, we find that the improper integral of the second kind

$$\int_0^1 \frac{1}{x^p} dx$$

exists for $p < 1$ and equals $1/(1-p)$ but does not exist for $p \geq 1$. Conversely, in Section 7.2.4 we observed that the improper integral of the first kind

$$\int_1^\infty \frac{1}{x^p} dx$$

exists for $p > 1$ and equals $1/(p-1)$ but does not exist for $p \leq 1$. Note that for $p = 1$ neither of the integrals exists.

Example 2: Calculate the following integral:

$$I = \int_{-e^2}^{e^2} \ln\left(x^2\right) dx.$$

Solution: A graph of the integrand is shown in Figure 7.2. First, we note that the integrand is not defined at $x = 0$. Therefore, we split the integral into two parts such that the undefined point marks once the upper and once the lower bound and we write the two integrals as a limit:

$$I = \int_{-e^2}^{e^2} \ln\left(x^2\right) dx = \lim_{\epsilon\to 0^-} \int_{-e^2}^{\epsilon} \ln\left(x^2\right) dx + \lim_{\epsilon\to 0^+} \int_{\epsilon}^{e^2} \ln\left(x^2\right) dx,$$

where $\epsilon \to 0^-$ means ϵ approaches 0 from below, i.e. $\epsilon < 0$ always holds. Second, the integrand is an even function and the integral runs over a symmetric domain. Hence we get

$$I = \lim_{\epsilon\to 0^+} 2 \int_{\epsilon}^{e^2} \ln\left(x^2\right) dx.$$

Check it!

The integral can be solved using the substitution $u = x^2$ followed by an integration by parts. This yields

$$I = \lim_{\epsilon\to 0^+} 2x\left(\ln\left(x^2\right) - 2\right)\Big|_{\epsilon}^{e^2} = 4 \lim_{\epsilon\to 0^+} x\left(\ln(x) - 1\right)\Big|_{\epsilon}^{e^2} = 4e^2 - \lim_{\epsilon\to 0^+} \epsilon(\ln(\epsilon) - 1).$$

The fact that the limit exists and converges to 0 can be seen by setting $\epsilon = 1/k$ and considering the limit $k \to \infty$:

$$\lim_{\epsilon\to 0^+} \epsilon(\ln(\epsilon) - 1) = \lim_{k\to\infty} \frac{1}{k}(\ln(\frac{1}{k}) - 1) = -\lim_{k\to\infty} \frac{1}{k}(\ln(k) + 1).$$

Now we can use either de l'Hôpital's rule or simply recognize that k grows much faster than $\ln k$ and hence the limit converges to 0. Thus, we find that the improper integral exists and is

$$I = 4e^2.$$

♢

7.6 L'Hôpital's rule

This section introduces a powerful method to evaluate tricky limits of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$. The rule is named after the French mathematician Guillaume de l'Hôpital, who published it in the 17^{th} century. For our purposes, the rule is often particularly useful to evaluate the limits that arise in improper integrals of the first (unbounded domain) and second kind (unbounded integrand).

Consider two functions, $f(x)$ and $g(x)$, and suppose that the following four prerequisites are satisfied:

(a) $f(x)$ and $g(x)$ are differentiable near $x = a$, but not necessarily at $x = a$.

(b) $g'(x) \neq 0$ for x near a but $x \neq a$.

(c) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

(d) and either

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or

(ii) $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$.

Then, l'Hôpital's rule states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{7.4}$$

Note: values of the limit a can include $\pm\infty$.

In *loose* terms, l'Hôpital's rule can be used if

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{is of type} \quad \frac{0}{0} \quad \text{or} \quad \pm\frac{\infty}{\infty}.$$

Let us now explore the power of l'Hôpital's rule through several examples.

Example 1: Calculate the following limit:

$$\lim_{x\to 0} \frac{\sin x}{e^x - 1}.$$

Solution: In this example l'Hôpital's rule can be used because $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0}(e^x - 1) = 0$. Thus,

$$\begin{aligned} \lim_{x\to 0} \frac{\sin x}{e^x - 1} &= \lim_{x\to 0} \frac{(\sin x)'}{(e^x - 1)'} = \lim_{x\to 0} \frac{\cos x}{e^x} \\ &= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} e^x} \qquad \text{(because of non-zero denominator)} \\ &= \frac{\cos 0}{1} = 1. \end{aligned}$$

◊

Example 2: Calculate the following limit:

$$\lim_{t\to\infty} \frac{t \ln t}{t^2 + 1}.$$

Solution: In this example, we can use l'Hôpital's rule because both the numerator and the denominator diverge: $\lim_{t\to\infty}(t \ln t) = \infty$ and $\lim_{t\to\infty}(t^2+1) = \infty$. Thus,

$$\lim_{t\to\infty} \frac{t\ln t}{t^2+1} = \lim_{t\to\infty} \frac{(t\ln t)'}{(t^2+1)'} = \lim_{t\to\infty} \frac{\ln t + 1}{2t}.$$

Unfortunately, the new limit that results from applying l'Hôpital's rule remains tricky. However, we can simply apply l'Hôpital's rule again because the prerequisites are still satisfied. In this case, the numerator and denominator still diverge: $\lim_{t\to\infty}(\ln t + 1) = \infty$ and $\lim_{t\to\infty} 2t = \infty$. Using l'Hôpital's rule again, we obtain

$$\lim_{t\to\infty} \frac{\ln t + 1}{2t} = \lim_{t\to\infty} \frac{(\ln t + 1)'}{(2t)'} = \lim_{t\to\infty} \frac{\frac{1}{t}}{2} = \frac{\lim\limits_{t\to\infty} \frac{1}{t}}{2} = 0.$$

Note: As long as *all* prerequisites of l'Hôpital's rule remain satisfied, the rule can be applied repeatedly. ♢

Example 3: Calculate the following limit:

$$\lim_{x\to 0+} \frac{\sin x}{x^2}.$$

Note: a plus-sign (+) added to the limit means that x *approaches the limit from above* (or *from the right*). In the present case, $x > 0$ always holds as x approaches zero. Later in this example we will see why this subtle point is important. In analogy, a minus-sign (−) added to the limit means that *the limit is approached from below* (or *from the left*). If no sign is added, it does not matter whether the limit is approached from above, below or even in an alternating manner.

Solution: Again, l'Hôpital's rule can be used because $\lim_{x\to 0} \sin x = 0$ and $\lim_{x\to 0} x^2 = 0$. Thus,

$$\lim_{x\to 0+} \frac{\sin x}{x^2} = \lim_{x\to 0+} \frac{(\sin x)'}{(x^2)'} = \lim_{x\to 0+} \frac{\cos x}{2x}.$$

Check it!

In order to evaluate this new limit, we might be tempted to apply l'Hôpital's rule again. Is this permissible? Stop for a moment and think about why or why not.

Of course, it is *not* permissible because the numerator $\lim_{x\to 0+} \cos x = 1$ and hence violates the prerequisites for applying l'Hôpital's rule. Ignoring the prerequisites and blindly applying l'Hôpital's rule again would yield an incorrect limit of zero. Instead, we get

$$\lim_{x\to 0+} \frac{\cos x}{2x} = \lim_{x\to 0+} \cos x \cdot \lim_{x\to 0+} \frac{1}{2x} = +\infty.$$

Hence, the limit does not exist, it diverges to $+\infty$.

Check it!

In order to see why it was important to approach the limit from above, $x \to 0+$, calculate the limit as x approaches zero from below:

$$\lim_{x\to 0-} \frac{\sin x}{x^2}.$$

In contrast to the above result, the limit is now $-\infty$ because $x < 0$ always holds as x approaches zero. ♢

Example 4: Calculate the following limit:

$$\lim_{a\to 0+} a \ln a.$$

Note: The limit of zero can only be approached from above, $a \to 0+$, because $\ln a$ is undefined for $a < 0$.

Solution: At first, this limit does not seem to have the correct form to apply l'Hôpital's rule. However, we can use a neat little trick and rewrite the limit as a quotient of two functions:

$$\lim_{a\to 0+} a \ln a = \lim_{a\to 0+} \frac{\ln a}{\frac{1}{a}}.$$

Once rewritten, it becomes apparent that l'Hôpital's rule can indeed be applied because again both the numerator and the denominator diverge: $\lim_{a\to 0+} \ln a = -\infty$ and $\lim_{a\to 0+} 1/a = \infty$. Thus,

$$\begin{aligned} \lim_{a\to 0+} \frac{\ln a}{\frac{1}{a}} &= \lim_{a\to 0+} \frac{(\ln a)'}{(\frac{1}{a})'} = \lim_{a\to 0+} \frac{\frac{1}{a}}{-\frac{1}{a^2}} \\ &= \lim_{a\to 0+} (-a) \quad = 0. \end{aligned}$$

♢

7.7 Summary

The main points of this chapter can be summarized as follows:

1. We reviewed the definition of an improper integral (type one) over an infinite domain:

$$\int_a^\infty f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. We computed some examples of improper integrals and discussed their convergence or divergence. We recalled (from earlier chapters) that

$$I = \int_0^\infty e^{-rt}\, dt \quad \text{converges,}$$

whereas

$$I = \int_1^\infty \frac{1}{x}\, dx \quad \text{diverges.}$$

3. More generally, we showed that

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{converges if } p > 1, \quad \text{diverges if } p \le 1.$$

4. We reviewed the definition of improper integrals (type two) for integrands that are unbounded at either end of the domain of integration, say $x = a$:

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f(x)\,dx.$$

5. If the integrand is not defined at one (or more) point(s) in the interior of the domain of integration, then the integral is split into two (or more) parts and we proceed as above.

6. In particular, we showed that

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \text{converges if } p < 1, \quad \text{diverges if } p \ge 1.$$

7. L'Hôpital's rule is a powerful tool to evaluate tricky limits that may arise for improper integrals of both kinds. It states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if $f(a) = g(a) = 0$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$ as well as some, more subtle prerequisites.

7.8 Exercises

Exercise 7.1 Consider the integral

$$A = \int_1^D \frac{1}{x^p}\,dx$$

(a) Sketch a region in the plane whose area represents this if (i) $p > 1$ and (ii) $p < 1$.

(b) Evaluate the integral for $p \neq 1$.

(c) How does the area A depend on the value of D in each of the cases (i) and (ii). Does the area increase without bound as D increases? Or does the area approach some constant?

(d) With this in mind, how might we try to understand an integral of the form

$$\int_1^\infty \frac{1}{x^p}\,dx$$

Exercise 7.2 Which of the following improper integrals converge? Give a reason in each case.

(a) $\displaystyle\int_1^\infty \frac{1}{x^{1.001}}\,dx$ (b) $\displaystyle\int_1^\infty x\,dx$ (c) $\displaystyle\int_1^\infty x^{-3}\,dx$

(d) $\displaystyle\int_0^\infty e^x\,dx$ (e) $\displaystyle\int_0^\infty e^{-2x}\,dx$ (f) $\displaystyle\int_0^\infty x\,e^{-x}\,dx$

Exercise 7.3 The gravitational force between two objects of mass m_1 and m_2 is $F = Gm_1m_2/r^2$ where r is the distance of separation. Initially the objects are a distance D apart. The work done in moving an object from position D to position x against a force F is defined as

$$W = \int_D^x F(r)\,dr.$$

Find the total work needed to move one of these objects infinitely far away.

Exercise 7.4 "Gabriel's Horn" is the surface of revolution formed by rotating the graph of the function $y = f(x) = 1/x$ about the x axis for $1 \le x < \infty$.

(a) Find the volume of air inside this shape and show that it is finite.

(b) When we cut a cross-section of this horn along the $x-y$-plane, we see a flat area which is wedged between the curves $y = 1/x$ and $y = -1/x$. Show that this "cross-sectional area" is infinite.

(c) The surface area of a surface of revolution generated by revolving the function $y = f(x)$ for $a \le x \le b$ about the x-axis is given by

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Write down an integral that would represent the surface area of "Gabriel's Horn".

(d) The integral in part (c) is not easy to evaluate explicitly – i.e. we cannot find an anti-derivative. However, we can show that it diverges. Set up a comparison that shows that the surface area of Gabriel's horn is infinite.

Exercise 7.5 Does the integral $\int_0^\infty \frac{\sin(x)}{x^4 + x^2 + 1}\, dx$ converge or diverge?

Exercise 7.6 Suppose that an airborne disease is introduced into a large population at time $t = 0$. At time $t \geqslant 0$, the rate at which this disease is spreading is $r(t) = 4000te^{-4t}$ new infections per day.

(a) After how long is this disease most infectious?

(b) How many people in total acquire the disease?

(c) What eventually happens to the rate of new infections?

Exercise 7.7 Does the integral $\int_0^5 \frac{x-1}{x^2 + x - 2}\, dx$ converge or diverge?

Exercise 7.8 Suppose that you place \$1,000,000 in an account that earns 5% annual interest, and you wish to withdraw z dollars from this account one year from now, $2z$ dollars two years from now, $3z$ dollars three years from now, and so on. What is the maximum value of z so that you never run out of money?

Exercise 7.9 For which values of p does the integral $\int_2^\infty \frac{1}{x(\ln x)^p}\, dx$ converge?

Exercise 7.10 Evaluate the following limits:

(a) $\lim_{x \to -\infty} xe^{4x}$ (b) $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$ (c) $\lim_{\theta \to \frac{\pi}{2}} \frac{\cos(\theta)}{\frac{\pi}{2} - \theta}$

Exercise 7.11 Determine the convergence of the following integrals. If they converge, find their values:

(a) $\int_0^\infty xe^{-x}\, dx$ (b) $\int_0^1 \frac{2x}{\sqrt{1 - x^2}}\, dx$ (c) $\int_0^1 \frac{1}{1 - x^3}\, dx$

Exercise 7.12 Convergence of an integral of the form $\int_{-\infty}^\infty f(x)\, dx$ is determined by splitting the integral up into two parts:

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^0 f(x)\, dx + \int_0^\infty f(x)\, dx.$$

The integral $\int_{-\infty}^{\infty} f(x)\ dx$ is said to converge if **both** of these two integrals converge. Can you come up with an example of a function $f(x)$ for which

$$\lim_{b\to\infty} \int_{-b}^{b} f(x)\ dx < \infty$$

but the integral $\int_{-\infty}^{\infty} f(x)\ dx$ diverges?

Exercise 7.13 Evaluate the integral $\int_0^{\pi} \frac{\cos\theta}{\sqrt{1-\sin^3\theta}}\, d\theta$.

Exercise 7.14 Evaluate the integral $\int_0^{\infty} e^{-x} \sin x\, dx$.

7.9 Solutions

Solution to 7.1

(a) See Figure 7.3

(b) $A = \dfrac{D^{1-p} - 1}{1 - p}$

(c) $p > 1$: A approaches a constant as D increases
$p < 1$: A increases with D.

(d) $p > 1$: $A = \dfrac{1}{p-1}$
$p < 1$: A DNE.

DNE: does not exist.

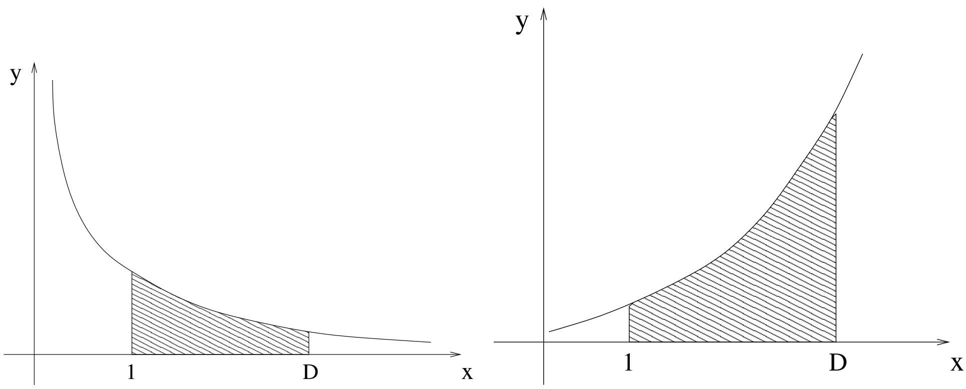

Figure 7.3. *Solution for problem* **??**

Solution to 7.2

(a) convergent, $p > 1$ (b) divergent, $p < 1$ (c) convergent, $p > 1$

(d) divergent, $\to \infty$ (e) convergent, $\dfrac{1}{2}$ (f) convergent, 1

Solution to 7.3 $W = G\, m_1\, m_2 \dfrac{1}{D}$

Solution to 7.4

(a) $V = \pi$

(b) $A \to \infty$

(c) $S = 2\pi \displaystyle\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx$

(d) $\dfrac{\sqrt{x^4+1}}{x^3} > \dfrac{1}{x}$ for $x > 1 \Rightarrow S \to \infty$

Solution to 7.5 The integral converges.

Solution to 7.6

(a) after 6 hours (b) 250 people (c) the rate tends to 0

Solution to 7.7 It converges: $\int_0^5 \frac{x-1}{x^2+x-2}\,dx = \int_0^5 \frac{1}{x+2}\,dx = \ln(7) - \ln(2)$.

Solution to 7.8 $z = 2500$ dollars

Solution to 7.9 It converges if $p > 1$ and diverges otherwise.

Solution to 7.10

(a) 0 (b) 1 (c) 1

Solution to 7.11

(a) it converges to 1 (b) it converges to 2 (c) it diverges

Solution to 7.12 One possible example is $f(x) = xe^{x^2}$; any odd function will do.

Solution to 7.13 $\displaystyle\int_0^{\pi} \frac{\cos\theta}{\sqrt{1-\sin^3\theta}}\,d\theta = 0.$

Solution to 7.14 $\displaystyle\int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{2}.$