Appendix A
Gamma and Beta Functions

A.1 A Useful Formula

The following formula is valid:

$$\int_{\mathbf{R}^n} e^{-|x|^2}\,dx = \left(\sqrt{\pi}\right)^n.$$

This is an immediate consequence of the corresponding one-dimensional identity

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

which is usually proved from its two-dimensional version by switching to polar coordinates:

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2}e^{-y^2}\,dy\,dx = 2\pi\int_0^{\infty} re^{-r^2}\,dr = \pi.$$

A.2 Definitions of $\Gamma(z)$ and $B(z,w)$

For a complex number z with $\operatorname{Re} z > 0$ define

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt.$$

$\Gamma(z)$ is called the gamma function. It follows from its definition that $\Gamma(z)$ is analytic on the right half-plane $\operatorname{Re} z > 0$.

Two fundamental properties of the gamma function are that

$$\Gamma(z+1) = z\Gamma(z) \qquad \text{and} \qquad \Gamma(n) = (n-1)!,$$

where z is a complex number with positive real part and $n \in \mathbf{Z}^+$. Indeed, integration by parts yields

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt = \left[\frac{t^z e^{-t}}{z}\right]_0^{\infty} + \frac{1}{z}\int_0^{\infty} t^z e^{-t}\,dt = \frac{1}{z}\Gamma(z+1).$$

Since $\Gamma(1) = 1$, the property $\Gamma(n) = (n-1)!$ for $n \in \mathbf{Z}^+$ follows by induction. Another important fact is that

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\,.$$

This follows easily from the identity

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}\,.$$

Next we define the beta function. Fix z and w complex numbers with positive real parts. We define

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \int_0^1 t^{w-1}(1-t)^{z-1}\,dt.$$

We have the following relationship between the gamma and the beta functions:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}\,,$$

when z and w have positive real parts.

The proof of this fact is as follows:

$$\begin{aligned}
\Gamma(z+w)B(z,w) &= \Gamma(z+w)\int_0^1 t^{w-1}(1-t)^{z-1}\,dt \\
&= \Gamma(z+w)\int_0^\infty u^{w-1}\left(\frac{1}{1+u}\right)^{z+w} du && t = u/(1+u) \\
&= \int_0^\infty\int_0^\infty u^{w-1}\left(\frac{1}{1+u}\right)^{z+w} v^{z+w-1}e^{-v}\,dv\,du \\
&= \int_0^\infty\int_0^\infty u^{w-1}s^{z+w-1}e^{-s(u+1)}\,ds\,du && s = v/(1+u) \\
&= \int_0^\infty s^z e^{-s}\int_0^\infty (us)^{w-1}e^{-su}\,du\,ds \\
&= \int_0^\infty s^{z-1}e^{-s}\Gamma(w)ds \\
&= \Gamma(z)\Gamma(w)\,.
\end{aligned}$$

A.3 Volume of the Unit Ball and Surface of the Unit Sphere

We denote by v_n the volume of the unit ball in $\mathbf{R}^n$ and by ω_{n-1} the surface area of the unit sphere $\mathbf{S}^{n-1}$. We have the following:

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

and

$$v_n = \frac{\omega_{n-1}}{n} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}\,.$$

The easy proofs are based on the formula in Appendix A.1. We have

$$\big(\sqrt{\pi}\big)^n = \int_{\mathbf{R}^n} e^{-|x|^2}\,dx = \omega_{n-1}\int_0^\infty e^{-r^2}\,r^{n-1}\,dr\,,$$

by switching to polar coordinates. Now change variables $t = r^2$ to obtain that

$$\pi^{\frac{n}{2}} = \tfrac{\omega_{n-1}}{2}\int_0^\infty e^{-t}\,t^{\frac{n}{2}-1}\,dt = \tfrac{\omega_{n-1}}{2}\Gamma\big(\tfrac{n}{2}\big)\,.$$

This proves the formula for the surface area of the unit sphere in $\mathbf{R}^n$.

To compute v_n, write again using polar coordinates

$$v_n = |B(0,1)| = \int_{|x|\le 1} 1\,dx = \int_{\mathbf{S}^{n-1}}\int_0^1 r^{n-1}\,dr\,d\theta = \frac{1}{n}\,\omega_{n-1}\,.$$

Here is another way to relate the volume to the surface area. Let $B(0,R)$ be the ball in $\mathbf{R}^n$ of radius $R > 0$ centered at the origin. Then the volume of the shell $B(0,R+h)\setminus B(0,R)$ divided by h tends to the surface area of $B(0,R)$ as $h \to 0$. In other words, the derivative of the volume of $B(0,R)$ with respect to the radius R is equal to the surface area of $B(0,R)$. Since the volume of $B(0,R)$ is $v_n R^n$, it follows that the surface area of $B(0,R)$ is $n\,v_n R^{n-1}$. Taking $R = 1$, we deduce $\omega_{n-1} = n\,v_n$.

A.4 Computation of Integrals Using Gamma Functions

Let $k_1,\dots,k_n$ be nonnegative even integers. The integral

$$\int_{\mathbf{R}^n} x_1^{k_1}\cdots x_n^{k_n} e^{-|x|^2}\,dx_1\cdots dx_n = \prod_{j=1}^n \int_{-\infty}^{+\infty} x_j^{k_j}\,e^{-x_j^2}\,dx_j = \prod_{j=1}^n \Gamma\Big(\frac{k_j+1}{2}\Big)$$

expressed in polar coordinates is equal to

$$\left(\int_{\mathbf{S}^{n-1}} \theta_1^{k_1}\cdots\theta_n^{k_n}\,d\theta\right)\int_0^\infty r^{k_1+\cdots+k_n}\,r^{n-1}e^{-r^2}\,dr\,,$$

where $\theta = (\theta_1,\dots,\theta_n)$. This leads to the identity

$$\int_{\mathbf{S}^{n-1}} \theta_1^{k_1}\cdots\theta_n^{k_n}\,d\theta = 2\Gamma\Big(\frac{k_1+\cdots+k_n+n}{2}\Big)^{-1}\prod_{j=1}^n \Gamma\Big(\frac{k_j+1}{2}\Big)\,.$$

Another classical integral that can be computed using gamma functions is the following:

$$\int_0^{\pi/2} (\sin\varphi)^a (\cos\varphi)^b \, d\varphi = \frac{1}{2} \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+b+2}{2})},$$

whenever a and b are complex numbers with $\operatorname{Re} a > -1$ and $\operatorname{Re} b > -1$.

Indeed, change variables $u = (\sin\varphi)^2$; then $du = 2(\sin\varphi)(\cos\varphi)d\varphi$, and the preceding integral becomes

$$\frac{1}{2}\int_0^1 u^{\frac{a-1}{2}}(1-u)^{\frac{b-1}{2}} du = \frac{1}{2} B\Big(\frac{a+1}{2}, \frac{b+1}{2}\Big) = \frac{1}{2} \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}{\Gamma(\frac{a+b+2}{2})}.$$

A.5 Meromorphic Extensions of $B(z,w)$ and $\Gamma(z)$

Using the identity $\Gamma(z+1) = z\Gamma(z)$, we can easily define a meromorphic extension of the gamma function on the whole complex plane starting from its known values on the right half-plane. We give an explicit description of the meromorphic extension of $\Gamma(z)$ on the whole plane. First write

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt$$

and observe that the second integral is an analytic function of z for all $z \in \mathbf{C}$. Write the first integral as

$$\int_0^1 t^{z-1} \left\{ e^{-t} - \sum_{j=0}^{N} \frac{(-t)^j}{j!} \right\} dt + \sum_{j=0}^{N} \frac{(-1)^j / j!}{z+j}.$$

The last integral converges when $\operatorname{Re} z > -N-1$, since the expression inside the curly brackets is $O(t^{N+1})$ as $t \to 0$. It follows that the gamma function can be defined to be an analytic function on $\operatorname{Re} z > -N-1$ except at the points $z = -j$, $j = 0, 1, \dots, N$, at which it has simple poles with residues $\frac{(-1)^j}{j!}$. Since N was arbitrary, it follows that the gamma function has a meromorphic extension on the whole plane.

In view of the identity

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)},$$

the definition of $B(z,w)$ can be extended to $\mathbf{C} \times \mathbf{C}$. It follows that $B(z,w)$ is a meromorphic function in each argument.

A.6 Asymptotics of $\Gamma(x)$ as $x \to \infty$

We now derive *Stirling's formula*:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2\pi x}} = 1\,.$$

First change variables $t = x + sx\sqrt{\frac{2}{x}}$ to obtain

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt = \left(\frac{x}{e}\right)^x \sqrt{2x} \int_{-\sqrt{x/2}}^{+\infty} \frac{\left(1 + s\sqrt{\frac{2}{x}}\right)^x}{e^{2s\sqrt{x/2}}} ds\,.$$

Setting $y = \sqrt{\frac{x}{2}}$, we obtain

$$\frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2x}} = \int_{-\infty}^{+\infty} \left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y} \chi_{(-y,\infty)}(s)\, ds.$$

To show that the last integral converges to $\sqrt{\pi}$ as $y \to \infty$, we need the following:
(1) The fact that

$$\lim_{y\to\infty} \left(\frac{\left(1+s/y\right)^y}{e^s}\right)^{2y} \to e^{-s^2},$$

which follows easily by taking logarithms and applying L'Hôpital's rule twice.
(2) The estimate, valid for $y \geq 1$,

$$\left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y} \leq \begin{cases} \dfrac{(1+s)^2}{e^s} & \text{when } s \geq 0, \\[2ex] e^{-s^2} & \text{when } -y < s < 0, \end{cases}$$

which can be easily checked using calculus. Using these facts, the Lebesgue dominated convergence theorem, the trivial fact that $\chi_{-y<s<\infty} \to 1$ as $y \to \infty$, and the identity in Appendix A.1, we obtain that

$$\begin{aligned} \lim_{x\to\infty} \frac{\Gamma(x+1)}{\left(\frac{x}{e}\right)^x \sqrt{2x}} &= \lim_{y\to\infty} \int_{-\infty}^{+\infty} \left(\frac{\left(1+\frac{s}{y}\right)^y}{e^s}\right)^{2y} \chi_{(-y,\infty)}(s)\, ds \\ &= \int_{-\infty}^{+\infty} e^{-s^2} ds \\ &= \sqrt{\pi}. \end{aligned}$$

A.7 Euler's Limit Formula for the Gamma Function

For n a positive integer and $\operatorname{Re} z > 0$ we consider the functions

$$\Gamma_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

We show that

$$\Gamma_n(z) = \frac{n!\,n^z}{z(z+1)\cdots(z+n)}$$

and we obtain *Euler's limit formula for the gamma function*

$$\lim_{n\to\infty} \Gamma_n(z) = \Gamma(z)\,.$$

We write $\Gamma(z) - \Gamma_n(z) = I_1(z) + I_2(z) + I_3(z)$, where

$$\begin{aligned} I_1(z) &= \int_n^\infty e^{-t} t^{z-1}\,dt\,,\\ I_2(z) &= \int_{n/2}^n \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt\,,\\ I_3(z) &= \int_0^{n/2} \left(e^{-t} - \left(1-\frac{t}{n}\right)^n\right) t^{z-1}\,dt\,. \end{aligned}$$

Obviously $I_1(z)$ tends to zero as $n\to\infty$. For I_2 and I_3 we have that $0 \le t < n$, and by the Taylor expansion of the logarithm we obtain

$$\log\left(1-\frac{t}{n}\right)^n = n\log\left(1-\frac{t}{n}\right) = -t - L\,,$$

where

$$L = \frac{t^2}{n}\left(\frac{1}{2} + \frac{1}{3}\frac{t}{n} + \frac{1}{4}\frac{t^2}{n^2} + \cdots\right).$$

It follows that

$$0 < e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t} - e^{-L}e^{-t} \le e^{-t}\,,$$

and thus $I_2(z)$ tends to zero as $n\to\infty$. For I_3 we have $t/n \le 1/2$, which implies that

$$L \le \frac{t^2}{n} \sum_{k=0}^{\infty} \frac{1}{(k+1)2^{k-1}} = \frac{t^2}{n} c\,.$$

Consequently, for $t/n \le 1/2$ we have

$$0 \le e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t}(1-e^{-L}) \le e^{-t}L \le e^{-t}\,\frac{ct^2}{n}\,.$$

Plugging this estimate into I_3, we deduce that

$$|I_3(z)| \le \frac{c}{n}\Gamma(\mathrm{Re}\,z + 2)\,,$$

which certainly tends to zero as $n\to\infty$.

Next, n integrations by parts give

$$\Gamma_n(z) = \frac{n}{nz}\,\frac{n-1}{n(z+1)}\,\frac{n-2}{n(z+2)}\cdots\frac{1}{n(z+n-1)}\int_0^n t^{z+n-1}\,dt = \frac{n!\,n^z}{z(z+1)\cdots(z+n)}\,.$$

This can be written as

$$1 = \Gamma_n(z)\,z\exp\left\{z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)\right\}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-z/k}\,.$$

Taking limits as $n \to \infty$, we obtain an *infinite product form of Euler's limit formula*,

$$1 = \Gamma(z)\,z\,e^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}\,,$$

where $\operatorname{Re} z > 0$ and γ is *Euler's constant*

$$\gamma = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\,.$$

The infinite product converges uniformly on compact subsets of the complex plane that excludes $z = 0, -1, -2, \dots$, and thus it represents a holomorphic function in this domain. This holomorphic function multiplied by $\Gamma(z)\,z\,e^{\gamma z}$ is equal to 1 on $\operatorname{Re} z > 0$ and by analytic continuation it must be equal to 1 on $\mathbf{C} \setminus \{0,-1,-2,\dots\}$. But $\Gamma(z)$ has simple poles, while the infinite product vanishes to order one at the nonpositive integers. We conclude that Euler's limit formula holds for all complex numbers z; consequently, $\Gamma(z)$ has no zeros and $\Gamma(z)^{-1}$ is entire.

An immediate consequence of Euler's limit formula is the identity

$$\frac{1}{|\Gamma(x+iy)|^2} = \frac{1}{|\Gamma(x)|^2}\prod_{k=0}^{\infty}\left(1+\frac{y^2}{(k+x)^2}\right),$$

which holds for x and y real with $x \notin \{0,-1,-2,\dots\}$. As a consequence we have that

$$|\Gamma(x+iy)| \le |\Gamma(x)|$$

and also that

$$\frac{1}{|\Gamma(x+iy)|} \le \frac{1}{|\Gamma(x)|}e^{C(x)|y|^2}\,,$$

where

$$C(x) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+x)^2}\,,$$

whenever $x \in \mathbf{R} \setminus \{0,-1,-2,\dots\}$ and $y \in \mathbf{R}$.

A.8 Reflection and Duplication Formulas for the Gamma Function

The *reflection formula* relates the values of the gamma function of a complex number z and its reflection about the point $1/2$ in the following way:

$$\frac{\sin(\pi z)}{\pi} = \frac{1}{\Gamma(z)} \frac{1}{\Gamma(1-z)}.$$

The *duplication formula* relates the entire functions $\Gamma(2z)^{-1}$ and $\Gamma(z)^{-1}$ as follows:

$$\frac{1}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \frac{\pi^{-\frac{1}{2}} 2^{2z-1}}{\Gamma(2z)}.$$

Both of these could be proved using Euler's limit formula. The reflection formula also uses the identity

$$\prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = \frac{\sin(\pi z)}{\pi z},$$

while the duplication formula makes use of the fact that

$$\lim_{n\to\infty} \frac{(n!)^2\, 2^{2n+1}}{(2n)!\, n^{1/2}} = 2\,\pi^{1/2}.$$

These and other facts related to the gamma function can be found in Olver [208].

Appendix B
Bessel Functions

B.1 Definition

We survey some basics from the theory of Bessel functions J_ν of complex order ν with $\operatorname{Re}\nu > -1/2$. We define the Bessel function J_ν of order ν by its *Poisson representation formula*

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} e^{its}(1-s^2)^\nu \, \frac{ds}{\sqrt{1-s^2}} \,,$$

where $\operatorname{Re}\nu > -1/2$ and $t \geq 0$. Although this definition is also valid when t is a complex number, for the applications we have in mind, it suffices to consider the case that t is real and nonnegative; in this case $J_\nu(t)$ is also a real number.

B.2 Some Basic Properties

Let us summarize a few properties of Bessel functions. We take $t > 0$.
(1) We have the following recurrence formula:

$$\frac{d}{dt}\big(t^{-\nu} J_\nu(t)\big) = -t^{-\nu} J_{\nu+1}(t), \qquad \operatorname{Re}\nu > -1/2.$$

(2) We also have the companion recurrence formula:

$$\frac{d}{dt}\big(t^{\nu} J_\nu(t)\big) = t^{\nu} J_{\nu-1}(t), \qquad \operatorname{Re}\nu > 1/2.$$

(3) $J_\nu(t)$ satisfies the differential equation:

$$t^2 \frac{d^2}{dt^2}(J_\nu(t)) + t\frac{d}{dt}(J_\nu(t)) + (t^2-\nu^2)J_\nu(t) = 0\,.$$

(4) If $\nu \in \mathbf{Z}^+$, then we have the following identity, which was taken by Bessel as the definition of J_ν for integer ν:

$$J_\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta} e^{-i\nu\theta}\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\sin\theta - \nu\theta)\, d\theta\,.$$

(5) For $\operatorname{Re}\nu > -1/2$ we have the following identity:

$$J_\nu(t) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{t}{2}\right)^\nu \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\nu+1)}\frac{t^{2j}}{(2j)!}\,.$$

(6) For $\operatorname{Re}\nu > 1/2$ the identity below is valid:

$$\frac{d}{dt}(J_\nu(t)) = \frac{1}{2}\big(J_{\nu-1}(t) - J_{\nu+1}(t)\big)\,.$$

We first verify property (1). We have

$$\begin{aligned}\frac{d}{dt}\big(t^{-\nu}J_\nu(t)\big) &= \frac{i}{2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} se^{its}(1-s^2)^{\nu-\frac{1}{2}}\,ds \\ &= \frac{i}{2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1}\frac{it}{2}e^{its}\frac{(1-s^2)^{\nu+\frac{1}{2}}}{\nu+\frac{1}{2}}\,ds \\ &= -t^{-\nu}J_{\nu+1}(t),\end{aligned}$$

where we integrated by parts and used the fact that $\Gamma(x+1) = x\Gamma(x)$. Property (2) can be proved similarly.

We proceed with the proof of property (3). A calculation using the definition of the Bessel function gives that the left-hand side of (3) is equal to

$$\frac{2^{-\nu}t^{\nu+1}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{+1} e^{ist}\Big((1-s^2)t + 2is(\nu+\tfrac{1}{2})\Big)(1-s^2)^{\nu-\frac{1}{2}}\,ds\,,$$

which in turn is equal to

$$-i\,\frac{2^{-\nu}t^{\nu+1}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{+1}\frac{d}{ds}\big(e^{ist}(1-s^2)^{\nu+\frac{1}{2}}\big)\,ds = 0\,.$$

Property (4) can be derived directly from (1). Define

$$G_\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\sin\theta}e^{-i\nu\theta}\,d\theta,$$

for $\nu = 0,1,2,\dots$ and $t > 0$. We can show easily that $G_0 = J_0$. If we had

$$\frac{d}{dt}\big(t^{-\nu}G_\nu(t)\big) = -t^{-\nu}G_{\nu+1}(t), \qquad t > 0,$$

for $\nu \in \mathbf{Z}^+$, we would immediately conclude that $G_\nu = J_\nu$ for $\nu \in \mathbf{Z}^+$. We have

$$\begin{aligned}
\frac{d}{dt}\big(t^{-\nu}G_\nu(t)\big) &= -t^{-\nu}\left(\frac{\nu}{t}G_\nu(t)-\frac{dG_\nu}{dt}(t)\right)\\
&= -t^{-\nu}\int_0^{2\pi}\frac{\nu}{2\pi t}e^{it\sin\theta}e^{-i\nu\theta}-\frac{1}{2\pi}\left(\frac{d}{dt}e^{it\sin\theta}\right)e^{-i\nu\theta}\,d\theta\\
&= -\frac{t^{-\nu}}{2\pi}\int_0^{2\pi} i\frac{d}{d\theta}\left(\frac{e^{it\sin\theta-i\nu\theta}}{t}\right)+(\cos\theta-i\sin\theta)e^{it\sin\theta}e^{-i\nu\theta}\,d\theta\\
&= -\frac{t^{-\nu}}{2\pi}\int_0^{2\pi} e^{it\sin\theta}e^{-i(\nu+1)\theta}\,d\theta\\
&= -t^{-\nu}G_{\nu+1}(t)\,.
\end{aligned}$$

For t real, the identity in (5) can be derived by inserting the expression

$$\sum_{j=0}^{\infty}(-1)^j\frac{(ts)^{2j}}{(2j)!}+i\sin(ts)$$

for e^{its} in the definition of the Bessel function $J_\nu(t)$ in Appendix B.1. Algebraic manipulations yield

$$\begin{aligned}
J_\nu(t) &= \frac{(t/2)^\nu}{\Gamma(\frac12)}\sum_{j=0}^{\infty}(-1)^j\frac{1}{\Gamma(\nu+\frac12)}\frac{t^{2j}}{(2j)!}2\int_0^1 s^{2j-1}(1-s^2)^{\nu-\frac12}s\,ds\\
&= \frac{(t/2)^\nu}{\Gamma(\frac12)}\sum_{j=0}^{\infty}(-1)^j\frac{1}{\Gamma(\nu+\frac12)}\frac{t^{2j}}{(2j)!}\frac{\Gamma(j+\frac12)\Gamma(\nu+\frac12)}{\Gamma(j+\nu+1)}\\
&= \frac{(t/2)^\nu}{\Gamma(\frac12)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(j+\frac12)}{\Gamma(j+\nu+1)}\frac{t^{2j}}{(2j)!}\,.
\end{aligned}$$

To derive property (6) we first multiply (1) by t^ν and (2) by $t^{-\nu}$; then we use the product rule for differentiation and we add the resulting expressions.

For further identities on Bessel functions, one may consult Watson's monograph [288].

B.3 An Interesting Identity

Let $\operatorname{Re}\mu > -\frac12$, $\operatorname{Re}\nu > -1$, and $t > 0$. Then the following identity is valid:

$$\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu\,ds = \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t)\,.$$

To prove this identity we use formula (5) in Appendix B.2. We have

$$\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu\,ds$$

$$= \frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\frac{1}{2})}\int_0^1 \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!} s^{2j+\mu+\mu}(1-s^2)^\nu s\,ds$$

$$= \frac{1}{2}\frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\int_0^1 u^{j+\mu}(1-u)^\nu\,du$$

$$= \frac{1}{2}\frac{\left(\frac{t}{2}\right)^\mu}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\frac{\Gamma(\mu+j+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+j+2)}$$

$$= \frac{2^\nu\Gamma(\nu+1)}{t^{\nu+1}}\frac{\left(\frac{t}{2}\right)^{\mu+\nu+1}}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+\nu+2)(2j)!}$$

$$= \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t)\,.$$

B.4 The Fourier Transform of Surface Measure on $\mathbf{S}^{n-1}$

Let $d\sigma$ denote surface measure on $\mathbf{S}^{n-1}$ for $n \geq 2$. Then the following is true:

$$\widehat{d\sigma}(\xi) = \int_{\mathbf{S}^{n-1}} e^{-2\pi i\xi\cdot\theta}\,d\theta = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|)\,.$$

To see this, use the result in Appendix D.3 to write

$$\begin{aligned}\widehat{d\sigma}(\xi) &= \int_{\mathbf{S}^{n-1}} e^{-2\pi i\xi\cdot\theta}\,d\theta \\ &= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{-1}^{+1} e^{-2\pi i|\xi|s}(1-s^2)^{\frac{n-2}{2}}\frac{ds}{\sqrt{1-s^2}} \\ &= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\frac{\Gamma(\frac{n-2}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{(\pi|\xi|)^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|) \\ &= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|)\,.\end{aligned}$$

B.5 The Fourier Transform of a Radial Function on $\mathbf{R}^n$

Let $f(x) = f_0(|x|)$ be a radial function defined on $\mathbf{R}^n$, where f_0 is defined on $[0,\infty)$. Then the Fourier transform of f is given by the formula

$$\widehat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n}{2}-1}(2\pi r|\xi|) r^{\frac{n}{2}}\, dr.$$

To obtain this formula, use polar coordinates to write

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbf{R}^n} f(x) e^{-2\pi i \xi\cdot x}\, dx \\
&= \int_0^\infty \int_{\mathbf{S}^{n-1}} f_0(r) e^{-2\pi i \xi\cdot r\theta} d\theta\, r^{n-1} dr \\
&= \int_0^\infty f_0(r) \widehat{d\sigma}(r\xi) r^{n-1} dr \\
&= \int_0^\infty f_0(r) \frac{2\pi}{(r|\xi|)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{n-1} dr \\
&= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n}{2}-1}(2\pi r|\xi|) r^{\frac{n}{2}}\, dr.
\end{aligned}$$

As an application we take $f(x) = \chi_{B(0,1)}$, where $B(0,1)$ is the unit ball in $\mathbf{R}^n$. We obtain

$$(\chi_{B(0,1)})\widehat{\ }(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|\xi|r) r^{\frac{n}{2}}\, dr = \frac{J_{\frac{n}{2}}(2\pi|\xi|)}{|\xi|^{\frac{n}{2}}},$$

in view of the result in Appendix B.3. More generally, for $\operatorname{Re}\lambda > -1$, let

$$m_\lambda(\xi) = \begin{cases} (1-|\xi|^2)^\lambda & \text{for } |\xi| \le 1, \\ 0 & \text{for } |\xi| > 1. \end{cases}$$

Then

$$m_\lambda^\vee(x) = \frac{2\pi}{|x|^{\frac{n-2}{2}}} \int_0^1 J_{\frac{n}{2}-1}(2\pi|x|r) r^{\frac{n}{2}} (1-r^2)^\lambda\, dr = \frac{\Gamma(\lambda+1)}{\pi^\lambda} \frac{J_{\frac{n}{2}+\lambda}(2\pi|x|)}{|x|^{\frac{n}{2}+\lambda}},$$

using again the identity in Appendix B.3.

B.6 Bessel Functions of Small Arguments

We seek the behavior of $J_k(r)$ as $r \to 0+$. We fix a complex number ν with $\operatorname{Re}\nu > -\frac{1}{2}$. Then we have the identity

$$J_\nu(r) = \frac{r^\nu}{2^\nu \Gamma(\nu+1)} + S_\nu(r),$$

where

$$S_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{+1} (e^{irt}-1)(1-t^2)^{\nu-\frac{1}{2}}\, dt$$

and S_ν satisfies

$$|S_\nu(r)| \le \frac{2^{-\operatorname{Re}\nu} r^{\operatorname{Re}\nu+1}}{(\operatorname{Re}\nu+1)\,|\Gamma(\nu+\frac{1}{2})|\,\Gamma(\frac{1}{2})}.$$

To prove this estimate we note that

$$\begin{aligned} J_\nu(r) &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}\,dt + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi}(\sin^2\phi)^{\nu-\frac{1}{2}}(\sin\phi)\,d\phi + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\nu+1)} + S_\nu(r), \end{aligned}$$

where we evaluated the last integral using the result in Appendix A.4. Using that $|e^{irt}-1| \le r|t|$, we deduce the assertion regarding the size of $|S_\nu(r)|$.

It follows from these facts and the estimate in Appendix A.7 that for $0 < r \le 1$ and $\operatorname{Re}\nu > -1/2$ we have

$$|J_\nu(r)| \le C_0\, e^{c_0\,|\operatorname{Im}\nu|^2}\, r^{\operatorname{Re}\nu},$$

where C_0 and c_0 are constants depending only on $\operatorname{Re}\nu$. Note that when $\operatorname{Re}\nu \ge 0$, the constant c_0 may be taken to be absolute (such as $c_0 = \pi^2$).

B.7 Bessel Functions of Large Arguments

For $r > 0$ and complex numbers ν with $\operatorname{Re}\nu > -1/2$ we prove the identity

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\left[ie^{-ir}\int_0^\infty e^{-rt}(t^2+2it)^{\nu-\frac{1}{2}}\,dt - ie^{ir}\int_0^\infty e^{-rt}(t^2-2it)^{\nu-\frac{1}{2}}\,dt\right].$$

Fix $0 < \delta < 1/10 < 10 < R < \infty$. We consider the region $\Omega_{\delta,R}$ in the complex plane whose boundary is the set consisting of the interval $[-1+\delta, 1-\delta]$ union a quarter circle centered at 1 of radius δ from $1-\delta$ to $1+i\delta$, union the line segments from $1+i\delta$ to $1+iR$, from $1+iR$ to $-1+iR$, and from $-1+iR$ to $-1+i\delta$, union a quarter circle centered at -1 of radius δ from $-1+i\delta$ to $-1+\delta$. This is a simply connected region on the interior of which the holomorphic function $(1-z^2)$ has no zeros. Since $\Omega_{\delta,R}$ is contained in the complement of the negative imaginary axis, there is a holomorphic branch of the logarithm such that $\log(t)$ is real, $\log(-t) = \log|t| + i\pi$, and $\log(it) = \log|t| + i\pi/2$ for $t > 0$. Since the function $\log(1-z^2)$ is well defined and holomorphic in $\Omega_{\delta,R}$, we may define the holomorphic function

$$(1-z^2)^{\nu-\frac{1}{2}} = e^{(\nu-\frac{1}{2})\log(1-z^2)}$$

for $z \in \Omega_{\delta,R}$. Since $e^{irz}(1-z^2)^{\nu-\frac{1}{2}}$ has no poles in $\Omega_{\delta,R}$, Cauchy's theorem yields

$$i\int_{\delta}^{R} e^{ir(1+it)}(t^2-2it)^{\nu-\frac{1}{2}}\,dt + \int_{-1+\delta}^{1-\delta} e^{irt}(1-t^2)^{\nu-\frac{1}{2}}\,dt \\ + i\int_{R}^{\delta} e^{ir(-1+it)}(t^2+2it)^{\nu-\frac{1}{2}}\,dt + E(\delta,R) = 0\,,$$

where $E(\delta,R)$ is the sum of the integrals over the two small quarter-circles of radius δ and the line segment from $1+iR$ to $-1+iR$. The first two of these integrals are bounded by constants times δ, the latter by a constant times $R^{2\mathrm{Re}\,\nu-1}e^{-rR}$; hence $E(\delta,R)\to 0$ as $\delta\to 0$ and $R\to\infty$. We deduce the identity

$$\int_{-1}^{+1} e^{irt}(1-t^2)^{\nu-\frac{1}{2}}\,dt = ie^{-ir}\int_0^{\infty} e^{-rt}(t^2+2it)^{\nu-\frac{1}{2}}\,dt - ie^{ir}\int_0^{\infty} e^{-rt}(t^2-2it)^{\nu-\frac{1}{2}}\,dt\,.$$

Estimating the two integrals on the right by putting absolute values inside and multiplying by the missing factor $r^{\nu}2^{-\nu}(\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2}))^{-1}$, we obtain

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\mathrm{Re}\,\nu}\,e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|}}{|\Gamma(\nu+\frac{1}{2})|\Gamma(\frac{1}{2})}\int_0^{\infty} e^{-rt}t^{\mathrm{Re}\,\nu-\frac{1}{2}}\big(\sqrt{t^2+4}\big)^{\mathrm{Re}\,\nu-\frac{1}{2}}\,dt\,,$$

since the absolute value of the argument of $t^2\pm 2it$ is at most $\pi/2$. When $\mathrm{Re}\,\nu>1/2$, we use the inequality $(\sqrt{t^2+4})^{\mathrm{Re}\,\nu-\frac{1}{2}} \le 2^{\mathrm{Re}\,\nu-\frac{3}{2}}\big(t^{\mathrm{Re}\,\nu-\frac{1}{2}}+2^{\mathrm{Re}\,\nu-\frac{1}{2}}\big)$ to get

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\mathrm{Re}\,\nu}\,e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|}}{|\Gamma(\nu+\frac{1}{2})|\Gamma(\frac{1}{2})}\,2^{\mathrm{Re}\,\nu-\frac{3}{2}}\left[\frac{\Gamma(2\mathrm{Re}\,\nu)}{r^{2\mathrm{Re}\,\nu}} + 2^{\mathrm{Re}\,\nu}\frac{\Gamma(\mathrm{Re}\,\nu+\frac{1}{2})}{r^{\mathrm{Re}\,\nu+\frac{1}{2}}}\right].$$

When $1/2\ge \mathrm{Re}\,\nu > -1/2$ we use that $\big(\sqrt{t^2+4}\big)^{\mathrm{Re}\,\nu-\frac{1}{2}}\le 1$ to deduce that

$$|J_\nu(r)| \le 2\,\frac{(r/2)^{\mathrm{Re}\,\nu}\,e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|}}{|\Gamma(\nu+\frac{1}{2})|\Gamma(\frac{1}{2})}\,\frac{\Gamma(\mathrm{Re}\,\nu+\frac{1}{2})}{r^{\mathrm{Re}\,\nu+\frac{1}{2}}}\,.$$

These estimates yield that for $\mathrm{Re}\,\nu>-1/2$ and $r\ge 1$ we have

$$|J_\nu(r)| \le C_0(\mathrm{Re}\,\nu)\,e^{\pi|\mathrm{Im}\,\nu|+\pi^2|\mathrm{Im}\,\nu|^2}\,r^{-1/2}$$

using the result in Appendix A.7. Here C_0 is a constant that depends only on $\mathrm{Re}\,\nu$.

B.8 Asymptotics of Bessel Functions

We obtain asymptotics for $J_\nu(r)$ as $r\to\infty$ whenever $\mathrm{Re}\,\nu>-1/2$. We have the following identity for $r>0$:

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + R_\nu(r),$$

where R_ν is given by

$$\begin{aligned} R_\nu(r) = {} & \frac{(2\pi)^{-\frac{1}{2}} r^\nu}{\Gamma(\nu+\frac{1}{2})} e^{i(r-\frac{\pi\nu}{2}-\frac{\pi}{4})} \int_0^\infty e^{-rt} t^{\nu+\frac{1}{2}} \left[(1+\tfrac{it}{2})^{\nu-\frac{1}{2}} - 1\right] \frac{dt}{t} \\ & + \frac{(2\pi)^{-\frac{1}{2}} r^\nu}{\Gamma(\nu+\frac{1}{2})} e^{-i(r-\frac{\pi\nu}{2}-\frac{\pi}{4})} \int_0^\infty e^{-rt} t^{\nu+\frac{1}{2}} \left[(1-\tfrac{it}{2})^{\nu-\frac{1}{2}} - 1\right] \frac{dt}{t} \end{aligned}$$

and satisfies $|R_\nu(r)| \le C_\nu r^{-3/2}$ whenever $r \ge 1$.

To see the validity of this identity we write

$$\begin{aligned} ie^{-ir}(t^2+2it)^{\nu-\frac{1}{2}} &= (2t)^{\nu-\frac{1}{2}} e^{-i(r-\frac{\nu\pi}{2}-\frac{\pi}{4})} (1-\tfrac{it}{2})^{\nu-\frac{1}{2}}, \\ -ie^{ir}(t^2-2it)^{\nu-\frac{1}{2}} &= (2t)^{\nu-\frac{1}{2}} e^{i(r-\frac{\nu\pi}{2}-\frac{\pi}{4})} (1+\tfrac{it}{2})^{\nu-\frac{1}{2}}. \end{aligned}$$

Inserting these expressions into the corresponding integrals in the formula proved in Appendix B.7, adding and subtracting 1 from each term $(1 \pm \frac{it}{2})^{\nu-\frac{1}{2}}$, and multiplying by the missing factor $(r/2)^\nu / \Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})$, we obtain the claimed identity

$$J_\nu(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + R_\nu(r).$$

It remains to estimate $R_\nu(r)$. We begin by noting that for a, b real with $a > -1$ we have the pair of inequalities

$$\begin{aligned} |(1 \pm iy)^{a+ib} - 1| &\le 3\,(|a|+|b|)\left(2^{\frac{a+1}{2}} e^{\frac{\pi}{2}|b|}\right) y && \text{when } 0 < y < 1, \\ |(1 \pm iy)^{a+ib} - 1| &\le (1+y^2)^{\frac{a}{2}} e^{\frac{\pi}{2}|b|} + 1 \le 2\left(2^{\frac{a+1}{2}} e^{\frac{\pi}{2}|b|}\right) y^a && \text{when } 1 \le y < \infty. \end{aligned}$$

The first inequality is proved by splitting into real and imaginary parts and applying the mean value theorem in the real part. Taking $\nu - \frac{1}{2} = a + ib$, $y = t/2$, and inserting these estimates into the integrals appearing in R_ν, we obtain

$$|R_\nu(r)| \le \frac{2^{\frac{1}{2}\mathrm{Re}\,\nu} 2^{\frac{1}{4}} e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|} r^{\mathrm{Re}\,\nu}}{(2\pi)^{1/2} |\Gamma(\nu+\frac{1}{2})|} \left[\frac{3\sqrt{2}|\nu|}{2} \int_0^2 e^{-rt} t^{\mathrm{Re}\,\nu+\frac{3}{2}} \frac{dt}{t} + \frac{2\sqrt{2}}{2^{\mathrm{Re}\,\nu}} \int_2^\infty e^{-rt} t^{2\mathrm{Re}\,\nu} \frac{dt}{t}\right].$$

It follows that for all $r > 0$ we have

$$\begin{aligned} |R_\nu(r)| &\le 2\,\frac{2^{\frac{1}{2}\mathrm{Re}\,\nu} e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|}}{|\Gamma(\nu+\frac{1}{2})|} \left[|\nu| \frac{\Gamma(\mathrm{Re}\,\nu+\frac{3}{2})}{r^{3/2}} + \frac{r^{-\mathrm{Re}\,\nu}}{2^{\mathrm{Re}\,\nu}} \int_{2r}^\infty e^{-t} t^{2\mathrm{Re}\,\nu} \frac{dt}{t}\right] \\ &\le 2\,\frac{2^{\frac{1}{2}\mathrm{Re}\,\nu} e^{\frac{\pi}{2}|\mathrm{Im}\,\nu|}}{|\Gamma(\nu+\frac{1}{2})|} \left[|\nu| \frac{\Gamma(\mathrm{Re}\,\nu+\frac{3}{2})}{r^{3/2}} + \frac{2^{\mathrm{Re}\,\nu}}{r^{\mathrm{Re}\,\nu}} \frac{\Gamma(2\mathrm{Re}\,\nu)}{e^r}\right], \end{aligned}$$

using that $e^{-t} \le e^{-t/2}e^{-r}$ for $t \ge 2r$. We conclude that for $r \ge 1$ and $\operatorname{Re}\nu > -1/2$ we have

$$|R_\nu(r)| \le C_0(\operatorname{Re}\nu)\,\frac{e^{\frac{\pi}{2}|\operatorname{Im}\nu|}\,(|\nu|+1)}{|\Gamma(\nu+\frac{1}{2})|}\,r^{-3/2},$$

where C_0 is a constant that depends only on $\operatorname{Re}\nu$. In view of the result in Appendix A.7, the last fraction is bounded by another constant depending on $\operatorname{Re}\nu$ times $e^{\pi^2(1+|\operatorname{Im}\nu|)^2}$.

Appendix C
Rademacher Functions

C.1 Definition of the Rademacher Functions

The Rademacher functions are defined on $[0,1]$ as follows: $r_0(t) = 1$; $r_1(t) = 1$ for $0 \le t \le 1/2$ and $r_1(t) = -1$ for $1/2 < t \le 1$; $r_2(t) = 1$ for $0 \le t \le 1/4$, $r_2(t) = -1$ for $1/4 < t \le 1/2$, $r_2(t) = 1$ for $1/2 < t \le 3/4$, and $r_2(t) = -1$ for $3/4 < t \le 1$; and so on. According to this definition, we have that $r_j(t) = \operatorname{sgn}(\sin(2^j \pi t))$ for $j = 0,1,2,\dots$. It is easy to check that the r_j's are mutually independent random variables on $[0,1]$. This means that for all functions f_j we have

$$\int_0^1 \prod_{j=0}^n f_j(r_j(t))\,dt = \prod_{j=0}^n \int_0^1 f_j(r_j(t))\,dt\,.$$

To see the validity of this identity, we write its right-hand side as

$$\begin{aligned} f_0(1)\prod_{j=1}^n \int_0^1 f_j(r_j(t))\,dt &= f_0(1)\prod_{j=1}^n \frac{f_j(1)+f_j(-1)}{2} \\ &= \frac{f_0(1)}{2^n} \sum_{S\subset\{1,2,\dots,n\}} \prod_{j\in S} f_j(1) \prod_{j\notin S} f_j(-1) \end{aligned}$$

and we observe that there is a one-to-one and onto correspondence between subsets S of $\{1,2,\dots,n\}$ and intervals $I_k = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$, $k = 0,1,\dots,2^n-1$, such that the restriction of the function $\prod_{j=1}^n f_j(r_j(t))$ on I_k is equal to

$$\prod_{j\in S} f_j(1) \prod_{j\notin S} f_j(-1)\,.$$

It follows that the last of the three equal displayed expressions is

$$f_0(1)\sum_{k=0}^{2^n-1} \int_{I_k} \prod_{j=1}^n f_j(r_j(t))\,dt = \int_0^1 \prod_{j=0}^n f_j(r_j(t))\,dt\,.$$

C.2 Khintchine's Inequalities

The following property of the Rademacher functions is of fundamental importance and with far-reaching consequences in analysis:

For any $0 < p < \infty$ and for any real-valued square summable sequences $\{a_j\}$ and $\{b_j\}$ we have

$$B_p\left(\sum_j |a_j+ib_j|^2\right)^{\frac{1}{2}} \le \left\|\sum_j (a_j+ib_j)r_j\right\|_{L^p([0,1])} \le A_p\left(\sum_j |a_j+ib_j|^2\right)^{\frac{1}{2}}$$

for some constants $0 < A_p, B_p < \infty$ that depend only on p.

These inequalities reflect the orthogonality of the Rademacher functions in L^p (especially when $p \neq 2$). Khintchine [155] was the first to prove a special form of this inequality, and he used it to estimate the asymptotic behavior of certain random walks. Later this inequality was systematically studied almost simultaneously by Littlewood [173] and by Paley and Zygmund [210], who proved the more general form stated previously. The foregoing inequalities are usually referred to by Khintchine's name.

C.3 Derivation of Khintchine's Inequalities

Both assertions in Appendix C.2 can be derived from an exponentially decaying distributional inequality for the function

$$F(t) = \sum_j (a_j+ib_j)r_j(t), \qquad t \in [0,1],$$

when a_j, b_j are square summable real numbers.

We first obtain a distributional inequality for the above function F under the following three assumptions:

(a) The sequence $\{b_j\}$ is identically zero.
(b) All but finitely many terms of the sequence $\{a_j\}$ are zero.
(c) The sequence $\{a_j\}$ satisfies $(\sum_j |a_j|^2)^{1/2} = 1$.

Let $\rho > 0$. Under assumptions (a), (b), and (c), independence gives

$$\begin{aligned}\int_0^1 e^{\rho \sum a_j r_j(t)}\,dt &= \prod_j \int_0^1 e^{\rho a_j r_j(t)}\,dt \\ &= \prod_j \frac{e^{\rho a_j}+e^{-\rho a_j}}{2} \\ &\le \prod_j e^{\frac{1}{2}\rho^2 a_j^2} = e^{\frac{1}{2}\rho^2 \sum a_j^2} = e^{\frac{1}{2}\rho^2},\end{aligned}$$

where we used the inequality $\frac{1}{2}(e^x+e^{-x}) \le e^{\frac{1}{2}x^2}$ for all real x, which can be checked using power series expansions. Since the same argument is also valid for $-\sum a_j r_j(t)$, we obtain that

$$\int_0^1 e^{\rho|F(t)|}\,dt \le 2e^{\frac{1}{2}\rho^2}.$$

From this it follows that

$$e^{\rho\alpha}|\{t\in[0,1]:\ |F(t)|>\alpha\}| \le \int_0^1 e^{\rho|F(t)|}\,dt \le 2e^{\frac{1}{2}\rho^2}$$

and hence we obtain the distributional inequality

$$d_F(\alpha)=|\{t\in[0,1]:\ |F(t)|>\alpha\}| \le 2e^{\frac{1}{2}\rho^2-\rho\alpha} = 2e^{-\frac{1}{2}\alpha^2},$$

by picking $\rho=\alpha$. The L^p norm of F can now be computed easily. Formula (1.1.6) gives

$$\big\|F\big\|_{L^p}^p = \int_0^\infty p\alpha^{p-1}d_F(\alpha)\,d\alpha \le \int_0^\infty p\alpha^{p-1}2e^{-\frac{\alpha^2}{2}}\,d\alpha = 2^{\frac{p}{2}}p\Gamma(p/2).$$

We have now proved that

$$\big\|F\big\|_{L^p} \le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\big\|F\big\|_{L^2}$$

under assumptions (a), (b), and (c).

We now dispose of assumptions (a), (b), and (c). Assumption (b) can be easily eliminated by a limiting argument and (c) by a scaling argument. To dispose of assumption (a), let a_j and b_j be real numbers. We have

$$\begin{aligned}
\Big\|\sum_j(a_j+ib_j)r_j\Big\|_{L^p} &\le \Big\||\sum_j a_jr_j|+|\sum_j b_jr_j|\Big\|_{L^p}\\
&\le \Big\|\sum_j a_jr_j\Big\|_{L^p}+\Big\|\sum_j b_jr_j\Big\|_{L^p}\\
&\le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\Big(\Big(\sum_j|a_j|^2\Big)^{\frac{1}{2}}+\Big(\sum_j|b_j|^2\Big)^{\frac{1}{2}}\Big)\\
&\le \sqrt{2}\,\big(p\Gamma(p/2)\big)^{\frac{1}{p}}\sqrt{2}\Big(\sum_j|a_j+ib_j|^2\Big)^{\frac{1}{2}}.
\end{aligned}$$

Let us now set $A_p=2\big(p\Gamma(p/2)\big)^{1/p}$ when $p>2$. Since we have the trivial estimate $\big\|F\big\|_{L^p}\le\big\|F\big\|_{L^2}$ when $0<p\le 2$, we obtain the required inequality $\big\|F\big\|_{L^p}\le A_p\big\|F\big\|_{L^2}$ with

$$A_p=\begin{cases}1 & \text{when } 0<p\le 2,\\ 2\,p^{\frac{1}{p}}\,\Gamma(p/2)^{\frac{1}{p}} & \text{when } 2<p<\infty.\end{cases}$$

Using Sterling's formula in Appendix A.6, we see that A_p is asymptotic to $\sqrt{p}$ as $p\to\infty$.

We now discuss the converse inequality $B_p\|F\|_{L^2} \le \|F\|_{L^p}$. It is clear that $\|F\|_{L^2} \le \|F\|_{L^p}$ when $p \ge 2$ and we may therefore take $B_p = 1$ for $p \ge 2$. Let us now consider the case $0 < p < 2$. Pick an s such that $2 < s < \infty$. Find a $0 < \theta < 1$ such that

$$\frac{1}{2} = \frac{1-\theta}{p} + \frac{\theta}{s}.$$

Then

$$\|F\|_{L^2} \le \|F\|_{L^p}^{1-\theta}\|F\|_{L^s}^{\theta} \le \|F\|_{L^p}^{1-\theta} A_s^{\theta}\|F\|_{L^2}^{\theta}.$$

It follows that

$$\|F\|_{L^2} \le A_s^{\frac{\theta}{1-\theta}}\|F\|_{L^p}.$$

We have now proved the inequality $B_p\|F\|_{L^2} \le \|F\|_{L^p}$ with

$$B_p = \begin{cases} 1 & \text{when } 2 \le p < \infty, \\ \sup\limits_{s>2} A_s^{-\frac{\frac{1}{p}-\frac{1}{2}}{\frac{1}{2}-\frac{1}{s}}} & \text{when } 0 < p < 2. \end{cases}$$

Observe that the function $s \to A_s^{-\left(\frac{1}{p}-\frac{1}{2}\right)/\left(\frac{1}{2}-\frac{1}{s}\right)}$ tends to zero as $s \to 2+$ and as $s \to \infty$. Hence it must attain its maximum for some $s = s(p)$ in the interval $(2,\infty)$. We see that $B_p \ge 16 \cdot 256^{-1/p}$ when $p < 2$ by taking $s = 4$.

It is worthwhile to mention that the best possible values of the constants A_p and B_p in Khintchine's inequality are known when $b_j = 0$. In this case Szarek [271] showed that the best possible value of B_1 is $1/\sqrt{2}$, and later Haagerup [116] found that when $b_j = 0$ the best possible values of A_p and B_p are the numbers

$$A_p = \begin{cases} 1 & \text{when } 0 < p \le 2, \\ 2^{\frac{1}{2}}\pi^{-\frac{1}{2p}}\Gamma(\frac{p+1}{2}) & \text{when } 2 < p < \infty, \end{cases}$$

and

$$B_p = \begin{cases} 2^{\frac{1}{2}-\frac{1}{p}} & \text{when } 0 < p \le p_0, \\ 2^{\frac{1}{2}}\pi^{-\frac{1}{2p}}\Gamma(\frac{p+1}{2}) & \text{when } p_0 < p < 2, \\ 1 & \text{when } 2 < p < \infty, \end{cases}$$

where $p_0 = 1.84742\ldots$ is the unique solution of the equation $2\Gamma(\frac{p+1}{2}) = \sqrt{\pi}$ in the interval $(1,2)$.

C.4 Khintchine's Inequalities for Weak Type Spaces

We note that the following weak type estimates are valid:

$$4^{-\frac{1}{p}} B_{\frac{p}{2}} \Big(\sum_j |a_j + ib_j|^2\Big)^{\frac{1}{2}} \le \Big\| \sum_j (a_j + ib_j) r_j \Big\|_{L^{p,\infty}} \le A_p \Big(\sum_j |a_j + ib_j|^2\Big)^{\frac{1}{2}}$$

for all $0 < p < \infty$.

Indeed, the upper estimate is a simple consequence of the fact that L^p is a subspace of $L^{p,\infty}$. For the converse inequality we use the fact that $L^{p,\infty}([0,1])$ is contained in $L^{p/2}([0,1])$ and we have (see Exercise 1.1.11)

$$\big\|F\big\|_{L^{p/2}} \le 4^{\frac{1}{p}} \big\|F\big\|_{L^{p,\infty}}.$$

Since any Lorentz space $L^{p,q}([0,1])$ can be sandwiched between $L^{2p}([0,1])$ and $L^{p/2}([0,1])$, similar inequalities hold for all Lorentz spaces $L^{p,q}([0,1])$, $0 < p < \infty$, $0 < q \le \infty$.

C.5 Extension to Several Variables

We first extend the inequality on the right in Appendix C.2 to several variables. For a positive integer n we let

$$F_n(t_1, \dots, t_n) = \sum_{j_1} \cdots \sum_{j_n} c_{j_1,\dots,j_n} r_{j_1}(t_1) \cdots r_{j_n}(t_n),$$

for $t_j \in [0,1]$, where $c_{j_1,\dots,j_n}$ is a sequence of complex numbers and F_n is a function defined on $[0,1]^n$.

For any $0 < p < \infty$ and for any complex-valued square summable sequence of n variables $\{c_{j_1,\dots,j_n}\}_{j_1,\dots,j_n}$, we have the following inequalities for F_n:

$$B_p^n \Big(\sum_{j_1} \cdots \sum_{j_n} |c_{j_1,\dots,j_n}|^2\Big)^{\frac{1}{2}} \le \big\|F_n\big\|_{L^p} \le A_p^n \Big(\sum_{j_1} \cdots \sum_{j_n} |c_{j_1,\dots,j_n}|^2\Big)^{\frac{1}{2}},$$

where A_p, B_p are the constants in Appendix C.2. The norms are over $[0,1]^n$.

The case $n = 2$ is indicative of the general case. For $p \ge 2$ we have

$$\begin{aligned}
\int_0^1 \int_0^1 |F_2(t_1,t_2)|^p \, dt_1 \, dt_2 &\le A_p^p \int_0^1 \Big(\sum_{j_1} \Big|\sum_{j_2} c_{j_1,j_2} r_{j_2}(t_2)\Big|^2\Big)^{\frac{p}{2}} dt_2 \\
&\le A_p^p \bigg(\sum_{j_1} \Big(\int_0^1 \Big|\sum_{j_2} c_{j_1,j_2} r_{j_2}(t_2)\Big|^p dt_2\Big)^{\frac{2}{p}}\bigg)^{\frac{p}{2}} \\
&\le A_p^{2p} \Big(\sum_{j_1}\sum_{j_2} |c_{j_1,j_n}|^2\Big)^{\frac{p}{2}},
\end{aligned}$$

where we used Minkowski's integral inequality (with exponent $p/2 \geq 1$) in the second inequality and the result in the case $n = 1$ twice.

The case $p < 2$ follows trivially from Hölder's inequality with constant $A_p = 1$. The reverse inequalities follow exactly as in the case of one variable. Replacing A_p by A_p^n in the argument, giving the reverse inequality in the case $n = 1$, we obtain the constant B_p^n.

Likewise one may extend the weak type inequalities of Appendix C.3 in several variables.

Appendix D
Spherical Coordinates

D.1 Spherical Coordinate Formula

Switching integration from spherical coordinates to Cartesian is achieved via the following identity:

$$\int_{R\mathbf{S}^{n-1}} f(x)\,d\sigma(x) = \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \int_{\varphi_{n-1}=0}^{2\pi} f(x(\varphi))J(n,R,\varphi)\,d\varphi_{n-1}\cdots d\varphi_1,$$

where

$$\begin{aligned}
x_1 &= R\cos\varphi_1\,,\\
x_2 &= R\sin\varphi_1\,\cos\varphi_2\,,\\
x_3 &= R\sin\varphi_1\,\sin\varphi_2\,\cos\varphi_3\,,\\
&\dots\\
x_{n-1} &= R\sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3\cdots\sin\varphi_{n-2}\,\cos\varphi_{n-1}\,,\\
x_n &= R\sin\varphi_1\,\sin\varphi_2\,\sin\varphi_3\cdots\sin\varphi_{n-2}\,\sin\varphi_{n-1}\,,
\end{aligned}$$

and $0 \le \varphi_1,\dots,\varphi_{n-2} \le \pi$, $0 \le \varphi_{n-1} = \theta \le 2\pi$,

$$x(\varphi) = (x_1(\varphi_1,\dots,\varphi_{n-1}),\dots,x_n(\varphi_1,\dots,\varphi_{n-1}))\,,$$

and

$$J(n,R,\varphi) = R^{n-1}(\sin\varphi_1)^{n-2}\cdots(\sin\varphi_{n-3})^2(\sin\varphi_{n-2})$$

is the Jacobian of the transformation.

D.2 A Useful Change of Variables Formula

The following formula is useful in computing integrals over the sphere $\mathbf{S}^{n-1}$ when $n \ge 2$. Let f be a function defined on $\mathbf{S}^{n-1}$. Then we have

$$\int_{R\mathbf{S}^{n-1}} f(x)\,d\sigma(x) = \int_{-R}^{+R} \int_{\sqrt{R^2-s^2}\,\mathbf{S}^{n-2}} f\big(s,\theta\big)\,d\theta\,\frac{R\,ds}{\sqrt{R^2-s^2}}\,.$$

To prove this formula, let $\varphi' = (\varphi_2,\dots,\varphi_{n-1})$ and

$$x' = x'(\varphi') = (\cos\varphi_2, \sin\varphi_2\cos\varphi_3, \dots, \sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}).$$

Using the change of variables in Appendix D.1 we express

$$\int_{R\mathbf{S}^{n-1}} f(x)\,d\sigma(x)$$

as the iterated integral

$$\int_{\varphi_1=0}^{\pi}\left[\int_{\varphi_2=0}^{\pi}\cdots\int_{\varphi_{n-1}=0}^{2\pi} f(R\cos\varphi_1, R\sin\varphi_1\, x'(\varphi'))J(n-1,1,\varphi')\,d\varphi'\right]\frac{R\,d\varphi_1}{(R\sin\varphi_1)^{2-n}},$$

and we can realize the expression inside the square brackets as

$$\int_{\mathbf{S}^{n-2}} f(R\cos\varphi_1, R\sin\varphi_1\, x')d\sigma(x').$$

Consequently,

$$\int_{R\mathbf{S}^{n-1}} f(x)\,d\sigma(x) = \int_{\varphi_1=0}^{\pi}\int_{\mathbf{S}^{n-2}} f(R\cos\varphi_1, R\sin\varphi_1\, x')\,d\sigma(x')R^{n-1}(\sin\varphi_1)^{n-2}d\varphi_1,$$

and the change of variables

$$s = R\cos\varphi_1, \qquad \varphi_1\in(0,\pi),$$

$$ds = -R\sin\varphi_1\,d\varphi_1, \qquad \sqrt{R^2-s^2} = R\sin\varphi_1,$$

yields

$$\int_{R\mathbf{S}^{n-1}} f(x)\,d\sigma(x) = \int_{-R}^{R}\left\{\int_{\mathbf{S}^{n-2}} f(s,\sqrt{R^2-s^2}\,\theta)\,d\theta\right\}\left(\sqrt{R^2-s^2}\right)^{n-2}\frac{R\,ds}{\sqrt{R^2-s^2}}.$$

Rescaling the sphere $\mathbf{S}^{n-2}$ to $\sqrt{R^2-s^2}\,\mathbf{S}^{n-2}$ yields the claimed identity.

D.3 Computation of an Integral over the Sphere

Let K be a function on the line. We use the result in Appendix D.2 to show that for $n \ge 2$ we have

$$\int_{\mathbf{S}^{n-1}} K(x\cdot\theta)\,d\theta = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_{-1}^{+1} K(s|x|)\left(\sqrt{1-s^2}\right)^{n-3}ds$$

when $x\in\mathbf{R}^n\setminus\{0\}$. Let $x' = x/|x|$ and pick a matrix $A\in O(n)$ such that $Ae_1 = x'$, where $e_1 = (1,0,\dots,0)$. We have

$$\begin{aligned}\int_{\mathbf{S}^{n-1}} K(x\cdot\theta)\,d\theta &= \int_{\mathbf{S}^{n-1}} K(|x|(x'\cdot\theta))\,d\theta\\ &= \int_{\mathbf{S}^{n-1}} K(|x|(Ae_1\cdot\theta))\,d\theta\\ &= \int_{\mathbf{S}^{n-1}} K(|x|(e_1\cdot A^{-1}\theta))\,d\theta\\ &= \int_{\mathbf{S}^{n-1}} K(|x|\theta_1)\,d\theta\\ &= \int_{-1}^{+1} K(|x|s)\omega_{n-2}\big(\sqrt{1-s^2}\,\big)^{n-2}\frac{ds}{\sqrt{1-s^2}}\\ &= \omega_{n-2}\int_{-1}^{+1} K(s|x|)\big(\sqrt{1-s^2}\,\big)^{n-3}\,ds,\end{aligned}$$

where $\omega_{n-2} = 2\pi^{\frac{n-1}{2}}\Gamma\big(\frac{n-1}{2}\big)^{-1}$ is the surface area of $\mathbf{S}^{n-2}$.

For example, we have

$$\int_{\mathbf{S}^{n-1}}\frac{d\theta}{|\xi\cdot\theta|^{\alpha}} = \omega_{n-2}\int_{-1}^{+1}\frac{1}{|s|^{\alpha}|\xi|^{\alpha}}(1-s^2)^{\frac{n-3}{2}}\,ds = \frac{1}{|\xi|^{\alpha}}\frac{2\pi^{\frac{n-1}{2}}\Gamma\big(\frac{1-\alpha}{2}\big)}{\Gamma\big(\frac{n-\alpha}{2}\big)},$$

and the integral converges only when $\operatorname{Re}\alpha < 1$.

D.4 The Computation of Another Integral over the Sphere

We compute the following integral for $n \geq 2$:

$$\int_{\mathbf{S}^{n-1}}\frac{d\theta}{|\theta-e_1|^{\alpha}},$$

where $e_1 = (1,0,\dots,0)$. Applying the formula in Appendix D.2, we obtain

$$\begin{aligned}\int_{\mathbf{S}^{n-1}}\frac{d\theta}{|\theta-e_1|^{\alpha}} &= \int_{-1}^{+1}\int_{\theta\in\sqrt{1-s^2}\,\mathbf{S}^{n-2}}\frac{d\theta}{(|s-1|^2+|\theta|^2)^{\frac{\alpha}{2}}}\frac{ds}{\sqrt{1-s^2}}\\ &= \int_{-1}^{+1}\omega_{n-2}\frac{(1-s^2)^{\frac{n-2}{2}}}{\big((1-s)^2+1-s^2\big)^{\frac{\alpha}{2}}}\frac{ds}{\sqrt{1-s^2}}\\ &= \frac{\omega_{n-2}}{2^{\frac{\alpha}{2}}}\int_{-1}^{+1}\frac{(1-s^2)^{\frac{n-3}{2}}}{(1-s)^{\frac{\alpha}{2}}}ds\\ &= \frac{\omega_{n-2}}{2^{\frac{\alpha}{2}}}\int_{-1}^{+1}(1-s)^{\frac{n-3-\alpha}{2}}(1+s)^{\frac{n-3}{2}}\,ds,\end{aligned}$$

which converges exactly when $\operatorname{Re}\alpha < n-1$.

D.5 Integration over a General Surface

Suppose that S is a hypersurface in $\mathbf{R}^n$ of the form $S=\{(u,\Phi(u)):\ u\in D\}$, where D is an open subset of $\mathbf{R}^{n-1}$ and Φ is a continuously differentiable mapping from D to $\mathbf{R}$. Let σ be the canonical surface measure on S. If g is a function on S, then we have

$$\int_S g(y)\,d\sigma(y)=\int_D g(x,\Phi(x))\Big(1+\sum_{j=1}^n|\partial_j\Phi(x)|^2\Big)^{\frac{1}{2}}\,dx\,.$$

Specializing to the sphere, we obtain

$$\int_{\mathbf{S}^{n-1}} g(\theta)\,d\theta=\int\limits_{\substack{\xi'\in\mathbf{R}^{n-1}\\|\xi'|<1}}\Big[g(\xi',\sqrt{1-|\xi'|^2})+g(\xi',-\sqrt{1-|\xi'|^2})\Big]\frac{d\xi'}{\sqrt{1-|\xi'|^2}}\,.$$

D.6 The Stereographic Projection

Define a map $\Pi:\mathbf{R}^n\to\mathbf{S}^n$ by the formula

$$\Pi(x_1,\dots,x_n)=\bigg(\frac{2x_1}{1+|x|^2},\dots,\frac{2x_n}{1+|x|^2},\frac{|x|^2-1}{1+|x|^2}\bigg).$$

It is easy to see that Π is a one-to-one map from $\mathbf{R}^n$ onto the sphere $\mathbf{S}^n$ minus the north pole $e_{n+1}=(0,\dots,0,1)$. Its inverse is given by the formula

$$\Pi^{-1}(\theta_1,\dots,\theta_{n+1})=\bigg(\frac{\theta_1}{1-\theta_{n+1}},\dots,\frac{\theta_n}{1-\theta_{n+1}}\bigg).$$

The Jacobian of the map is verified to be

$$J_\Pi(x)=\Big(\frac{2}{1+|x|^2}\Big)^n,$$

and the following change of variables formulas are valid:

$$\int_{\mathbf{S}^n}F(\theta)\,d\theta=\int_{\mathbf{R}^n}F(\Pi(x))J_\Pi(x)\,dx$$

and

$$\int_{\mathbf{R}^n}F(x)\,dx=\int_{\mathbf{S}^n}F(\Pi^{-1}(\theta))\,J_{\Pi^{-1}}(\theta)\,d\theta\,,$$

where

$$J_{\Pi^{-1}}(\theta)=\frac{1}{J_\Pi(\Pi^{-1}(\theta))}=\bigg(\frac{|\theta_1|^2+\dots+|\theta_n|^2+|1-\theta_{n+1}|^2}{2|1-\theta_{n+1}|^2}\bigg)^n.$$

Another interesting formula about the stereographic projection Π is

$$|\Pi(x) - \Pi(y)| = 2|x-y|(1+|x|^2)^{-1/2}(1+|y|^2)^{-1/2},$$

for all x, y in $\mathbf{R}^n$.

Appendix E
Some Trigonometric Identities and Inequalities

The following inequalities are valid for t real:

$$
\begin{aligned}
0 < t < \frac{\pi}{2} &\implies \sin(t) < t < \tan(t)\,,\\
0 < |t| < \frac{\pi}{2} &\implies \frac{2}{\pi} < \frac{\sin(t)}{t} < 1\,,\\
-\infty < t < +\infty &\implies |\sin(t)| \le |t|\,,\\
-\infty < t < +\infty &\implies |1-\cos(t)| \le \frac{|t|^2}{2}\,,\\
-\infty < t < +\infty &\implies |1-e^{it}| \le |t|\,,\\
|t| \le \frac{\pi}{2} &\implies |\sin(t)| \ge \frac{2|t|}{\pi}\,,\\
|t| \le \pi &\implies |1-\cos(t)| \ge \frac{2|t|^2}{\pi^2}\,,\\
|t| \le \pi &\implies |1-e^{it}| \ge \frac{2|t|}{\pi}\,.
\end{aligned}
$$

The following sum to product formulas are valid:

$$
\begin{aligned}
\sin(a)+\sin(b) &= 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right),\\
\sin(a)-\sin(b) &= 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right),\\
\cos(a)+\cos(b) &= 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right),\\
\cos(a)-\cos(b) &= -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right).
\end{aligned}
$$

The following identities are also easily proved:

$$
\begin{aligned}
\sum_{k=1}^{N}\cos(kx) &= -\frac{1}{2} + \frac{\sin((N+\frac{1}{2})x)}{2\sin(\frac{x}{2})}\,,\\
\sum_{k=1}^{N}\sin(kx) &= \frac{\cos(\frac{x}{2}) - \cos((N+\frac{1}{2})x)}{2\sin(\frac{x}{2})}\,.
\end{aligned}
$$

Appendix F
Summation by Parts

Let $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ be two sequences of complex numbers. Then for $N \geq 1$ we have

$$\sum_{k=0}^{N} a_k b_k = A_N b_N - \sum_{k=0}^{N-1} A_k (b_{k+1} - b_k),$$

where

$$A_k = \sum_{j=0}^{k} a_j \,.$$

More generally we have

$$\sum_{k=M}^{N} a_k b_k = A_N b_N - A_{M-1} b_M - \sum_{k=M}^{N-1} A_k (b_{k+1} - b_k)\,,$$

whenever $0 \leq M \leq N$, where $A_{-1} = 0$ and

$$A_k = \sum_{j=0}^{k} a_j$$

for $k \geq 0$.

Appendix G
Basic Functional Analysis

A quasinorm is a nonnegative functional $\|\cdot\|$ on a vector space X that satisfies $\|x+y\|_X \le K(\|x\|_X + \|y\|_X)$ for some $K \ge 0$ and all $x,y \in X$ and also $\|\lambda x\|_X = |\lambda|\|x\|_X$ for all scalars λ. When $K=1$, the quasinorm is called a norm. A quasi-Banach space is a quasinormed space that is complete with respect to the topology generated by the quasinorm. The proofs of the following theorems can be found in several books including Albiac and Kalton [1], Kalton Peck and Roberts [150], and Rudin [230].

The Hahn–Banach theorem. Let X be a normed space and X_0 a subspace. Every bounded linear functional Λ_0 on X_0 has a bounded extension Λ on X with the same norm. In addition, if Λ_0 is subordinate to a positively homogeneous subadditive functional P, then Λ may be chosen to have the same property.

Banach–Alaoglou theorem. Let X be a quasi-Banach space and let X^* be the space of all bounded linear functionals on X. Then the unit ball of X^* is weak* compact.

Open mapping theorem. Suppose that X and Y are quasi-Banach spaces and T is a bounded surjective linear map from X onto Y. Then there exists a constant $K < \infty$ such that for all $x \in X$ we have

$$\|x\|_X \le K\|T(x)\|_Y .$$

Closed graph theorem. Suppose that X and Y are quasi-Banach spaces and T is a linear map from X to Y whose graph is a closed set, i.e., whenever $x_k, x \in X$ and $(x_k, T(x_k)) \mapsto (x,y)$ in $X \times Y$ for some $y \in Y$, then $T(x) = y$. Then T is a bounded linear map from X to Y.

Uniform boundedness principle. Suppose that X is a quasi-Banach space, Y is a quasinormed space and $(T_\alpha)_{\alpha\in I}$ is a family of bounded linear maps from X to Y such that for all $x \in X$ there exists a $C_x < \infty$ such that

$$\sup_{\alpha\in I} \|T_\alpha(x)\|_Y \le C_x .$$

Then there exists a constant $K < \infty$ such that

$$\sup_{\alpha\in I} \|T_\alpha\|_{X\to Y} \le K .$$

Appendix H
The Minimax Lemma

Minimax type results are used in the theory of games and have their origin in the work of Von Neumann [286]. Much of the theory in this subject is based on convex analysis techniques. For instance, this is the case with the next proposition, which is needed in the "difficult" inequality in the proof of the minimax lemma. We refer to Fan [87] for a general account of minimax results. The following exposition is based on the simple presentation in Appendix A2 of [98].

Minimax Lemma. *Let A, B be convex subsets of certain vector spaces. Assume that a topology is defined in B for which it is a compact Hausdorff space and assume that there is a function* $\Phi : A \times B \to \mathbf{R} \bigcup \{+\infty\}$ *that satisfies the following:*
(a) $\Phi(\,.\,,b)$ *is a concave function on A for each* $b \in B$*,*
(b) $\Phi(a,\,.\,)$ *is a convex function on B for each* $a \in A$*,*
(c) $\Phi(a,\,.\,)$ *is lower semicontinuous on B for each* $a \in A$*.*

Then the following identity holds:

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b)\,.$$

To prove the lemma we need the following proposition:

Proposition. *Let B be a convex compact subset of a vector space and suppose that* $g_j : B \to \mathbf{R} \bigcup \{+\infty\}$*,* $j = 1,2,\ldots,n$*, are convex and lower semicontinuous. If*

$$\max_{1\le j\le n} g_j(b) > 0 \quad \textit{for all} \quad b \in B\,,$$

then there exist nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ *such that*

$$\lambda_1 g_1(b) + \lambda_2 g_2(b) + \cdots + \lambda_n g_n(b) > 0 \quad \textit{for all} \quad b \in B\,.$$

Proof. We first consider the case $n = 2$. Define subsets of B

$$B_1 = \{b \in B :\ g_1(b) \le 0\}, \quad B_2 = \{b \in B :\ g_2(b) \le 0\}\,.$$

If $B_1 = \emptyset$, we take $\lambda_1 = 1$ and $\lambda_2 = 0$, and we similarly deal with the case $B_2 = \emptyset$. If B_1 and B_2 are nonempty, then they are closed and thus compact. The hypothesis of the proposition implies that $g_2(b) > 0 \ge g_1(b)$ for all $b \in B_1$. Therefore, the function $-g_1(b)/g_2(b)$ is well defined and upper semicontinuous on B_1 and thus attains its maximum. The same is true for $-g_2(b)/g_1(b)$ defined on B_2. We set

$$\mu_1 = \max_{b \in B_1} \frac{-g_1(b)}{g_2(b)} \geq 0, \qquad \mu_2 = \max_{b \in B_2} \frac{-g_2(b)}{g_1(b)} \geq 0.$$

We need to find $\lambda > 0$ such that $\lambda g_1(b) + g_2(b) > 0$ for all $b \in B$. This is clearly satisfied if $b \notin B_1 \bigcup B_2$, while for $b_1 \in B_1$ and $b_2 \in B_2$ we have

$$\begin{aligned} \lambda g_1(b_1) + g_2(b_1) \ &\geq\ (1 - \lambda \mu_1) g_2(b_1), \\ \lambda g_1(b_2) + g_2(b_2) \ &\geq\ (\lambda - \mu_2) g_1(b_2). \end{aligned}$$

Therefore, it suffices to find a $\lambda > 0$ such that $1 - \lambda\mu_1 > 0$ and $\lambda - \mu_2 > 0$. Such a λ exists if and only if $\mu_1\mu_2 < 1$. To prove that $\mu_1\mu_2 < 1$, we can assume that $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Then we take $b_1 \in B_1$ and $b_2 \in B_2$, for which the maxima μ_1 and μ_2 are attained, respectively. Then we have

$$\begin{aligned} g_1(b_1) + \mu_1 g_2(b_1) \ &= 0, \\ g_1(b_2) + \frac{1}{\mu_2} g_2(b_2) \ &= 0. \end{aligned}$$

But $g_1(b_1) < 0 < g_1(b_2)$; thus taking $b_\theta = \theta b_1 + (1-\theta) b_2$ for some θ in $(0,1)$, we have

$$g_1(b_\theta) \leq \theta g_1(b_1) + (1-\theta) g_1(b_2) = 0.$$

Considering the same convex combination of the last displayed equations and using this identity, we obtain that

$$\mu_1\mu_2\theta g_2(b_1) + (1-\theta) g_2(b_2) = 0.$$

The hypothesis of the proposition implies that $g_2(b_\theta) > 0$ and the convexity of g_2:

$$\theta g_2(b_1) + (1-\theta) g_2(b_2) > 0.$$

Since $g_2(b_1) > 0$, we must have $\mu_1\mu_2 g_2(b_1) < g_2(b_1)$, which gives $\mu_1\mu_2 < 1$. This proves the required claim and completes the case $n = 2$.

We now use induction to prove the proposition for arbitrary n. Assume that the result has been proved for $n-1$ functions. Consider the subset of B

$$B_n = \{b \in B :\ g_n(b) \leq 0\}.$$

If $B_n = \emptyset$, we choose $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n = 1$. If B_n is not empty, then it is compact and convex and we can restrict $g_1, g_2, \ldots, g_{n-1}$ to B_n. Using the induction hypothesis, we can find $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \geq 0$ such that

$$g_0(b) = \lambda_1 g_1(b) + \lambda_2 g_2(b) + \cdots + \lambda_{n-1} g_{n-1}(b) > 0$$

for all $b \in B_n$. Then g_0 and g_n are convex lower semicontinuous functions on B, and $\max(g_0(b), g_n(b)) > 0$ for all $b \in B$. Using the case $n = 2$, which was first proved, we can find $\lambda_0, \lambda_n \geq 0$ such that for all $b \in B$ we have

$$\begin{aligned}0 &< \lambda_0 g_0(b) + \lambda_n g_n(b) \\ &= \lambda_0\lambda_1 g_1(b) + \lambda_0\lambda_2 g_2(b) + \cdots + \lambda_0\lambda_{n-1} g_{n-1}(b) + \lambda_n g_n(b).\end{aligned}$$

This establishes the case of n functions and concludes the proof of the induction and hence of the proposition. □

We now turn to the proof of the minimax lemma.

Proof. The fact that the left-hand side in the required conclusion of the minimax lemma is at least as big as the right-hand side is obvious. We can therefore concentrate on the converse inequality. In doing this we may assume that the right-hand side is finite. Without loss of generality we can subtract a finite constant from $\Phi(a,b)$, and so we can also assume that

$$\sup_{a\in A}\min_{b\in B}\Phi(a,b) = 0\,.$$

Then, by hypothesis *(c)* of the minimax lemma, the subsets

$$B_a = \{b\in B:\ \Phi(a,b)\le 0\}, \qquad a\in A$$

of B are closed and nonempty, and we show that they satisfy the finite intersection property. Indeed, suppose that

$$B_{a_1}\cap B_{a_2}\cap\cdots\cap B_{a_n} = \emptyset$$

for some $a_1,a_2,\ldots,a_n\in A$. We write $g_j(b) = \Phi(a_j,b)$, $j = 1,2,\ldots,n$, and we observe that the conditions of the previous proposition are satisfied. Therefore we can find $\lambda_1,\lambda_2,\ldots,\lambda_n\ge 0$ such that for all $b\in B$ we have

$$\lambda_1\Phi(a_1,b) + \lambda_2\Phi(a_2,b) + \cdots + \lambda_n\Phi(a_n,b) > 0\,.$$

For simplicity we normalize the λ_j's by setting $\lambda_1+\lambda_2+\cdots+\lambda_n = 1$. If we set $a_0 = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$, the concavity hypothesis (a) gives

$$\Phi(a_0,b) > 0$$

for all $b\in B$, contradicting the fact that $\sup_{a\in A}\min_{b\in B}\Phi(a,b) = 0$. Therefore, the family of closed subsets $\{B_a\}_{a\in A}$ of B satisfies the finite intersection property. The compactness of B now implies $\bigcap_{a\in A} B_a \ne \emptyset$. Take $b_0\in\bigcap_{a\in A} B_a$. Then $\Phi(a,b_0)\le 0$ for every $a\in A$, and therefore

$$\min_{b\in B}\sup_{a\in A}\Phi(a,b) \le \sup_{a\in A}\Phi(a,b_0) \le 0$$

as required. □

Appendix I
The Schur Lemma

Schur's lemma provides sufficient conditions for linear operators to be bounded on L^p. Moreover, for positive operators it provides necessary and sufficient such conditions. We discuss these situations.

I.1 The Classical Schur Lemma

We begin with an easy situation. Suppose that $K(x,y)$ is a locally integrable function on a product of two σ-finite measure spaces $(X,\mu)\times(Y,\nu)$, and let T be a linear operator given by

$$T(f)(x) = \int_Y K(x,y)f(y)\,d\nu(y)$$

when f is bounded and compactly supported. It is a simple consequence of Fubini's theorem that for almost all $x \in X$ the integral defining T converges absolutely. The following lemma provides a sufficient criterion for the L^p boundedness of T.

Lemma. *Suppose that a locally integrable function $K(x,y)$ satisfies*

$$\sup_{x\in X}\int_Y |K(x,y)|\,d\nu(y) = A < \infty,$$
$$\sup_{y\in Y}\int_X |K(x,y)|\,d\mu(x) = B < \infty.$$

Then the operator T extends to a bounded operator from $L^p(Y)$ to $L^p(X)$ with norm $A^{1-\frac{1}{p}}B^{\frac{1}{p}}$ for $1 \le p \le \infty$.

Proof. The second condition gives that T maps L^1 to L^1 with bound B, while the first condition gives that T maps L^∞ to L^∞ with bound A. It follows by the Riesz–Thorin interpolation theorem that T maps L^p to L^p with bound $A^{1-\frac{1}{p}}B^{\frac{1}{p}}$. □

This lemma can be improved significantly when the operators are assumed to be positive.

I.2 Schur's Lemma for Positive Operators

We have the following necessary and sufficient condition for the L^p boundedness of positive operators.

Lemma. *Let (X,μ) and (Y,ν) be two σ-finite measure spaces, where μ and ν are positive measures, and suppose that $K(x,y)$ is a nonnegative measurable function on $X\times Y$. Let $1<p<\infty$ and $0<A<\infty$. Let T be the linear operator*

$$T(f)(x)=\int_Y K(x,y)f(y)\,d\nu(y)$$

and T^t its transpose operator

$$T^t(g)(y)=\int_X K(x,y)g(x)\,d\mu(x)\,.$$

To avoid trivialities, we assume that there is a compactly supported, bounded, and positive ν-a.e. function h_1 on Y such that $T(h_1)>0$ μ-a.e. Then the following are equivalent:

(i) T maps $L^p(Y)$ to $L^p(X)$ with norm at most A.
(ii) For all $B>A$ there is a measurable function h on Y that satisfies $0<h<\infty$ ν-a.e., $0<T(h)<\infty$ μ-a.e., and such that

$$T^t\big(T(h)^{\frac{p}{p'}}\big)\le B^p\,h^{\frac{p}{p'}}\,.$$

(iii) For all $B>A$ there are measurable functions u on X and v on Y such that $0<u<\infty$ μ-a.e., $0<v<\infty$ ν-a.e., and such that

$$\begin{aligned} T(u^{p'}) &\le B\,v^{p'},\\ T^t(v^p) &\le B\,u^p. \end{aligned}$$

Proof. First we assume (ii) and we prove (iii). Define u,v by the equations $v^{p'}=T(h)$ and $u^{p'}=Bh$ and observe that (iii) holds for this choice of u and v. Moreover, observe that $0<u,v<\infty$ a.e. with respect to the measures μ and ν, respectively.

Next we assume (iii) and we prove (i). For g in $L^{p'}(X)$ we have

$$\int_X T(f)(x)\,g(x)\,d\mu(x)=\int_X\int_Y K(x,y)f(y)g(x)\frac{v(x)}{u(y)}\frac{u(y)}{v(x)}\,d\nu(y)\,d\mu(x).$$

We now apply Hölder's inequality with exponents p and p' to the functions

$$f(y)\frac{v(x)}{u(y)}\qquad\text{and}\qquad g(x)\frac{u(y)}{v(x)}$$

with respect to the measure $K(x,y)\,d\nu(y)\,d\mu(x)$ on $X\times Y$. Since

$$\left(\int_Y\int_X f(y)^p\frac{v(x)^p}{u(y)^p}K(x,y)\,d\mu(x)\,d\nu(y)\right)^{\frac{1}{p}}\le B^{\frac{1}{p}}\left\|f\right\|_{L^p(Y)}$$

and

$$\left(\int_X \int_Y g(x)^{p'} \frac{u(y)^{p'}}{v(x)^{p'}} K(x,y)\, d\nu(y)\, d\mu(x)\right)^{\frac{1}{p'}} \le B^{\frac{1}{p'}} \big\|g\big\|_{L^{p'}(X)},$$

we conclude that

$$\left|\int_X T(f)(x) g(x)\, d\mu(x)\right| \le B^{\frac{1}{p}+\frac{1}{p'}} \|f\|_{L^p(Y)} \big\|g\big\|_{L^{p'}(X)}.$$

Taking the supremum over all g with $L^{p'}(X)$ norm 1, we obtain

$$\big\|T(f)\big\|_{L^p(X)} \le B\big\|f\big\|_{L^p(Y)}.$$

Since B was any number greater than A, we conclude that

$$\big\|T\big\|_{L^p(Y)\to L^p(X)} \le A\,,$$

which proves (i).

We finally assume (i) and we prove (ii). Without loss of generality, take here $A = 1$ and $B > 1$. Define a map $S:\ L^p(Y) \to L^p(Y)$ by setting

$$S(f)(y) = \big(T^t\big(T(f)^{\frac{p}{p'}}\big)\big)^{\frac{p'}{p}}(y).$$

We observe two things. First, $f_1 \le f_2$ implies $S(f_1) \le S(f_2)$, which is an easy consequence of the fact that the same monotonicity is valid for T. Next, we observe that $\big\|f\big\|_{L^p} \le 1$ implies that $\big\|S(f)\big\|_{L^p} \le 1$ as a consequence of the boundedness of T on L^p (with norm at most 1).

Construct a sequence h_n, $n = 1,2,\dots$, by induction as follows. Pick $h_1 > 0$ on Y as in the hypothesis of the theorem such that $T(h_1) > 0$ μ-a.e. and such that $\big\|h_1\big\|_{L^p} \le B^{-p'}(B^{p'} - 1)$. (The last condition can be obtained by multiplying h_1 by a small constant.) Assuming that h_n has been defined, we define

$$h_{n+1} = h_1 + \frac{1}{B^{p'}} S(h_n).$$

We check easily by induction that we have the monotonicity property $h_n \le h_{n+1}$ and the fact that $\big\|h_n\big\|_{L^p} \le 1$. We now define

$$h(x) = \sup_n h_n(x) = \lim_{n\to\infty} h_n(x).$$

Fatou's lemma gives that $\big\|h\big\|_{L^p} \le 1$, from which it follows that $h < \infty$ ν-a.e. Since $h \ge h_1 > 0$ ν-a.e., we also obtain that $h > 0$ ν-a.e.

Next we use the Lebesgue dominated convergence theorem to obtain that $h_n \to h$ in $L^p(Y)$. Since T is bounded on L^p, it follows that $T(h_n) \to T(h)$ in $L^p(X)$. It follows that $T(h_n)^{\frac{p}{p'}} \to T(h)^{\frac{p}{p'}}$ in $L^{p'}(X)$. Our hypothesis gives that T^t maps $L^{p'}(X)$

to $L^{p'}(Y)$ with norm at most 1. It follows $T^t\big(T(h_n)^{\frac{p}{p'}}\big) \to T^t\big(T(h)^{\frac{p}{p'}}\big)$ in $L^{p'}(Y)$. Raising to the power $\frac{p'}{p}$, we obtain that $S(h_n) \to S(h)$ in $L^p(Y)$.

It follows that for some subsequence n_k of the integers we have $S(h_{n_k}) \to S(h)$ a.e. in Y. Since the sequence $S(h_n)$ is increasing, we conclude that the entire sequence $S(h_n)$ converges almost everywhere to $S(h)$. We use this information in conjunction with $h_{n+1} = h_1 + \frac{1}{B^{p'}} S(h_n)$. Indeed, letting $n \to \infty$ in this identity, we obtain

$$h = h_1 + \frac{1}{B^{p'}} S(h)\,.$$

Since $h_1 > 0$ ν-a.e. it follows that $S(h) \le B^{p'} h$ ν-a.e., which proves the required estimate in (ii).

It remains to prove that $0 < T(h) < \infty$ μ-a.e. Since $\big\|h\big\|_{L^p} \le 1$ and T is L^p bounded, it follows that $\big\|T(h)\big\|_{L^p} \le 1$, which implies that $T(h) < \infty$ μ-a.e. We also have $T(h) \ge T(h_1) > 0$ μ-a.e. □

I.3 An Example

Consider the Hilbert operator

$$T(f)(x) = \int_0^\infty \frac{f(y)}{x+y}\,dy\,,$$

where $x \in (0,\infty)$. The operator T takes measurable functions on $(0,\infty)$ to measurable functions on $(0,\infty)$. We claim that T maps $L^p(0,\infty)$ to itself for $1 < p < \infty$; precisely, we have the estimate

$$\int_0^\infty T(f)(x)\,g(x)\,dx \le \frac{\pi}{\sin(\pi/p)} \big\|f\big\|_{L^p(0,\infty)} \big\|g\big\|_{L^{p'}(0,\infty)}\,.$$

To see this we use Schur's lemma. We take

$$u(x) = v(x) = x^{-\frac{1}{pp'}}\,.$$

We have that

$$T(u^{p'})(x) = \int_0^\infty \frac{y^{-\frac{1}{p}}}{x+y}\,dy = x^{-\frac{1}{p}} \int_0^\infty \frac{t^{-\frac{1}{p}}}{1+t}\,dt = B(\tfrac{1}{p'},\tfrac{1}{p})\,v(x)^{p'}\,,$$

where B is the usual beta function and the last identity follows from the change of variables $s = (1+t)^{-1}$. Now an easy calculation yields

$$B(\tfrac{1}{p'},\tfrac{1}{p}) = \frac{\pi}{\sin(\pi/p)}\,,$$

so the lemma in Appendix I.2 gives that $\|T\|_{L^p \to L^p} \le \frac{\pi}{\sin(\pi/p)}$. The sharpness of this constant follows by considering the sequence of functions

$$h_\varepsilon(x) = \begin{cases} x^{-\frac{1}{p}+\varepsilon} & \text{when } x < 1, \\ x^{-\frac{1}{p}-\varepsilon} & \text{when } x \ge 1, \end{cases}$$

which satisfies

$$\lim_{\varepsilon \to 0} \frac{\|T(h_\varepsilon)\|_{L^p(0,\infty)}}{\|h_\varepsilon\|_{L^p(0,\infty)}} = \frac{\pi}{\sin(\pi/p)}.$$

We make some comments related to the history of Schur's lemma. Schur [237] first proved a matrix version of the lemma in Appendix I.1 when $p = 2$. Precisely, Schur's original version was the following: If $K(x,y)$ is a positive decreasing function in both variables and satisfies

$$\sup_m \sum_n K(m,n) + \sup_n \sum_m K(m,n) < \infty,$$

then

$$\sum_m \sum_n a_{mn} K(m,n) b_{mn} \le C \|a\|_{\ell^2} \|b\|_{\ell^2}.$$

Hardy–Littlewood and Pólya [121] extended this result to L^p for $1 < p < \infty$ and disposed of the condition that K be a decreasing function. Aronszajn, Mulla, and Szeptycki [9] proved that (iii) implies (i) in the lemma of Appendix I.2. Gagliardo in [97] proved the converse direction that (i) implies (iii) in the same lemma. The case $p = 2$ was previously obtained by Karlin [151]. Condition (ii) was introduced by Howard and Schep [131], who showed that it is equivalent to (i) and (iii). A multilinear analogue of the lemma in Appendix I.2 was obtained by Grafakos and Torres [113]; the easy direction (iii) implies (i) was independently observed by Bekollé, Bonami, Peloso, and Ricci [17]. See also Cwikel and Kerman [65] for an alternative multilinear formulation of the Schur lemma.

The case $p = p' = 2$ of the application in Appendix I.3 is a continuous version of Hilbert's double series theorem. The discrete version was first proved by Hilbert in his lectures on integral equations (published by Weyl [290]) without a determination of the exact constant. This exact constant turns out to be π, as discovered by Schur [237]. The extension to other p's (with sharp constants) is due to Hardy and M. Riesz and published by Hardy [120].

Appendix J
The Whitney Decomposition of Open Sets in $\mathbf{R}^n$

An arbitrary open set in $\mathbf{R}^n$ can be decomposed as a union of disjoint cubes whose lengths are proportional to their distance from the boundary of the open set. See, for instance, Figure J.1 when the open set is the unit disk in $\mathbf{R}^2$. For a given cube Q in $\mathbf{R}^n$, we denote by $\ell(Q)$ its length.

Proposition. *Let Ω be an open nonempty proper subset of $\mathbf{R}^n$. Then there exists a family of closed cubes $\{Q_j\}_j$ such that*

(a) $\bigcup_j Q_j = \Omega$ and the Q_j's have disjoint interiors.
(b) $\sqrt{n}\,\ell(Q_j) \le \operatorname{dist}(Q_j, \Omega^c) \le 4\sqrt{n}\,\ell(Q_j)$.
(c) If the boundaries of two cubes Q_j and Q_k touch, then

$$\frac{1}{4} \le \frac{\ell(Q_j)}{\ell(Q_k)} \le 4\,.$$

(d) For a given Q_j there exist at most 12^n Q_k's that touch it.

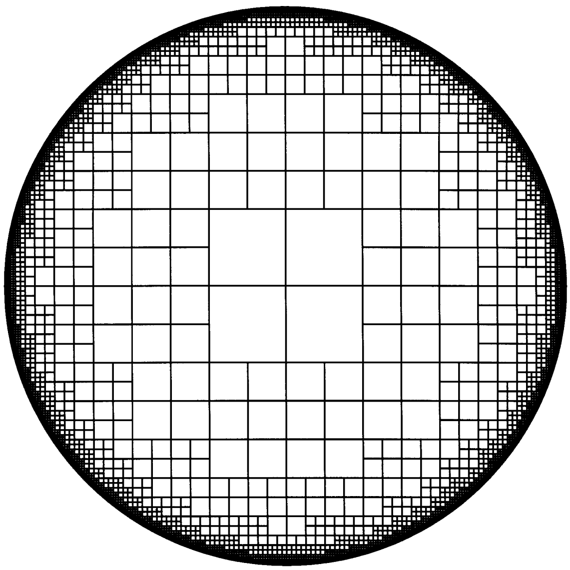

Fig. J.1 The Whitney decomposition of the unit disk.

Proof. Let $\mathscr{D}_k$ be the collection of all dyadic cubes of the form

$$\{(x_1,\dots,x_n)\in\mathbf{R}^n:\ m_j 2^{-k}\le x_j<(m_j+1)2^{-k}\},$$

where $m_j\in\mathbf{Z}$. Observe that each cube in $\mathscr{D}_k$ gives rise to 2^n cubes in $\mathscr{D}_{k+1}$ by bisecting each side.

Write the set Ω as the union of the sets

$$\Omega_k=\{x\in\Omega:\ 2\sqrt{n}2^{-k}<\operatorname{dist}(x,\Omega^c)\le 4\sqrt{n}2^{-k}\}$$

over all $k\in\mathbf{Z}$. Let $\mathscr{F}'$ be the set of all cubes Q in $\mathscr{D}_k$ for some $k\in\mathbf{Z}$ such that $Q\cap\Omega_k\neq\emptyset$. We show that the collection $\mathscr{F}'$ satisfies property (b). Let $Q\in\mathscr{F}'$ and pick $x\in\Omega_k\cap Q$ for some $k\in\mathbf{Z}$. Observe that

$$\sqrt{n}2^{-k}\le\operatorname{dist}(x,\Omega^c)-\sqrt{n}\,\ell(Q)\le\operatorname{dist}(Q,\Omega^c)\le\operatorname{dist}(x,\Omega^c)\le 4\sqrt{n}2^{-k},$$

which proves (b).

Next we observe that

$$\bigcup_{Q\in\mathscr{F}'}Q=\Omega\,.$$

Indeed, every Q in $\mathscr{F}'$ is contained in Ω (since it has positive distance from its complement) and every $x\in\Omega$ lies in some Ω_k and in some dyadic cube in $\mathscr{D}_k$.

The problem is that the cubes in the collection $\mathscr{F}'$ may not be disjoint. We have to refine the collection $\mathscr{F}'$ by eliminating those cubes that are contained in some other cubes in the collection. Recall that two dyadic cubes have disjoint interiors or else one contains the other. For every cube Q in $\mathscr{F}'$ we can therefore consider the unique *maximal* cube $Q^{\max}$ in $\mathscr{F}'$ that contains it. Two different such maximal cubes must have disjoint interiors by maximality. Now set $\mathscr{F}=\{Q^{\max}:\ Q\in\mathscr{F}'\}$.

The collection of cubes $\{Q_j\}_j=\mathscr{F}$ clearly satisfies (a) and (b), and we now turn our attention to the proof of (c). Observe that if Q_j and Q_k in $\mathscr{F}$ touch then

$$\sqrt{n}\,\ell(Q_j)\le\operatorname{dist}(Q_j,\Omega^c)\le\operatorname{dist}(Q_j,Q_k)+\operatorname{dist}(Q_k,\Omega^c)\le 0+4\sqrt{n}\,\ell(Q_k),$$

which proves (c). To prove (d), observe that any cube Q in $\mathscr{D}_k$ is touched by exactly 3^n-1 other cubes in $\mathscr{D}_k$. But each cube Q in $\mathscr{D}_k$ can contain at most 4^n cubes of $\mathscr{F}$ of length at least one-quarter of the length of Q. This fact combined with (c) yields (d). □

The following observation is a consequence of the result just proved: Let $\mathscr{F}=\{Q_j\}_j$ be the Whitney decomposition of a proper open subset Ω of $\mathbf{R}^n$. Fix $0<\varepsilon<1/4$ and denote by Q_k^* the cube with the same center as Q_k but with side length $(1+\varepsilon)$ times that of Q_k. Then Q_k and Q_j touch if and only if Q_k^* and Q_j intersect. Consequently, every point in Ω is contained in at most 12^n cubes Q_k^*.

Appendix K
Smoothness and Vanishing Moments

K.1 The Case of No Cancellation

Let $a,b \in \mathbf{R}^n$, $\mu,\nu \in \mathbf{R}$, and $M,N > n$. Set

$$I(a,\mu,M;b,\nu,N) = \int_{\mathbf{R}^n} \frac{2^{\mu n}}{(1+2^{\mu}|x-a|)^M} \frac{2^{\nu n}}{(1+2^{\nu}|x-b|)^N}\, dx\,.$$

Then we have

$$I(a,\mu,M;b,\nu,N) \le C_0 \frac{2^{\min(\mu,\nu)n}}{\big(1+2^{\min(\mu,\nu)}|a-b|\big)^{\min(M,N)}}\,,$$

where

$$C_0 = v_n\left(\frac{M4^N}{M-n} + \frac{N4^M}{N-n}\right)$$

and v_n is the volume of the unit ball in $\mathbf{R}^n$.

To prove this estimate, first observe that

$$\int_{\mathbf{R}^n} \frac{dx}{(1+|x|)^M} \le \frac{v_n M}{M-n}\,.$$

Without loss of generality, assume that $\nu \le \mu$. Consider the cases $2^{\nu}|a-b| \le 1$ and $2^{\nu}|a-b| \ge 1$. In the case $2^{\nu}|a-b| \le 1$ we use the estimate

$$\frac{2^{\nu n}}{(1+2^{\nu}|x-a|)^N} \le 2^{\nu n} \le \frac{2^{\nu n}2^{\min(M,N)}}{(1+2^{\nu}|a-b|)^{\min(M,N)}}\,,$$

and the result is a consequence of the estimate

$$I(a,\mu,M;b,\nu,N) \le \frac{2^{\nu n}2^{\min(M,N)}}{(1+2^{\nu}|a-b|)^{\min(M,N)}} \int_{\mathbf{R}^n} \frac{2^{\mu n}}{(1+2^{\mu}|x-a|)^M}\, dx\,.$$

In the case $2^{\nu}|a-b| \ge 1$ let H_a and H_b be the two half-spaces, containing the points a and b, respectively, formed by the hyperplane perpendicular to the line segment $[a,b]$ at its midpoint. Split the integral over $\mathbf{R}^n$ as the integral over H_a and the integral over H_b. For $x \in H_a$ use that $|x-b| \ge \frac{1}{2}|a-b|$. For $x \in H_b$ use a similar inequality and the fact that $2^{\nu}|a-b| \ge 1$ to obtain

$$\frac{2^{\mu n}}{(1+2^{\mu}|x-a|)^M} \le \frac{2^{\mu n}}{(2^{\mu}\frac{1}{2}|a-b|)^M} \le \frac{4^M 2^{(\nu-\mu)(M-n)} 2^{\nu n}}{(1+2^{\nu}|a-b|)^M}.$$

The required estimate follows.

K.2 The Case of Cancellation

Let $a,b \in \mathbf{R}^n$, $M,N > 0$, and L a nonnegative integer. Suppose that ϕ_μ and ϕ_ν are two functions on $\mathbf{R}^n$ that satisfy

$$\begin{aligned} |(\partial_x^\alpha \phi_\mu)(x)| &\le \frac{A_\alpha 2^{\mu n} 2^{\mu L}}{(1+2^{\mu}|x-x_\mu|)^M}, \qquad \text{for all } |\alpha| = L, \\ |\phi_\nu(x)| &\le \frac{B 2^{\nu n}}{(1+2^{\nu}|x-x_\nu|)^N}, \end{aligned}$$

for some A_α and B positive, and

$$\int_{\mathbf{R}^n} \phi_\nu(x) x^\beta \, dx = 0 \qquad \text{for all } |\beta| \le L-1,$$

where the last condition is supposed to be vacuous when $L = 0$. Suppose that $N > M+L+n$ and that $\nu \ge \mu$. Then we have

$$\left| \int_{\mathbf{R}^n} \phi_\mu(x)\phi_\nu(x)\, dx \right| \le C_{00} \frac{2^{\mu n} 2^{-(\nu-\mu)L}}{(1+2^{\mu}|x_\mu - x_\nu|)^M},$$

where

$$C_{00} = v_n \frac{N-L-M}{N-L-M-n} B \sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!}.$$

To prove this statement, we subtract the Taylor polynomial of order $L-1$ of ϕ_μ at the point x_ν from the function $\phi_\mu(x)$ and use the remainder theorem to control the required integral by

$$B \sum_{|\alpha|=L} \frac{A_\alpha}{\alpha!} \int_{\mathbf{R}^n} \frac{|x-x_\nu|^L 2^{\mu n} 2^{\mu L}}{(1+2^{\mu}|\xi_x - x_\mu|)^M} \frac{2^{\nu n}}{(1+2^{\nu}|x-x_\nu|)^N}\, dx,$$

for some ξ_x on the segment joining x_ν to x. Using $\nu \ge \mu$ and the triangle inequality, we obtain

$$\frac{1}{1+2^{\mu}|\xi_x - x_\mu|} \le \frac{1+2^{\nu}|x-x_\nu|}{1+2^{\mu}|x_\mu - x_\nu|}.$$

We insert this estimate in the last integral and we use that $N > L+M+n$ to deduce the required conclusion.

K.3 The Case of Three Factors

Given three numbers a,b,c we denote by $\operatorname{med}(a,b,c)$ the number with the property $\min(a,b,c) \le \operatorname{med}(a,b,c) \le \max(a,b,c)$.

Let $x_\nu, x_\mu, x_\lambda \in \mathbf{R}^n$. Suppose that ψ_ν, ψ_μ, ψ_λ are functions defined on $\mathbf{R}^n$ such that for all $N > n$ sufficiently large there exist constants $A_\nu, A_\mu, A_\lambda < \infty$ such that

$$\begin{aligned} |\psi_\nu(x)| &\le A_\nu \frac{2^{\nu n/2}}{(1+2^\nu|x-x_\nu|)^N}, \\ |\psi_\mu(x)| &\le A_\mu \frac{2^{\mu n/2}}{(1+2^\mu|x-x_\mu|)^N}, \\ |\psi_\lambda(x)| &\le A_\lambda \frac{2^{\lambda n/2}}{(1+2^\lambda|x-x_\lambda|)^N}, \end{aligned}$$

for all $x \in \mathbf{R}^n$. Then the following estimate is valid:

$$\begin{aligned} &\int_{\mathbf{R}^n} |\psi_\nu(x)|\,|\psi_\mu(x)|\,|\psi_\lambda(x)|\,dx \\ \le &\frac{C_{N,n} A_\nu A_\mu A_\lambda\, 2^{-\max(\mu,\nu,\lambda)n/2}\, 2^{\operatorname{med}(\mu,\nu,\lambda)n/2}\, 2^{\min(\mu,\nu,\lambda)n/2}}{((1+2^{\min(\nu,\mu)}|x_\nu - x_\mu|)(1+2^{\min(\mu,\lambda)}|x_\mu - x_\lambda|)(1+2^{\min(\lambda,\nu)}|x_\lambda - x_\nu|))^N} \end{aligned}$$

for some constant $C_{N,n} > 0$ independent of the remaining parameters.

Analogous estimates hold if some of these factors are assumed to have cancellation and the others vanishing moments. See the article of Grafakos and Torres [114] for precise statements of these results and applications. Similar estimates with m factors, $m \in \mathbf{Z}^+$, are studied in Bényi and Tzirakis [21].

Glossary

$A \subseteq B$	A is a subset of B (not necessarily a proper subset)
$A \subsetneqq B$	A is a proper subset of B
A^c	the complement of a set A
χ_E	the characteristic function of the set E
d_f	the distribution function of a function f
f^*	the decreasing rearrangement of a function f
$f_n \uparrow f$	f_n increases monotonically to a function f
$\mathbf{Z}$	the set of all integers
$\mathbf{Z}^+$	the set of all positive integers $\{1, 2, 3, \dots\}$
$\mathbf{Z}^n$	the n-fold product of the integers
$\mathbf{R}$	the set of real numbers
$\mathbf{R}^+$	the set of positive real numbers
$\mathbf{R}^n$	the Euclidean n-space
$\mathbf{Q}$	the set of rationals
$\mathbf{Q}^n$	the set of n-tuples with rational coordinates
$\mathbf{C}$	the set of complex numbers
$\mathbf{C}^n$	the n-fold product of complex numbers
$\mathbf{T}$	the unit circle identified with the interval $[0, 1]$
$\mathbf{T}^n$	the n-dimensional torus $[0, 1]^n$
$\lvert x \rvert$	$\sqrt{\lvert x_1\rvert^2 + \cdots + \lvert x_n\rvert^2}$ when $x = (x_1, \dots, x_n) \in \mathbf{R}^n$
$\mathbf{S}^{n-1}$	the unit sphere $\{x \in \mathbf{R}^n : \lvert x\rvert = 1\}$

e_j	the vector $(0,\dots,0,1,0,\dots,0)$ with 1 in the jth entry and 0 elsewhere
$\log t$	the logarithm with base e of $t>0$
$\log_a t$	the logarithm with base a of $t>0$ $(1\neq a>0)$
$\log^+ t$	$\max(0,\log t)$ for $t>0$
$[t]$	the integer part of the real number t
$x\cdot y$	the quantity $\sum_{j=1}^n x_j y_j$ when $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$
$B(x,R)$	the ball of radius R centered at x in $\mathbf{R}^n$
ω_{n-1}	the surface area of the unit sphere $\mathbf{S}^{n-1}$
v_n	the volume of the unit ball $\{x\in\mathbf{R}^n:\ \lvert x\rvert<1\}$
$\lvert A\rvert$	the Lebesgue measure of the set $A\subseteq\mathbf{R}^n$
dx	Lebesgue measure
$\mathrm{Avg}_B f$	the average $\frac{1}{\lvert B\rvert}\int_B f(x)\,dx$ of f over the set B
$\langle f,g\rangle$	the real inner product $\int_{\mathbf{R}^n} f(x)g(x)\,dx$
$\langle f\mid g\rangle$	the complex inner product $\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx$
$\langle u,f\rangle$	the action of a distribution u on a function f
p'	the number $p/(p-1)$, whenever $0<p\neq 1<\infty$
$1'$	the number ∞
∞'	the number 1
$f=O(g)$	means $\lvert f(x)\rvert\leq M\lvert g(x)\rvert$ for some M for x near x_0
$f=o(g)$	means $\lvert f(x)\rvert\,\lvert g(x)\rvert^{-1}\to 0$ as $x\to x_0$
A^t	the transpose of the matrix A
A^*	the conjugate transpose of a complex matrix A
A^{-1}	the inverse of the matrix A
$O(n)$	the space of real matrices satisfying $A^{-1}=A^t$
$\lVert T\rVert_{X\to Y}$	the norm of the (bounded) operator $T:X\to Y$
$A\approx B$	means that there exists a $c>0$ such that $c^{-1}\leq\frac{B}{A}\leq c$
$\lvert\alpha\rvert$	indicates the size $\lvert\alpha_1\rvert+\cdots+\lvert\alpha_n\rvert$ of a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$
$\partial_j^m f$	the mth partial derivative of $f(x_1,\dots,x_n)$ with respect to x_j
$\partial^\alpha f$	$\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} f$
$\mathscr{C}^k$	the space of functions f with $\partial^\alpha f$ continuous for all $\lvert\alpha\rvert\leq k$

$\mathscr{C}_0$	the space of continuous functions with compact support
$\mathscr{C}_{00}$	the space of continuous functions that vanish at infinity
$\mathscr{C}_0^\infty$	the space of smooth functions with compact support
$\mathscr{D}$	the space of smooth functions with compact support
$\mathscr{S}$	the space of Schwartz functions
$\mathscr{C}^\infty$	the space of smooth functions $\bigcup_{k=1}^\infty \mathscr{C}^k$
$\mathscr{D}'(\mathbf{R}^n)$	the space of distributions on $\mathbf{R}^n$
$\mathscr{S}'(\mathbf{R}^n)$	the space of tempered distributions on $\mathbf{R}^n$
$\mathscr{E}'(\mathbf{R}^n)$	the space of distributions with compact support on $\mathbf{R}^n$
$\mathscr{P}$	the set of all complex-valued polynomials of n real variables
$\mathscr{S}'(\mathbf{R}^n)/\mathscr{P}$	the space of tempered distributions on $\mathbf{R}^n$ modulo polynomials
$\ell(Q)$	the side length of a cube Q in $\mathbf{R}^n$
∂Q	the boundary of a cube Q in $\mathbf{R}^n$
$L^p(X,\mu)$	the Lebesgue space over the measure space (X,μ)
$L^p(\mathbf{R}^n)$	the space $L^p(\mathbf{R}^n, \lvert\cdot\rvert)$
$L^{p,q}(X,\mu)$	the Lorentz space over the measure space (X,μ)
$L^p_{\text{loc}}(\mathbf{R}^n)$	the space of functions that lie in $L^p(K)$ for any compact set K in $\mathbf{R}^n$
$\lvert d\mu\rvert$	the total variation of a finite Borel measure μ on $\mathbf{R}^n$
$\mathscr{M}(\mathbf{R}^n)$	the space of all finite Borel measures on $\mathbf{R}^n$
$\mathscr{M}_p(\mathbf{R}^n)$	the space of L^p Fourier multipliers, $1 \le p \le \infty$
$\mathscr{M}^{p,q}(\mathbf{R}^n)$	the space of translation-invariant operators that map $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$
$\lVert\mu\rVert_{\mathscr{M}}$	$\int_{\mathbf{R}^n} \lvert d\mu\rvert$ the norm of a finite Borel measure μ on $\mathbf{R}^n$
$\mathcal{M}$	the centered Hardy–Littlewood maximal operator with respect to balls
M	the uncentered Hardy–Littlewood maximal operator with respect to balls
$\mathcal{M}_c$	the centered Hardy–Littlewood maximal operator with respect to cubes
M_c	the uncentered Hardy–Littlewood maximal operator with respect to cubes
$\mathcal{M}_\mu$	the centered maximal operator with respect to a measure μ
M_μ	the uncentered maximal operator with respect to a measure μ
M_s	the strong maximal operator
M_d	the dyadic maximal operator

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Index

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(*continued from page ii*)

137 AXLER/BOURDON/RAMEY. Harmonic Function Theory. 2nd ed.
138 COHEN. A Course in Computational Algebraic Number Theory.
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