CHAPTER 8

SEQUENCES OF FUNCTIONS

In previous chapters we have often made use of sequences of real numbers. In this chapter we shall consider sequences whose terms are *functions* rather than real numbers. Sequences of functions arise naturally in real analysis and are especially useful in obtaining approximations to a given function and defining new functions from known ones.

In Section 8.1 we will introduce two different notions of convergence for a sequence of functions: pointwise convergence and uniform convergence. The latter type of convergence is very important, and will be the main focus of our attention. The reason for this focus is the fact that, as is shown in Section 8.2, uniform convergence "preserves" certain properties in the sense that if each term of a uniformly convergent sequence of functions possesses these properties, then the limit function also possesses the properties.

In Section 8.3 we will apply the concept of uniform convergence to define and derive the basic properties of the exponential and logarithmic functions. Section 8.4 is devoted to a similar treatment of the trigonometric functions.

Section 8.1 Pointwise and Uniform Convergence

Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$; we shall say that (f_n) is a **sequence of functions** on A to $\mathbb{R}$. Clearly, for each $x \in A$, such a sequence gives rise to a sequence of real numbers, namely the sequence

$$(1) \qquad (f_n(x)),$$

obtained by evaluating each of the functions at the point x. For certain values of $x \in A$ the sequence (1) may converge, and for other values of $x \in A$ this sequence may diverge. For each $x \in A$ for which the sequence (1) converges, there is a uniquely determined real number $\lim(f_n(x))$. In general, the value of this limit, when it exists, will depend on the choice of the point $x \in A$. Thus, there arises in this way a function whose domain consists of all numbers $x \in A$ for which the sequence (1) converges.

8.1.1 Definition Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the **limit on** A_0 **of the sequence** (f_n). When such a function f exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

It follows from Theorem 3.1.4 that, except for a possible modification of the domain A_0, the limit function is uniquely determined. Ordinarily we choose A_0 to be the largest set possible; that is, we take A_0 to be the set of all $x \in A$ for which the sequence (1) is convergent in $\mathbb{R}$.

In order to symbolize that the sequence (f_n) converges on A_0 to f, we sometimes write

$$f = \lim(f_n) \quad \text{on} \quad A_0, \quad \text{or} \quad f_n \to f \quad \text{on} \quad A_0.$$

Sometimes, when f_n and f are given by formulas, we write

$$f(x) = \lim f_n(x) \quad \text{for} \quad x \in A_0, \quad \text{or} \quad f_n(x) \to f(x) \quad \text{for} \quad x \in A_0.$$

8.1.2 Examples **(a)** $\lim(x/n) = 0$ for $x \in \mathbb{R}$.

For $n \in \mathbb{N}$, let $f_n(x) := x/n$ and let $f(x) := 0$ for $x \in \mathbb{R}$. By Example 3.1.6(a), we have $\lim(1/n) = 0$. Hence it follows from Theorem 3.2.3 that

$$\lim(f_n(x)) = \lim(x/n) = x \lim(1/n) = x \cdot 0 = 0$$

for all $x \in \mathbb{R}$. (See Figure 8.1.1.)

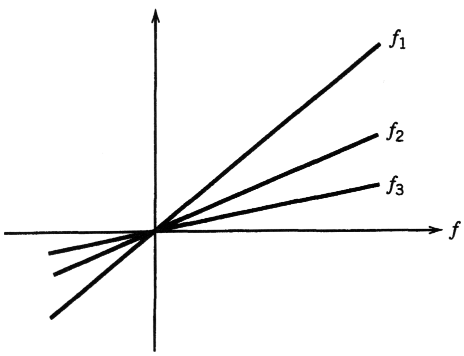

Figure 8.1.1 $f_n(x) = x/n$

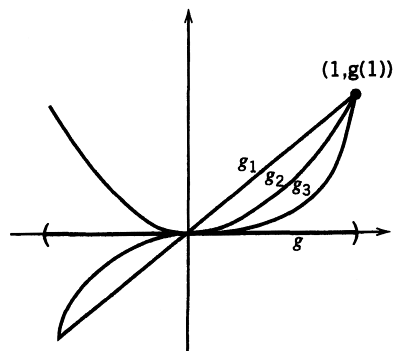

Figure 8.1.2 $g_n(x) = x^n$

(b) $\lim(x^n)$.

Let $g_n(x) := x^n$ for $x \in \mathbb{R}, n \in \mathbb{N}$. (See Figure 8.1.2.) Clearly, if $x = 1$, then the sequence $(g_n(1)) = (1)$ converges to 1. It follows from Example 3.1.11(b) that $\lim(x^n) = 0$ for $0 \leq x < 1$ and it is readily seen that this is also true for $-1 < x < 0$. If $x = -1$, then $g_n(-1) = (-1)^n$, and it was seen in Example 3.2.8(b) that the sequence is divergent. Similarly, if $|x| > 1$, then the sequence (x^n) is not bounded, and so it is not convergent in $\mathbb{R}$. We conclude that if

$$g(x) := \begin{cases} 0 & \text{for} \quad -1 < x < 1, \\ 1 & \text{for} \quad x = 1, \end{cases}$$

then the sequence (g_n) converges to g on the set $(-1, 1]$.

(c) $\lim((x^2 + nx)/n) = x$ for $x \in \mathbb{R}$.

Let $h_n(x) := (x^2 + nx)/n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $h(x) := x$ for $x \in \mathbb{R}$. (See Figure 8.1.3.) Since we have $h_n(x) = (x^2/n) + x$, it follows from Example 3.1.6(a) and Theorem 3.2.3 that $h_n(x) \to x = h(x)$ for all $x \in \mathbb{R}$.

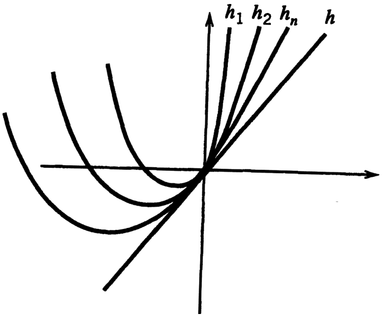

Figure 8.1.3 $h_n(x) = (x^2 + nx)/n$

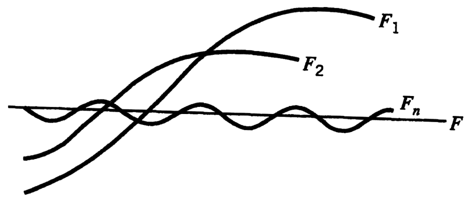

Figure 8.1.4 $F_n(x) = \sin(nx + n)/n$

(d) $\lim((1/n)\sin(nx + n)) = 0$ for $x \in \mathbb{R}$.

Let $F_n(x) := (1/n)\sin(nx + n)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $F(x) := 0$ for $x \in \mathbb{R}$. (See Figure 8.1.4.) Since $|\sin y| \leq 1$ for all $y \in \mathbb{R}$ we have

(2) $$|F_n(x) - F(x)| = \left|\frac{1}{n}\sin(nx + n)\right| \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$. Therefore it follows that $\lim(F_n(x)) = 0 = F(x)$ for all $x \in \mathbb{R}$. The reader should note that, given any $\varepsilon > 0$, if n is sufficiently large, then $|F_n(x) - F(x)| < \varepsilon$ for all values of x simultaneously! □

Partly to reinforce Definition 8.1.1 and partly to prepare the way for the important notion of uniform convergence, we reformulate Definition 8.1.1 as follows.

8.1.3 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then*

(3) $$|f_n(x) - f(x)| < \varepsilon.$$

We leave it to the reader to show that this is equivalent to Definition 8.1.1. We wish to emphasize that the value of $K(\varepsilon, x)$ will depend, in general, on *both* $\varepsilon > 0$ and $x \in A_0$. The reader should confirm the fact that in Examples 8.1.2(a–c), the value of $K(\varepsilon, x)$ required to obtain an inequality such as (3) does depend on both $\varepsilon > 0$ and $x \in A_0$. The intuitive reason for this is that the convergence of the sequence is "significantly faster" at some points than it is at others. However, in Example 8.1.2(d), as we have seen in inequality (2), if we choose n sufficiently large, we can make $|F_n(x) - F(x)| < \varepsilon$ for *all* values of $x \in \mathbb{R}$. It is precisely this rather subtle difference that distinguishes between the notion of the "pointwise convergence" of a sequence of functions (as defined in Definition 8.1.1) and the notion of "uniform convergence."

Uniform Convergence

8.1.4 Definition A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$ (depending on ε but **not** on $x \in A_0$) such that if $n \geq K(\varepsilon)$, then

(4) $$|f_n(x) - f(x)| < \varepsilon \qquad \text{for all} \quad x \in A_0.$$

In this case we say that the sequence (f_n) is **uniformly convergent on** A_0. Sometimes we write

$$f_n \rightrightarrows f \quad \text{on} \quad A_0, \qquad \text{or} \qquad f_n(x) \rightrightarrows f(x) \qquad \text{for} \quad x \in A_0.$$

It is an immediate consequence of the definitions that if the sequence (f_n) is uniformly convergent on A_0 to f, then this sequence also converges pointwise on A_0 to f in the sense of Definition 8.1.1. That the converse is not always true is seen by a careful examination of Examples 8.1.2(a–c); other examples will be given below.

It is sometimes useful to have the following necessary and sufficient condition for a sequence (f_n) *to fail* to converge uniformly on A_0 to f.

8.1.5 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that*

$$(5) \qquad \left|f_{n_k}(x_k) - f(x_k)\right| \geq \varepsilon_0 \qquad \text{for all} \quad k \in \mathbb{N}.$$

The proof of this result requires only that the reader negate Definition 8.1.4; we leave this to the reader as an important exercise. We now show how this result can be used.

8.1.6 Examples **(a)** Consider Example 8.1.2(a). If we let $n_k := k$ and $x_k := k$, then $f_{n_k}(x_k) = 1$ so that $\left|f_{n_k}(x_k) - f(x_k)\right| = |1 - 0| = 1$. Therefore the sequence (f_n) does not converge uniformly on $\mathbb{R}$ to f.

(b) Consider Example 8.1.2(b). If $n_k := k$ and $x_k := \left(\frac{1}{2}\right)^{1/k}$, then

$$\left|g_{n_k}(x_k) - g(x_k)\right| = \left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2}.$$

Therefore the sequence (g_n) does not converge uniformly on $(-1, 1]$ to g.

(c) Consider Example 8.1.2(c). If $n_k := k$ and $x_k := -k$, then $h_{n_k}(x_k) = 0$ and $h(x_k) = -k$ so that $|h_{n_k}(x_k) - h(x_k)| = k$. Therefore the sequence (h_n) does not converge uniformly on $\mathbb{R}$ to h. □

The Uniform Norm

In discussing uniform convergence, it is often convenient to use the notion of the uniform norm on a set of bounded functions.

8.1.7 Definition If $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ is a function, we say that φ is **bounded on** A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded we define the **uniform norm of** φ **on** A by

$$(6) \qquad \|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}.$$

Note that it follows that if $\varepsilon > 0$, then

$$(7) \qquad \|\varphi\|_A \leq \varepsilon \qquad \Longleftrightarrow \qquad |\varphi(x)| \leq \varepsilon \qquad \text{for all} \quad x \in A.$$

8.1.8 Lemma *A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $\|f_n - f\|_A \to 0$.*

Proof. ($\Rightarrow$) If (f_n) converges uniformly on A to f, then by Definition 8.1.4, given any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $n \geq K(\varepsilon)$ and $x \in A$ then

$$|f_n(x) - f(x)| \leq \varepsilon.$$

From the definition of supremum, it follows that $||f_n - f||_A \leq \varepsilon$ whenever $n \geq K(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary this implies that $||f_n - f||_A \to 0$.

($\Leftarrow$) If $||f_n - f||_A \to 0$, then given $\varepsilon > 0$ there is a natural number $H(\varepsilon)$ such that if $n \geq H(\varepsilon)$ then $||f_n - f||_A \leq \varepsilon$. It follows from (7) that $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq H(\varepsilon)$ and $x \in A$. Therefore (f_n) converges uniformly on A to f. Q.E.D.

We now illustrate the use of Lemma 8.1.8 as a tool in examining a sequence of bounded functions for uniform convergence.

8.1.9 Examples **(a)** We cannot apply Lemma 8.1.8 to the sequence in Example 8.1.2(a) since the function $f_n(x) - f(x) = x/n$ is not bounded on $\mathbb{R}$.

For the sake of illustration, let $A := [0, 1]$. Although the sequence (x/n) did not converge uniformly on $\mathbb{R}$ to the zero function, we shall show that the convergence is uniform on A. To see this, we observe that

$$||f_n - f||_A = \sup\{|x/n - 0| : 0 \leq x \leq 1\} = \frac{1}{n}$$

so that $||f_n - f||_A \to 0$. Therefore (f_n) is uniformly convergent on A to f.

(b) Let $g_n(x) := x^n$ for $x \in A := [0, 1]$ and $n \in \mathbb{N}$, and let $g(x) := 0$ for $0 \leq x < 1$ and $g(1) := 1$. The functions $g_n(x) - g(x)$ are bounded on A and

$$||\, g_n - g||_A = \sup\begin{Bmatrix} x^n & \text{for} \quad 0 \leq x < 1 \\ 0 & \text{for} \quad x = 1 \end{Bmatrix} = 1$$

for any $n \in \mathbb{N}$. Since $||\, g_n - g||_A$ does *not* converge to 0, we infer that the sequence (g_n) does *not* converge uniformly on A to g.

(c) We cannot apply Lemma 8.1.8 to the sequence in Example 8.1.2(c) since the function $h_n(x) - h(x) = x^2/n$ is not bounded on $\mathbb{R}$.

Instead, let $A := [0, 8]$ and consider

$$||\, h_n - h||_A = \sup\{x^2/n : 0 \leq x \leq 8\} = 64/n.$$

Therefore, the sequence (h_n) converges uniformly on A to h.

(d) If we refer to Example 8.1.2(d), we see from (2) that $||\, F_n - F||_{\mathbb{R}} \leq 1/n$. Hence (F_n) converges uniformly on $\mathbb{R}$ to F.

(e) Let $G(x) := x^n(1 - x)$ for $x \in A := [0, 1]$. Then the sequence $(G_n(x))$ converges to $G(x) := 0$ for each $x \in A$. To calculate the uniform norm of $G_n - G = G_n$ on A, we find the derivative and solve

$$G_n'(x) = x^{n-1}(n - (n + 1)x) = 0$$

to obtain the point $x_n := n/(n + 1)$. This is an interior point of [0, 1], and it is easily verified by using the First Derivative Test 6.2.8 that G_n attains a maximum on [0, 1] at x_n. Therefore, we obtain

$$||\, G_n||_A = G_n(x_n) = (1 + 1/n)^{-n} \cdot \frac{1}{n + 1},$$

which converges to $(1/e) \cdot 0 = 0$. Thus we see that convergence is uniform on A. □

By making use of the uniform norm, we can obtain a necessary and sufficient condition for uniform convergence that is often useful.

8.1.10 Cauchy Criterion for Uniform Convergence *Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in $\mathbb{N}$ such that for all $m, n \geq H(\varepsilon)$, then $||f_m - f_n||_A \leq \varepsilon$.* □

Proof. $(\Rightarrow)$ If $f_n \rightrightarrows f$ on A, then given $\varepsilon > 0$ there exists a natural number $K\left(\frac{1}{2}\varepsilon\right)$ such that if $n \geq K\left(\frac{1}{2}\varepsilon\right)$ then $||f_n - f||_A \leq \frac{1}{2}\varepsilon$. Hence, if both $m, n \geq K\left(\frac{1}{2}\varepsilon\right)$, then we conclude that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\,\varepsilon = \varepsilon$$

for all $x \in A$. Therefore $||f_m - f_n||_A \leq \varepsilon$ for $m, n \geq K\left(\frac{1}{2}\varepsilon\right) =: H(\varepsilon)$.

$(\Leftarrow)$ Conversely, suppose that for $\varepsilon > 0$ there is $H(\varepsilon)$ such that if $m, n \geq H(\varepsilon)$, then $||f_m - f_n||_A \leq \varepsilon$. Therefore, for each $x \in A$ we have

$$(8) \qquad |f_m(x) - f_n(x)| \leq ||f_m - f_n||_A \leq \varepsilon \qquad \text{for} \quad m, n \geq H(\varepsilon).$$

It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$; therefore, by Theorem 3.5.5, it is a convergent sequence. We define $f : A \to \mathbb{R}$ by

$$f(x) := \lim(f_n(x)) \qquad \text{for} \quad x \in A.$$

If we let $n \to \infty$ in (8), it follows from Theorem 3.2.6 that for each $x \in A$ we have

$$|f_m(x) - f(x)| \leq \varepsilon \qquad \text{for} \quad m \geq H(\varepsilon).$$

Therefore the sequence (f_n) converges uniformly on A to f. Q.E.D.

Exercises for Section 8.1

1. Show that $\lim(x/(x+n)) = 0$ for all $x \in \mathbb{R}, x \geq 0$.
2. Show that $\lim(nx/(1+n^2x^2)) = 0$ for all $x \in \mathbb{R}$.
3. Evaluate $\lim(nx/(1+nx))$ for $x \in \mathbb{R}$, $x \geq 0$.
4. Evaluate $\lim(x^n/(1+x^n))$ for $x \in \mathbb{R}$, $x \geq 0$.
5. Evaluate $\lim((\sin nx)/(1+nx))$ for $x \in \mathbb{R}$, $x \geq 0$.
6. Show that $\lim(\text{Arctan } nx) = (\pi/2)\text{sgn } x$ for $x \in \mathbb{R}$.
7. Evaluate $\lim(e^{-nx})$ for $x \in \mathbb{R}, x \geq 0$.
8. Show that $\lim(xe^{-nx}) = 0$ for $x \in \mathbb{R}$, $x \geq 0$.
9. Show that $\lim(x^2e^{-nx}) = 0$ and that $\lim(n^2x^2e^{-nx}) = 0$ for $x \in \mathbb{R}$, $x \geq 0$.
10. Show that $\lim((\cos \pi x)^{2n})$ exists for all $x \in \mathbb{R}$. What is its limit?
11. Show that if $a > 0$, then the convergence of the sequence in Exercise 1 is uniform on the interval $[0, a]$, but is not uniform on the interval $[0, \infty)$.
12. Show that if $a > 0$, then the convergence of the sequence in Exercise 2 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.
13. Show that if $a > 0$, then the convergence of the sequence in Exercise 3 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.

14. Show that if $0 < b < 1$, then the convergence of the sequence in Exercise 4 is uniform on the interval $[0, b]$, but is not uniform on the interval $[0, 1]$.
15. Show that if $a > 0$, then the convergence of the sequence in Exercise 5 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.
16. Show that if $a > 0$, then the convergence of the sequence in Exercise 6 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $(0, \infty)$.
17. Show that if $a > 0$, then the convergence of the sequence in Exercise 7 is uniform on the interval $[a, \infty)$, but is not uniform on the interval $[0, \infty)$.
18. Show that the convergence of the sequence in Exercise 8 is uniform on $[0, \infty)$.
19. Show that the sequence $(x^2 e^{-nx})$ converges uniformly on $[0, \infty)$.
20. Show that if $a > 0$, then the sequence $(n^2 x^2 e^{-nx})$ converges uniformly on the interval $[a, \infty)$, but that it does not converge uniformly on the interval $[0, \infty)$.
21. Show that if (f_n), (g_n) converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.
22. Show that if $f_n(x) := x + 1/n$ and $f(x) := x$ for $x \in \mathbb{R}$, then (f_n) converges uniformly on $\mathbb{R}$ to f, but the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$. (Thus the product of uniformly convergent sequences of functions may not converge uniformly.)
23. Let (f_n), (g_n) be sequences of bounded functions on A that converge uniformly on A to f, g, respectively. Show that $(f_n g_n)$ converges uniformly on A to fg.
24. Let (f_n) be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on the interval $[-M, M]$, show that the sequence $(g \circ f_n)$ converges uniformly to $g \circ f$ on A.

Section 8.2 Interchange of Limits

It is often useful to know whether the limit of a sequence of functions is a continuous function, a differentiable function, or a Riemann integrable function. Unfortunately, it is not always the case that the limit of a sequence of functions possesses these useful properties.

8.2.1 Examples **(a)** Let $g_n(x) := x^n$ for $x \in [0, 1]$ and $n \in \mathbb{N}$. Then, as we have noted in Example 8.1.2(b), the sequence (g_n) converges pointwise to the function

$$g(x) := \begin{cases} 0 & \text{for} \quad 0 \le x < 1, \\ 1 & \text{for} \quad x = 1. \end{cases}$$

Although all of the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$. Recall that it was shown in Example 8.1.6(b) that this sequence does not converge uniformly to g on $[0, 1]$.

(b) Each of the functions $g_n(x) = x^n$ in part (a) has a continuous derivative on [0,1]. However, the limit function g does not have a derivative at $x = 1$, since it is not continuous at that point.

(c) Let $f_n : [0, 1] \to \mathbb{R}$ be defined for $n \ge 2$ by

$$f_n(x) := \begin{cases} n^2 x & \text{for} \quad 0 \le x \le 1/n, \\ -n^2(x - 2/n) & \text{for} \quad 1/n \le x \le 2/n, \\ 0 & \text{for} \quad 2/n \le x \le 1. \end{cases}$$

(See Figure 8.2.1.) It is clear that each of the functions f_n is continuous on [0, 1]; hence it is Riemann integrable. Either by means of a direct calculation, or by referring to the significance of the integral as an area, we obtain

$$\int_0^1 f_n(x)dx = 1 \quad \text{for} \quad n \geq 2.$$

The reader may show that $f_n(x) \to 0$ for all $x \in [0, 1]$; hence the limit function f vanishes identically and is continuous (and hence integrable), and $\int_0^1 f(x)dx = 0$. Therefore we have the uncomfortable situation that:

$$\int_0^1 f(x)dx = 0 \neq 1 = \lim \int_0^1 f_n(x)dx.$$

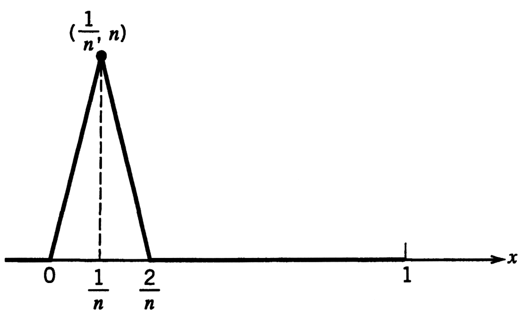

Figure 8.2.1 Example 8.2.1(c)

(d) Those who consider the functions f_n in part (c) to be "artificial" may prefer to consider the sequence (h_n) defined by $h_n(x) := 2nxe^{-nx^2}$ for $x \in [0, 1]$, $n \in \mathbb{N}$. Since $h_n = H'_n$, where $H_n(x) := -e^{-nx^2}$, the Fundamental Theorem 7.3.1 gives

$$\int_0^1 h_n(x)dx = H_n(1) - H_n(0) = 1 - e^{-n}.$$

It is an exercise to show that $h(x) := \lim(h_n(x)) = 0$ for all $x \in [0, 1]$; hence

$$\int_0^1 h(x)dx \neq \lim \int_0^1 h_n(x)dx. \qquad \square$$

Although the extent of the discontinuity of the limit function in Example 8.2.1 (a) is not very great, it is evident that more complicated examples can be constructed that will produce more extensive discontinuity. In any case, we must abandon the hope that the limit of a convergent sequence of continuous [respectively, differentiable, integrable] functions will be continuous [respectively, differentiable, integrable].

It will now be seen that the additional hypothesis of uniform convergence is sufficient to guarantee that the limit of a sequence of continuous functions is continuous. Similar results will also be established for sequences of differentiable and integrable functions.

Interchange of Limit and Continuity

8.2.2 Theorem *Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.*

Proof. By hypothesis, given $\varepsilon > 0$ there exists a natural number $H := H\left(\frac{1}{3}\varepsilon\right)$ such that if $n \geq H$ then $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $x \in A$. Let $c \in A$ be arbitrary; we will show that f is continuous at c. By the Triangle Inequality we have

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)| \\ &\leq \tfrac{1}{3}\varepsilon + |f_H(x) - f_H(c)| + \tfrac{1}{3}\varepsilon. \end{aligned}$$

Since f_H is continuous at c, there exists a number $\delta := \delta\left(\frac{1}{3}\varepsilon, c, f_H\right) > 0$ such that if $|x - c| < \delta$ and $x \in A$, then $|f_H(x) - f_H(c)| < \frac{1}{3}\varepsilon$. Therefore, if $|x - c| < \delta$ and $x \in A$, then we have $|f(x) - f(c)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the continuity of f at the arbitrary point $c \in A$. (See Figure 8.2.2.) Q.E.D.

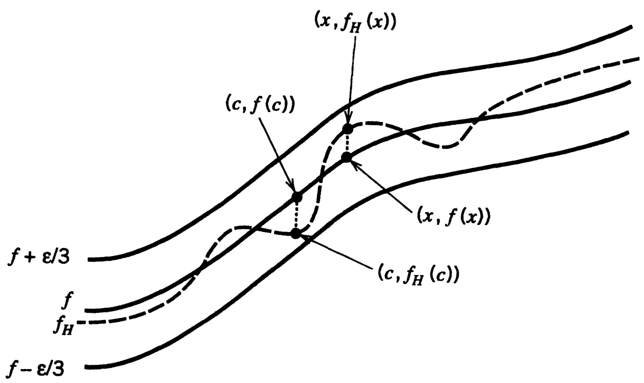

Figure 8.2.2 $|f(x) - f(c)| < \varepsilon$

Remark Although the uniform convergence of the sequence of continuous functions is sufficient to guarantee the continuity of the limit function, it is *not* necessary. (See Exercise 2.)

Interchange of Limit and Derivative

We mentioned in Section 6.1 that Weierstrass showed that the function defined by the series

$$f(x) := \sum_{k=0}^{\infty} 2^{-k} \cos\left(3^k x\right)$$

is continuous at every point but does not have a derivative at any point in $\mathbb{R}$. By considering the partial sums of this series, we obtain a sequence of functions (f_n) that possess a derivative at every point and are uniformly convergent to f. Thus, even though the sequence of differentiable functions (f_n) is uniformly convergent, it does not follow that the limit function is differentiable. (See Exercises 9 and 10.)

We now show that if the *sequence of derivatives* $\left(f'_n\right)$ is uniformly convergent, then all is well. If one adds the hypothesis that the derivatives are continuous, then it is possible to give a short proof, based on the integral. (See Exercise 11.) However, if the derivatives are not assumed to be continuous, a somewhat more delicate argument is required.

8.2.3 Theorem *Let $J \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $(f_n(x_0))$ converges, and that the sequence $\left(f'_n\right)$ of derivatives exists on J and converges uniformly on J to a function g.*

Then the sequence (f_n) converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.

Proof. Let $a < b$ be the endpoints of J and let $x \in J$ be arbitrary. If $m, n \in \mathbb{N}$, we apply the Mean Value Theorem 6.2.4 to the difference $f_m - f_n$ on the interval with endpoints x_0, x. We conclude that there exists a point y (depending on m, n) such that

$$f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (x - x_0)\{f'_m(y) - f'_n(y)\}.$$

Hence we have

$$(1) \qquad \|f_m - f_n\|_J \leq |f_m(x_0) - f_n(x_0)| + (b - a)\|f'_m - f'_n\|_J.$$

From Theorem 8.1.10, it follows from (1) and the hypotheses that $(f_n(x_0))$ is convergent and that (f'_n) is uniformly convergent on J, that (f_n) is uniformly convergent on J. We denote the limit of the sequence (f_n) by f. Since the f_n are all continuous and the convergence is uniform, it follows from Theorem 8.2.2 that f is continuous on J.

To establish the existence of the derivative of f at a point $c \in J$, we apply the Mean Value Theorem 6.2.4 to $f_m - f_n$ on an interval with end points c, x. We conclude that there exists a point z (depending on m, n) such that

$$\{f_m(x) - f_n(x)\} - \{f_m(c) - f_n(c)\} = (x - c)\{f'_m(z) - f'_n(z)\}.$$

Hence, if $x \neq c$, we have

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \|f'_m - f'_n\|_J.$$

Since (f'_n) converges uniformly on J, if $\varepsilon > 0$ is given there exists $H(\varepsilon)$ such that if $m, n \geq H(\varepsilon)$ and $x \neq c$, then

$$(2) \qquad \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \varepsilon.$$

If we take the limit in (2) with respect to m and use Theorem 3.2.6, we have

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \varepsilon.$$

provided that $x \neq c$, $n \geq H(\varepsilon)$. Since $g(c) = \lim\big(f'_n(c)\big)$, there exists $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$, then $|f'_n(c) - g(c)| < \varepsilon$. Now let $K := \sup\{H(\varepsilon), N(\varepsilon)\}$. Since $f'_K(c)$ exists, there exists $\delta_K(\varepsilon) > 0$ such that if $0 < |x - c| < \delta_K(\varepsilon)$, then

$$\left| \frac{f_K(x) - f_K(c)}{x - c} - f'_K(c) \right| < \varepsilon.$$

Combining these inequalities, we conclude that if $0 < |x - c| < \delta_K(\varepsilon)$, then

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $f'(c)$ exists and equals $g(c)$. Since $c \in J$ is arbitrary, we conclude that $f' = g$ on J. Q.E.D.

Interchange of Limit and Integral

We have seen in Example 8.2.1(c) that if (f_n) is a sequence $\mathcal{R}[a, b]$ that converges on $[a, b]$ to a function f in $\mathcal{R}[a, b]$, then it need not happen that

$$(3) \qquad \int_a^b f = \lim_{n \to \infty} \int_a^b f_n.$$

We will now show that *uniform convergence* of the sequence is sufficient to guarantee that this equality holds.

8.2.4 Theorem *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that (f_n) converges **uniformly** on $[a, b]$ to f. Then $f \in \mathcal{R}[a,b]$ and* (3) *holds.*

Proof. It follows from the Cauchy Criterion 8.1.10 that given $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m > n \geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x) - f_n(x) \leq \varepsilon \quad \text{for} \quad x \in [a,b].$$

Theorem 7.1.5 implies that

$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, the sequence $(\int_a^b f_m)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges to some number, say $A \in \mathbb{R}$.

We now show $f \in \mathcal{R}[a,b]$ with integral A. If $\varepsilon > 0$ is given, let $K(\varepsilon)$ be such that if $m > K(\varepsilon)$, then $|f_m(x) - f(x)| < \varepsilon$ for all $x \in [a,b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a, b]$ and if $m > K(\varepsilon)$, then

$$\begin{aligned}
|S(f_m;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| &= \left|\sum_{i=1}^n \{f_m(t_i) - f(t_i)\}(x_i - x_{i-1})\right| \\
&\leq \sum_{i=1}^n |f_m(t_i) - f(t_i)|(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b-a).
\end{aligned}$$

We now choose $r \geq K(\varepsilon)$ such that $|\int_a^b f_r - A| < \varepsilon$ and we let $\delta_{r,\varepsilon} > 0$ be such that $|\int_a^b f_r - S(f_r;\dot{\mathcal{P}})| < \varepsilon$ whenever $||\dot{\mathcal{P}}|| < \delta_{r,\varepsilon}$. Then we have

$$\begin{aligned}
|S(f;\dot{\mathcal{P}}) - A| \leq\ & |S(f;\dot{\mathcal{P}}) - S(f_r;\dot{\mathcal{P}})| + \left|S(f_r;\dot{\mathcal{P}}) - \int_a^b f_r\right| + \left|\int_a^b f_r - A\right| \\
\leq\ & \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2).
\end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}[a,b]$ and $\int_a^b f = A$. Q.E.D.

The hypothesis of uniform convergence is a very stringent one and restricts the utility of this result. In Section 10.4 we will obtain some far-reaching generalizations of Theorem 8.2.4. For the present, we will state a result that does not require the uniformity of the convergence, but does require that the limit function be Riemann integrable. The proof is omitted.

8.2.5 Bounded Convergence Theorem *Let (f_n) be* a *sequence in $\mathcal{R}[a,b]$ that converges on $[a, b]$ to a function $f \in \mathcal{R}[a,b]$. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ for all $x \in [a,b]$, $n \in \mathbb{N}$. Then equation* (3) *holds.*

Dini's Theorem

We will end this section with a famous theorem due to Ulisse Dini (1845–1918) that gives a partial converse to Theorem 8.2.2 when the sequence is monotone. We will present a proof using nonconstant gauges (see Section 5.5).

8.2.6 Dini's Theorem *Suppose that (f_n) is a monotone sequence of continuous functions on $I := [a, b]$ that converges on I to a continuous function f. Then the convergence of the sequence is uniform.*

Proof. We suppose that the sequence (f_n) is decreasing and let $g_m := f_m - f$. Then (g_m) is a decreasing sequence of continuous functions converging on I to the 0-function. We will show that the convergence is uniform on I.

Given $\varepsilon > 0$, $t \in I$, there exists $m_{\varepsilon,t} \in \mathbb{N}$ such that $0 \le g_{m_{\varepsilon,t}}(t) < \varepsilon/2$. Since $g_{m_{\varepsilon,t}}$ is continuous at t, there exists $\delta_\varepsilon(t) > 0$ such that $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$ for all $x \in I$ satisfying $|x - 1| \le \delta_\varepsilon(t)$. Thus, δ_ε is a gauge on I, and if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition, we set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots, m_{\varepsilon,t_n}\}$. If $m \ge M_\varepsilon$ and $x \in I$, then (by Lemma 5.5.3) there exists an index i with $|x - t_i| \le \delta_\varepsilon(t_i)$ and hence

$$0 \le g_m(x) \le g_{m,t_i}(x) < \varepsilon.$$

Therefore, the sequence (g_m) converges uniformly to the 0-function. Q.E.D.

It will be seen in the exercises that we cannot drop any one of the three hypotheses: (i) the functions f_n are continuous, (ii) the limit function f is continuous, (iii) I is a closed bounded interval.

Exercises for Section 8.2

1. Show that the sequence $(x^n/(1 + x^n))$ does not converge uniformly on $[0, 2]$ by showing that the limit function is not continuous on $[0, 2]$.
2. Prove that the sequence in Example 8.2.1(c) is an example of a sequence of continuous functions that converges nonuniformly to a continuous limit.
3. Construct a sequence of functions on $[0, 1]$ each of which is discontinuous at every point of $[0, 1]$ and which converges uniformly to a function that is continuous at every point.
4. Suppose (f_n) is a sequence of continuous functions on an interval I that converges uniformly on I to a function f. If $(x_n) \subseteq I$ converges to $x_0 \in I$, show that $\lim(f_n(x_n)) = f(x_0)$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous on $\mathbb{R}$ and let $f_n(x) := f(x + 1/n)$ for $x \in \mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to f.
6. Let $f_n(x) := 1/(1 + x)^n$ for $x \in [0, 1]$. Find the pointwise limit f of the sequence (f_n) on $[0, 1]$. Does (f_n) converge uniformly to f on $[0, 1]$?
7. Suppose the sequence (f_n) converges uniformly to f on the set A, and suppose that each f_n is bounded on A. (That is, for each n there is a constant M_n such that $|f_n(x)| \le M_n$ for all $x \in A$.) Show that the function f is bounded on A.
8. Let $f_n(x) := nx/(1 + nx^2)$ for $x \in A := [0, \infty)$. Show that each f_n is bounded on A, but the pointwise limit f of the sequence is not bounded on A. Does (f_n) converge uniformly to f on A?

9. Let $f_n(x) := x^n/n$ for $x \in [0, 1]$. Show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on $[0, 1]$, and that the sequence (f_n') converges on $[0, 1]$ to a function g, but that $g(1) \neq f'(1)$.
10. Let $g_n(x) := e^{-nx}/n$ for $x \geq 0, n \in \mathbb{N}$. Examine the relation between $\lim(g_n)$ and $\lim(g'_n)$.
11. Let $I := [a, b]$ and let (f_n) be a sequence of functions on $I \to \mathbb{R}$ that converges on I to f. Suppose that each derivative f_n' is continuous on I and that the sequence (f_n') is uniformly convergent to g on I. Prove that $f(x) - f(a) = \int_a^x g(t)dt$ and that $f'(x) = g(x)$ for all $x \in I$.
12. Show that $\lim \int_1^2 e^{-nx^2} dx = 0$.
13. If $a > 0$, show that $\lim \int_a^\pi (\sin nx)/(nx)dx = 0$. What happens if $a = 0$?
14. Let $f_n(x) := nx/(1 + nx)$ for $x \in [0, 1]$. Show that (f_n) converges nonuniformly to an integrable function f and that $\int_0^1 f(x)dx = \lim \int_0^1 f_n(x)dx$.
15. Let $g_n(x) := nx(1 - x)^n$ for $x \in [0, 1]$, $n \in \mathbb{N}$. Discuss the convergence of (g_n) and $(\int_0^1 g_n dx)$.
16. Let $\{r_1, r_2, \ldots, r_n \ldots\}$ be an enumeration of the rational numbers in $I := [0, 1]$, and let $f_n : I \to \mathbb{R}$ be defined to be 1 if $x = r_1, \ldots, r_n$ and equal to 0 otherwise. Show that f_n is Riemann integrable for each $n \in \mathbb{N}$, that $f_1(x) \leq f_2(x) \leq \cdots \leq f_n(x) \leq \cdots$, and that $f(x) := \lim(f_n(x))$ is the Dirichlet function, which is not Riemann integrable on [0, 1].
17. Let $f_n(x) := 1$ for $x \in (0, 1/n)$ and $f_n(x) := 0$ elsewhere in [0, 1]. Show that (f_n) is a decreasing sequence of discontinuous functions that converges to a continuous limit function, but the convergence is not uniform on [0, 1].
18. Let $f_n(x) := x^n$ for $x \in [0, 1]$, $n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functions that converges to a function that is not continuous, but the convergence is not uniform on $[0, 1]$.
19. Let $f_n(x) := x/n$ for $x \in [0, \infty)$, $n \in \mathbb{N}$. Show that (f_n) is a decreasing sequence of continuous functions that converges to a continuous limit function, but the convergence is not uniform on $[0, \infty)$.
20. Give an example of a decreasing sequence (f_n) of continuous functions on [0, 1) that converges to a continuous limit function, but the convergence is not uniform on [0, 1).

Section 8.3 The Exponential and Logarithmic Functions

We will now introduce the exponential and logarithmic functions and will derive some of their most important properties. In earlier sections of this book we assumed some familiarity with these functions for the purpose of discussing examples. However, it is necessary at some point to place these important functions on a firm foundation in order to establish their existence and determine their basic properties. We will do that here. There are several alternative approaches one can take to accomplish this goal. We will proceed by first proving the existence of a function that has *itself* as derivative. From this basic result, we obtain the main properties of the exponential function. The logarithm function is then introduced as the inverse of the exponential function, and this inverse relation is used to derive the properties of the logarithm function.

The Exponential Function

We begin by establishing the key existence result for the exponential function.

8.3.1 Theorem *There exists a function $E : \mathbb{R} \to \mathbb{R}$ such that:*

(i) $E'(x) = E(x)$ *for all* $x \in \mathbb{R}$.

(ii) $E(0) = 1$.

Proof. We inductively define a sequence (E_n) of continuous functions as follows:

(1) $$E_1(x) := 1 + x,$$

(2) $$E_{n+1}(x) := 1 + \int_0^x E_n(t)dt,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Clearly E_1 is continuous on $\mathbb{R}$ and hence is integrable over any bounded interval. If E_n has been defined and is continuous on $\mathbb{R}$, then it is integrable over any bounded interval, so that E_{n+1} is well-defined by the above formula. Moreover, it follows from the Fundamental Theorem (Second Form) 7.3.5 that E_{n+1} is differentiable at any point $x \in \mathbb{R}$ and that

(3) $$E'_{n+1}(x) = E_n(x) \qquad \text{for} \quad n \in \mathbb{N}.$$

An Induction argument (which we leave to the reader) shows that

(4) $$E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \qquad \text{for} \quad x \in \mathbb{R}.$$

Let $A > 0$ be given; then if $|x| \le A$ and $m > n > 2A$, we have

(5) $$\begin{aligned} |E_m(x) - E_n(x)| &= \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right| \\ &\le \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n} \right)^{m-n-1} \right] \\ &< \frac{A^{n+1}}{(n+1)!} 2. \end{aligned}$$

Since $\lim(A^n/n!) = 0$, it follows that the sequence (E_n) converges uniformly on the interval $[-A, A]$ where $A > 0$ is arbitrary. In particular this means that $(E_n(x))$ converges for each $x \in \mathbb{R}$. We define $E : \mathbb{R} \to \mathbb{R}$ by

$$E(x) := \lim E_n(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

Since each $x \in \mathbb{R}$ is contained inside some interval $[-A, A]$, it follows from Theorem 8.2.2 that E is continuous at x. Moreover, it is clear from (1) and (2) that $E_n(0) = 1$ for all $n \in \mathbb{N}$. Therefore $E(0) = 1$, which proves (ii).

On any interval $[-A, A]$ we have the uniform convergence of the sequence (E_n). In view of (3), we also have the uniform convergence of the sequence (E'_n) of derivatives. It therefore follows from Theorem 8.2.3 that the limit function E is differentiable on $[-A, A]$ and that

$$E'(x) = \lim(E'_n(x)) = \lim(E_{n-1}(x)) = E(x)$$

for all $x \in [-A, A]$. Since $A > 0$ is arbitrary, statement (i) is established. Q.E.D.

8.3.2 Corollary *The function E has a derivative of every order and $E^{(n)}(x) = E(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.*

Proof. If $n = 1$, the statement is merely property (i). It follows for arbitrary $n \in \mathbb{N}$ by Induction. Q.E.D.

8.3.3 Corollary *If* $x > 0$, *then* $1 + x < E(x)$.

Proof. It is clear from (4) that if $x > 0$, then the sequence $(E_n(x))$ is strictly increasing. Hence $E_1(x) < E(x)$ for all $x > 0$. Q.E.D.

It is next shown that the function E, whose existence was established in Theorem 8.3.1, is unique.

8.3.4 Theorem *The function* $E : \mathbb{R} \to \mathbb{R}$ *that satisfies* (i) *and* (ii) *of Theorem* 8.3.1 *is unique.*

Proof. Let E_1 and E_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy properties (i) and (ii) of Theorem 8.3.1 and let $F := E_1 - E_2$. Then

$$F'(x) = E_1'(x) - E_2'(x) = E_1(x) - E_2(x) = F(x)$$

for all $x \in \mathbb{R}$ and

$$F(0) = E_1(0) - E_2(0) = 1 - 1 = 0.$$

It is clear (by Induction) that F has derivatives of all orders and indeed that $F^{(n)}(x) = F(x)$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be arbitrary, and let I_x be the closed interval with endpoints $0, x$. Since F is continuous on I_x, there exists $K > 0$ such that $|F(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to F on the interval I_x and use the fact that $F^{(k)}(0) = F(0) = 0$ for all $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$\begin{aligned} F(x) &= F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}}{(n-1)!}x^{n-1} + \frac{F^{(n)}(c_n)}{n!}x^n \\ &= \frac{F(c_n)}{n!}x^n. \end{aligned}$$

Therefore we have

$$|F(x)| \le \frac{K|x|^n}{n!} \qquad \text{for all} \quad n \in \mathbb{N}.$$

But since $\lim(|x|/n!) = 0$, we conclude that $F(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $E_1(x) - E_2(x) = F(x) = 0$ for all $x \in \mathbb{R}$. Q.E.D.

The standard terminology and notation for the function E (which we now know exists and is unique) is given in the following definition.

8.3.5 Definition The unique function $E : \mathbb{R} \to \mathbb{R}$, such that $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$, is called the **exponential function**. The number $e := E(1)$ is called **Euler's number**. We will frequently write

$$\exp(x) := E(x) \qquad \text{or} \quad e^x := E(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

The number e can be obtained as a limit, and thereby approximated, in several different ways. [See Exercises 1 and 10, and Example 3.3.6.]

The use of the notation e^x for $E(x)$ is justified by property (v) in the next theorem, where it is noted that if r is a rational number, then $E(r)$ and e^r coincide. (Rational exponents were discussed in Section 5.6.) Thus, the function E can be viewed as extending the idea of exponentiation from rational numbers to arbitrary real numbers. For a definition of a^x for $a > 0$ and arbitrary $x \in \mathbb{R}$, see Definition 8.3.10.

8.3.6 Theorem *The exponential function satisfies the following properties:*

(iii) $E(x) \neq 0$ *for all* $x \in \mathbb{R}$;
(iv) $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$;
(v) $E(r) = e^r$ *for all* $r \in \mathbb{Q}$.

Proof. (iii) Let $\alpha \in \mathbb{R}$ be such that $E(\alpha) = 0$, and let J_α be the closed interval with endpoints $0, \alpha$. Let $K \geq |E(t)|$ for all $t \in J_\alpha$. Taylor's Theorem 6.4.1 implies that for each $n \in \mathbb{N}$ there exists a point $c_n \in J_\alpha$ such that

$$\begin{aligned} 1 = E(0) = E(\alpha) &+ \frac{E'(\alpha)}{1!}(-\alpha) + \cdots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(-\alpha)^{n-1} \\ &+ \frac{E^{(n)}(\alpha)}{(n)!}(-\alpha)^n = \frac{E(c_n)}{n!}(-\alpha)^n. \end{aligned}$$

Thus we have $0 < 1 \leq (K/n!)|\alpha|^n$ for $n \in \mathbb{N}$. But since $\lim(|\alpha|^n/n!) = 0$, this is a contradiction.

(iv) Let y be fixed; by (iii) we have $E(y) \neq 0$. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) := \frac{E(x+y)}{E(y)} \quad \text{for} \quad x \in \mathbb{R}.$$

Evidently we have $G'(x) = E'(x+y)/E(y) = E(x+y)/E(y) = G(x)$ for all $x \in \mathbb{R}$, and $G(0) = E(0+y)/E(y) = 1$. It follows from the uniqueness of E, proved in Theorem 8.3.4, that $G(x) = E(x)$ for all $x \in \mathbb{R}$. Hence $E(x+y) = E(x)E(y)$ for all $x \in \mathbb{R}$. Since $y \in \mathbb{R}$ is arbitrary, we obtain (iv).

(v) It follows from (iv) and Induction that if $n \in \mathbb{N}$, $x \in \mathbb{R}$, then

$$E(nx) = E(x)^n.$$

If we let $x = 1/n$, this relation implies that

$$e = E(1) = E\left(n \cdot \frac{1}{n}\right) = \left(E\left(\frac{1}{n}\right)\right)^n,$$

whence it follows that $E(1/n) = e^{1/n}$. Also we have $E(-m) = 1/E(m) = 1/e^m = e^{-m}$ for $m \in \mathbb{N}$. Therefore, if $m \in \mathbb{Z}$, $n \in \mathbb{N}$, we have

$$E(m/n) = (E(1/n))^m = (e^{1/n})^m = e^{m/n}.$$

This establishes (v). Q.E.D.

8.3.7 Theorem *The exponential function E is strictly increasing on $\mathbb{R}$ and has range equal to $\{y \in \mathbb{R} : y > 0\}$. Further, we have*

(vi) $\lim_{x \to -\infty} E(x) = 0$ and $\lim_{x \to \infty} E(x) = \infty$.

Proof. We know that $E(0) = 1 > 0$ and $E(x) \neq 0$ for all $x \in \mathbb{R}$. Since E is continuous on $\mathbb{R}$, it follows from Bolzano's Intermediate Value Theorem 5.3.7 that $E(x) > 0$ for all $x \in \mathbb{R}$. Therefore $E'(x) = E(x) > 0$ for $x \in \mathbb{R}$, so that E is strictly increasing on $\mathbb{R}$.

It follows from Corollary 8.3.3 that $2 < e$ and that $\lim_{x\to\infty} E(x) = \infty$. Also, if $z > 0$, then since $0 < E(-z) = 1/E(z)$ it follows that $\lim_{x\to-\infty} E(x) = 0$. Therefore, by the Intermediate Value Theorem 5.3.7, every $y \in \mathbb{R}$ with $y > 0$ belongs to the range of E. Q.E.D.

The Logarithm Function

We have seen that the exponential function E is a strictly increasing differentiable function with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$. (See Figure 8.3.1.) It follows that $\mathbb{R}$ has an inverse function.

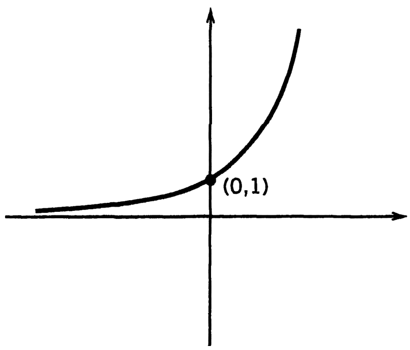

Figure 8.3.1 Graph of E

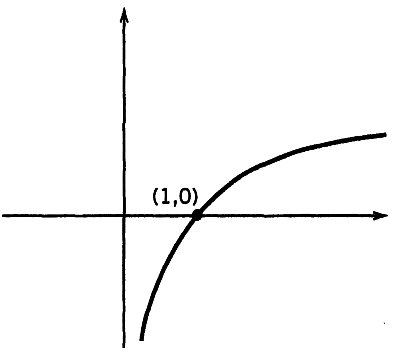

Figure 8.3.2 Graph of L

8.3.8 Definition The function inverse to $E : \mathbb{R} \to \mathbb{R}$ is called the **logarithm** (or the **natural logarithm**). (See Figure 8.3.2.) It will be denoted by L, or by ln.

Since E and L are inverse functions, we have

$$(L \circ E)(x) = x \qquad \text{for all} \quad x \in \mathbb{R}$$

and

$$(E \circ L)(y) = y \qquad \text{for all} \quad y \in \mathbb{R},\ y > 0.$$

These formulas may also be written in the form

$$\ln e^x = x, \qquad e^{\ln y} = y.$$

8.3.9 Theorem *The logarithm is a strictly increasing function L with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$. The derivative of L is given by*

(vii) $L'(x) = 1/x$ *for* $x > 0$.
The logarithm satisfies the functional equation

(viii) $L(xy) = L(x) + L(y)$ *for* $x > 0, y > 0$.
Moreover, we have

(ix) $L(1) = 0$ *and* $L(e) = 1$,

(x) $L(x^r) = rL(x)$ *for* $x > 0, r \in \mathbb{Q}$.

(xi) $\lim_{x\to 0+} L(x) = -\infty$ *and* $\lim_{x\to\infty} L(x) = \infty$.

Proof. That L is strictly increasing with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$ follows from the fact that E is strictly increasing with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$.

(vii) Since $E'(x) = E(x) > 0$, it follows from Theorem 6.1.9 that L is differentiable on $(0, \infty)$ and that

$$L'(x) = \frac{1}{(E' \circ L)(x)} = \frac{1}{(E \circ L)(x)} = \frac{1}{x} \qquad \text{for} \quad x \in (0, \infty).$$

(viii) If $x > 0$, $y > 0$, let $u := L(x)$ and $v := L(y)$. Then we have $x = E(u)$ and $y = E(v)$. It follows from property (iv) of Theorem 8.3.6 that

$$xy = E(u)E(v) = E(u + v),$$

so that $L(xy) = (L \circ E)(u + v) = u + v = L(x) + L(y)$. This establishes (viii).

The properties in (ix) follow from the relations $E(0) = 1$ and $E(1) = e$.

(x) This result follows from (viii) and Mathematical Induction for $n \in \mathbb{N}$, and is extended to $r \in \mathbb{Q}$ by arguments similar to those in the proof of 8.3.6(v).

To establish property (xi), we first note that since $2 < e$, then $\lim(e^n) = \infty$ and $\lim(e^{-n}) = 0$. Since $L(e^n) = n$ and $L(e^{-n}) = -n$ it follows from the fact that L is strictly increasing that

$$\lim_{x\to\infty} L(x) = \lim L(e^n) = \infty \quad \text{and} \quad \lim_{x\to 0+} L(x) = \lim L(e^{-n}) = -\infty. \qquad \text{Q.E.D.}$$

Power Functions

In Definition 5.6.6, we discussed the power function $x \mapsto x^r$, $x > 0$, where r is a rational number. By using the exponential and logarithm functions, we can extend the notion of power functions from rational to arbitrary real powers.

8.3.10 Definition If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha := e^{\alpha \ln x} = E(\alpha L(x)).$$

The function $x \mapsto x^\alpha$ for $x > 0$ is called the **power function** with exponent α.

Note If $x > 0$ and $\alpha = m/n$ where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, then we defined $x^\alpha := (x^m)^{1/n}$ in Section 5.6. Hence we have $\ln x^\alpha = \alpha \ln x$, whence $x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}$. Hence Definition 8.3.10 is consistent with the definition given in Section 5.6.

We now state some properties of the power functions. Their proofs are immediate consequences of the properties of the exponential and logarithm functions and will be left to the reader.

8.3.11 Theorem *If $\alpha \in \mathbb{R}$ and x, y belong to $(0, \infty)$, then:*

(a) $1^\alpha = 1$, **(b)** $x^\alpha > 0$,
(c) $(xy)^\alpha = x^\alpha y^\alpha$; **(d)** $(x/y)^\alpha = x^\alpha / y^\alpha$.

8.3.12 Theorem *If α, $\beta \in \mathbb{R}$ and $x \in (0, \infty)$, then:*

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$ **(b)** $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$;
(c) $x^{-\alpha} = 1/x^\alpha$; **(d)** *if $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$.*

The next result concerns the differentiability of the power functions.

8.3.13 Theorem *Let $\alpha \in \mathbb{R}$. Then the function $x \mapsto x^\alpha$ on $(0, \infty)$ to $\mathbb{R}$ is continuous and differentiable, and*

$$Dx^\alpha = \alpha x^{\alpha-1} \qquad \text{for} \quad x \in (0, \infty).$$

Proof. By the Chain Rule we have

$$\begin{aligned} Dx^\alpha &= De^{\alpha \ln x} = e^{\alpha \ln x} \cdot D(\alpha \ln x) \\ &= x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1} \qquad \text{for} \quad x \in (0, \infty). \end{aligned}$$

Q.E.D.

It will be seen in an exercise that if $\alpha > 0$, the power function $x \mapsto x^\alpha$ is strictly increasing on $(0, \infty)$ to $\mathbb{R}$, and that if $\alpha < 0$, the function $x \mapsto x^\alpha$ is strictly decreasing. (What happens if $\alpha = 0$?)

The graphs of the functions $x \mapsto x^\alpha$ on $(0, \infty)$ to $\mathbb{R}$ are similar to those in Figure 5.6.8.

The Function $\log_a$

If $a > 0$, $a \neq 1$, it is sometimes useful to define the function $\log_a$.

8.3.14 Definition Let $a > 0$, $a \neq 1$. We define

$$\log_a(x) := \frac{\ln x}{\ln a} \qquad \text{for} \quad x \in (0, \infty).$$

For $x \in (0, \infty)$, the number $\log_a(x)$ is called the **logarithm of x to the base** a. The case $a = e$ yields the logarithm (or natural logarithm) function of Definition 8.3.8. The case $a = 10$ gives the base 10 logarithm (or common logarithm) function $\log_{10}$ often used in computations. Properties of the functions $\log_a$ will be given in the exercises.

Exercises for Section 8.3

1. Show that if $x > 0$ and if $n > 2x$, then
$$\left| e^x - \left(1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!}\right) \right| < \frac{2x^{n+1}}{(n+1)!}$$
Use this formula to show that $2\frac{2}{3} < e < 2\frac{3}{4}$, hence e is not an integer.
2. Calculate e correct to five decimal places.
3. Show that if $0 \le x \le a$ and $n \in \mathbb{N}$, then
$$1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} \le e^x \le 1 + \frac{x}{1!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{e^a x^n}{n!}.$$
4. Show that if $n \ge 2$, then
$$0 < en! - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)n! < \frac{e}{n+1} < 1.$$
Use this inequality to prove that e is not a rational number.
5. If $x \ge 0$ and $n \in \mathbb{N}$, show that
$$\frac{1}{x+1} = 1 - x + x^2 - x^3 + \cdots + (-x)^{n-1} + \frac{(-x)^n}{1+x}.$$
Use this to show that
$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}\,dt$$

and that

$$\left|\ln(x+1) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \le \frac{x^{n+1}}{n+1}.$$

6. Use the formula in the preceding exercise to calculate ln 1.1 and ln 1.4 accurate to four decimal places. How large must one choose n in this inequality to calculate ln 2 accurate to four decimal places?

7. Show that $\ln(e/2) = 1 - \ln 2$. Use this result to calculate ln 2 accurate to four decimal places.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $K \in \mathbb{R}$ such that $f(x) = Ke^x$ for all $x \in \mathbb{R}$.

9. Let $a_k > 0$ for $k = 1, \ldots, n$ and let $A := (a_1 + \cdots + a_n)/n$ be the arithmetic mean of these numbers. For each k, put $x_k := a_k/A - 1$ in the inequality $1 + x \le e^x$. Multiply the resulting terms to prove the Arithmetic–Geometric Mean Inequality

$$(a_1 \cdots a_n)^{1/n} \le \frac{1}{n}(a_1 + \cdots + a_n). \tag{6}$$

Moreover, show that equality holds in (6) if and only if $a_1 = a_2 = \cdots = a_n$.

10. Evaluate $L'(1)$ by using the sequence $(1 + 1/n)$ and the fact that $e = \lim\big((1 + 1/n)^n\big)$.

11. Establish the assertions in Theorem 8.3.11.

12. Establish the assertions in Theorem 8.3.12.

13. (a) Show that if $\alpha > 0$, then the function $x \mapsto x^\alpha$ is strictly increasing on $(0, \infty)$ to $\mathbb{R}$ and that $\lim_{x\to 0+} x^\alpha = 0$ and $\lim_{x\to\infty} x^\alpha = \infty$.

    (b) Show that if $\alpha < 0$, then the function $x \mapsto x^\alpha$ is strictly decreasing on $(0, \infty)$ to $\mathbb{R}$ and that $\lim_{x\to 0+} x^\alpha = \infty$ and $\lim_{x\to\infty} x^\alpha = 0$.

14. Prove that if $a > 0$, $a \ne 1$, then $a^{\log_a x} = x$ for all $x \in (0, \infty)$ and $\log_a(a^y) = y$ for all $y \in \mathbb{R}$. Therefore the function $x \mapsto \log_a x$ on $(0, \infty)$ to $\mathbb{R}$ is inverse to the function $y \mapsto a^y$ on $\mathbb{R}$.

15. If $a > 0$, $a \ne 1$, show that the function $x \mapsto \log_a x$ is differentiable on $(0, \infty)$ and that $D \log_a x = 1/(x \ln a)$ for $x \in (0, \infty)$.

16. If $a > 0$, $a \ne 1$, and x and y belong to $(0, \infty)$, prove that $\log_a(xy) = \log_a x + \log_a y$.

17. If $a > 0$, $a \ne 1$, and $b > 0$, $b \ne 1$, show that

$$\log_a x = \left(\frac{\ln b}{\ln a}\right)\log_b x \qquad \text{for} \quad x \in (0, \infty).$$

In particular, show that $\log_{10} x = (\ln e/\ln 10) \ln x = (\log_{10} e) \ln x$ for $x \in (0, \infty)$.

Section 8.4 The Trigonometric Functions

Along with the exponential and logarithmic functions, there is another very important collection of transcendental functions known as the "trigonometric functions." These are the sine, cosine, tangent, cotangent, secant, and cosecant functions. In elementary courses, they are usually introduced on a geometric basis in terms of either triangles or the unit circle. In this section, we introduce the trigonometric functions in an analytical manner and then establish some of their basic properties. In particular, the various properties of the trigonometric functions that were used in examples in earlier parts of this book will be derived rigorously in this section.

It suffices to deal with the sine and cosine since the other four trigonometric functions are defined in terms of these two. Our approach to the sine and cosine is similar in spirit to our approach to the exponential function in that we first establish the existence of functions that satisfy certain differentiation properties.

8.4.1 Theorem *There exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that*

(i) $C''(x) = -C(x)$ *and* $S''(x) = -S(x)$ *for all* $x \in \mathbb{R}$.

(ii) $C(0) = 1, C'(0) = 0,$ *and* $S(0) = 0, S'(0) = 1$.

Proof. We define the sequences (C_n) and (S_n) of continuous functions inductively as follows:

(1) $$C_1(x) := 1, \qquad S_1(x) := x,$$

(2) $$S_n(x) := \int_0^x C_n(t)\,dt,$$

(3) $$C_{n+1}(x) := 1 - \int_0^x S_n(t)\,dt,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.

One sees by Induction that the functions C_n and S_n are continuous on $\mathbb{R}$ and hence they are integrable over any bounded interval; thus these functions are well-defined by the above formulas. Moreover, it follows from the Fundamental Theorem 7.3.5 that S_n and C_{n+1} are differentiable at every point and that

(4) $$S_n'(x) = C_n(x) \quad \text{and} \quad C_{n+1}'(x) = -S_n(x) \quad \text{for} \quad n \in \mathbb{N}, x \in \mathbb{R}.$$

Induction arguments (which we leave to the reader) show that

$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!},$$

$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let $A > 0$ be given. Then if $|x| \le A$ and $m > n > 2A$, we have that (since $A/2n < 1/4$):

(5) $$\begin{aligned} |C_m(x) - C_n(x)| &= \left| \frac{x^{2n}}{(2n)!} - \frac{x^{2n+2}}{(2n+2)!} + \cdots \pm \frac{x^{2m-2}}{(2m-2)!} \right| \\ &\le \frac{A^{2n}}{(2n)!} \left[1 + \left(\frac{A}{2n}\right)^2 + \cdots + \left(\frac{A}{2n}\right)^{2m-2n-2} \right] \\ &< \frac{A^{2n}}{(2n)!} \left(\frac{16}{15}\right). \end{aligned}$$

Since $\lim(A^{2n}/(2n)!) = 0$, the sequence (C_n) converges uniformly on the interval $[-A, A]$, where $A > 0$ is arbitrary. In particular, this means that $(C_n(x))$ converges for each $x \in \mathbb{R}$. We define $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) := \lim C_n(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that C is continuous on $\mathbb{R}$ and, since $C_n(0) = 1$ for all $n \in \mathbb{N}$, that $C(0) = 1$.

If $|x| \le A$ and $m \ge n > 2A$, it follows from (2) that

$$S_m(x) - S_n(x) = \int_0^x \{C_m(t) - C_n(t)\}\,dt.$$

If we use (5) and Corollary 7.3.15, we conclude that

$$|S_m(x) - S_n(x)| \le \frac{A^{2n}}{(2n)!}\left(\frac{16}{15}A\right),$$

whence the sequence (S_n) converges uniformly on $[-A, A]$. We define $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) := \lim S_n(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that S is continuous on $\mathbb{R}$ and, since $S_n(0) = 0$ for all $n \in \mathbb{N}$, that $S(0) = 0$.

Since $C_n'(x) = -S_{n-1}(x)$ for $n > 1$, it follows from the above that the sequence (C_n') converges uniformly on $[-A, A]$. Hence by Theorem 8.2.3, the limit function C is differentiable on $[-A, A]$ and

$$C'(x) = \lim C'_n(x) = \lim(-S_{n-1}(x)) = -S(x) \qquad \text{for} \quad x \in [-A, A].$$

Since $A > 0$ is arbitrary, we have

(6) $$C'(x) = -S(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

A similar argument, based on the fact that $S_n'(x) = C_n(x)$, shows that S is differentiable on $\mathbb{R}$ and that

(7) $$S'(x) = C(x) \qquad \text{for all} \quad x \in \mathbb{R}.$$

It follows from (6) and (7) that

$$C''(x) = -(S(x))' = -C(x) \qquad \text{and} \quad S''(x) = (C(x))' = -S(x)$$

for all $x \in \mathbb{R}$. Moreover, we have

$$C'(0) = -S(0) = 0, \qquad S'(0) = C(0) = 1.$$

Thus statements (i) and (ii) are proved. Q.E.D.

8.4.2 Corollary *If C, S are the functions in Theorem* 8.4.1, *then*

(iii) $C'(x) = -S(x)$ *and* $S'(x) = C(x)$ *for* $x \in \mathbb{R}$.

Moreover, these functions have derivatives of all orders.

Proof. The formulas (iii) were established in (6) and (7). The existence of the higher order derivatives follows by Induction. Q.E.D.

8.4.3 Corollary *The functions C and S satisfy the Pythagorean Identity:*

(iv) $(C(x))^2 + (S(x))^2 = 1$ for $x \in \mathbb{R}$.

Proof. Let $f(x) := (C(x))^2 + (S(x))^2$ for $x \in \mathbb{R}$, so that

$$f'(x) = 2C(x)(-S(x)) + 2S(x)(C(x)) = 0 \qquad \text{for} \quad x \in \mathbb{R}.$$

Thus it follows that $f(x)$ is a constant for all $x \in \mathbb{R}$. But since $f(0) = 1 + 0 = 1$, we conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. Q.E.D.

We next establish the uniqueness of the functions C and S.

8.4.4 Theorem *The functions C and S satisfying properties* (i) *and* (ii) *of Theorem* 8.4.1 *are unique.*

Proof. Let C_1 and C_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy $C_j''(x) = -C_j(x)$ for all $x \in \mathbb{R}$ and $C_j(0) = 1$, $C_j'(0) = 0$ for $j = 1, 2$. If we let $D := C_1 - C_2$, then $D''(x) = -D(x)$ for $x \in \mathbb{R}$ and $D(0) = 0$ and $D^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

Now let $x \in \mathbb{R}$ be arbitrary, and let I_x be the interval with endpoints $0, x$. Since $D = C_1 - C_2$ and $T := S_1 - S_2 = C_2' - C_1'$ are continuous on I_x, there exists $K > 0$ such that $|D(t)| \le K$ and $|T(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to D on I_x and use the fact that $D(0) = 0$, $D^{(k)}(0) = 0$ for $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$\begin{aligned} D(x) &= D(0) + \frac{D'(0)}{1!}x + \cdots + \frac{D^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{D^{(n)}(c_n)}{n!}x^n \\ &= \frac{D^{(n)}(c_n)}{n!}x^n. \end{aligned}$$

Now either $D^{(n)}(c_n) = \pm D(c_n)$ or $D^{(n)}(c_n) = \pm T(c_n)$. In either case we have

$$|D(x)| \le \frac{K|x|^n}{n!}.$$

But since $\lim\,(|x|^n/n!) = 0$, we conclude that $D(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $C_1(x) - C_2(x) = 0$ for all $x \in \mathbb{R}$.

A similar argument shows that if S_1 and S_2 are two functions on $\mathbb{R} \to \mathbb{R}$ such that $S_j''(x) = -S_j(x)$ for all $x \in \mathbb{R}$ and $S_j(0) = 0$, $S_j'(0) = 1$ for $j = 1, 2$, then we have $S_1(x) = S_2(x)$ for all $x \in \mathbb{R}$. Q.E.D.

Now that existence and uniqueness of the functions C and S have been established, we shall give these functions their familiar names.

8.4.5 Definition The unique functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that $C''(x) = -C(x)$ and $S''(x) = -S(x)$ for all $x \in \mathbb{R}$ and $C(0) = 1$, $C'(0) = 0$, and $S(0) = 0$, $S'(0) = 1$, are called the **cosine function** and the **sine function,** respectively. We ordinarily write

$$\cos x := C(x) \qquad \text{and} \qquad \sin x := S(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

The differentiation properties in (i) of Theorem 8.4.1 do not by themselves lead to uniquely determined functions. We have the following relationship.

8.4.6 Theorem *If $f : \mathbb{R} \to \mathbb{R}$ is such that*

$$f''(x) = -f(x) \qquad \textit{for} \quad x \in \mathbb{R},$$

then there exist real numbers α, β such that

$$f(x) = \alpha C(x) + \beta S(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

Proof. Let $g(x) := f(0)C(x) + f'(0)S(x)$ for $x \in \mathbb{R}$. It is readily seen that $g''(x) = -g(x)$ and that $g(0) = f(0)$, and since

$$g'(x) = -f(0)S(x) + f'(0)C(x),$$

that $g'(0) = f'(0)$. Therefore the function $h := f - g$ is such that $h''(x) = -h(x)$ for all $x \in \mathbb{R}$ and $h(0) = 0$, $h'(0) = 0$. Thus it follows from the proof of the preceding theorem that $h(x) = 0$ for all $x \in \mathbb{R}$. Therefore $f(x) = g(x)$ for all $x \in \mathbb{R}$. Q.E.D.

We shall now derive a few of the basic properties of the cosine and sine functions.

8.4.7 Theorem *The function C is even and S is odd in the sense that*

(v) $C(-x) = C(x)$ *and* $S(-x) = -S(x)$ *for* $x \in \mathbb{R}$.

If $x, y \in \mathbb{R}$, then we have the "addition formulas"

(vi) $C(x+y) = C(x)C(y) - S(x)S(y), \quad S(x+y) = S(x)C(y) + C(x)S(y).$

Proof. (v) If $\varphi(x) := C(-x)$ for $x \in \mathbb{R}$, then a calculation shows that $\varphi''(x) = -\varphi(x)$ for $x \in \mathbb{R}$. Moreover, $\varphi(0) = 1$ and $\varphi'(0) = 0$ so that $\varphi = C$. Hence, $C(-x) = C(x)$ for all $x \in \mathbb{R}$. In a similar way one shows that $S(-x) = -S(x)$ for all $x \in \mathbb{R}$.

(vi) Let $y \in \mathbb{R}$ be given and let $f(x) := C(x+y)$ for $x \in \mathbb{R}$. A calculation shows that $f''(x) = -f(x)$ for $x \in \mathbb{R}$. Hence, by Theorem 8.4.6, there exists real numbers α, β such that

$$f(x) = C(x+y) = \alpha C(x) + \beta S(x) \quad \text{and}$$
$$f'(x) = -S(x+y) = -\alpha S(x) + \beta C(x)$$

for $x \in \mathbb{R}$. If we let $x = 0$, we obtain $C(y) = \alpha$ and $-S(y) = \beta$, whence the first formula in (vi) follows. The second formula is proved similarly.

Q.E.D.

The following inequalities were used earlier (for example, in 4.2.8).

8.4.8 Theorem *If $x \in \mathbb{R}$, $x \geq 0$, then we have*

(vii) $-x \leq S(x) \leq x$; **(viii)** $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$; **(x)** $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Proof. Corollary 8.4.3 implies that $-1 \leq C(t) \leq 1$ for $t \in \mathbb{R}$, so that if $x \geq 0$, then

$$-x \leq \int_0^x C(t)dt \leq x,$$

whence we have (vii). If we integrate (vii), we obtain

$$-\tfrac{1}{2}x^2 \leq \int_0^x S(t)\,dt \leq \tfrac{1}{2}x^2,$$

whence we have

$$-\tfrac{1}{2}x^2 \leq -C(x) + 1 \leq \tfrac{1}{2}x^2.$$

Thus we have $1 - \frac{1}{2}x^2 \leq C(x)$, which implies (viii).

Inequality (ix) follows by integrating (viii), and (x) follows by integrating (ix). Q.E.D.

The number π is obtained via the following lemma.

8.4.9 Lemma *There exists a root γ of the cosine function in the interval $(\sqrt{2}, \sqrt{3})$. Moreover $C(x) > 0$ for $x \in [0, \gamma)$. The number 2γ is the smallest positive root of S.*

Proof. Inequality (x) of Theorem 8.4.8 implies that C has a root between the positive root $\sqrt{2}$ of $x^2 - 2 = 0$ and the smallest positive root of $x^4 - 12x^2 + 24 = 0$, which is $\sqrt{6 - 2\sqrt{3}} < \sqrt{3}$. We let γ be the smallest such root of C.

It follows from the second formula in (vi) with $x = y$ that $S(2x) = 2S(x)C(x)$. This relation implies that $S(2\gamma) = 0$, so that 2γ is a positive root of S. The same relation implies that if $2\delta > 0$ is the smallest positive root of S, then $C(\delta) = 0$. Since γ is the smallest positive root of C, we have $\delta = \gamma$. Q.E.D.

8.4.10 Definition Let $\pi := 2\gamma$ denote the smallest positive root of S.

Note The inequality $\sqrt{2} < \gamma < \sqrt{6 - 2\sqrt{3}}$ implies that $2.828 < \pi < 3.185$.

8.4.11 Theorem *The functions C and S have period 2π in the sense that*

(xi) $C(x + 2\pi) = C(x)$ *and* $S(x + 2\pi) = S(x)$ *for* $x \in \mathbb{R}$.

Moreover we have

(xii) $S(x) = C\left(\frac{1}{2}\pi - x\right) = -C\left(x + \frac{1}{2}\pi\right), \quad C(x) = S\left(\frac{1}{2}\pi - x\right) = S\left(x + \frac{1}{2}\pi\right)$ *for all* $x \in \mathbb{R}$.

Proof. (xi) Since $S(2x) = 2S(x)C(x)$ and $S(\pi) = 0$, then $S(2\pi) = 0$. Further, if $x = y$ in (vi), we obtain $C(2x) = (C(x))^2 - (S(x))^2$. Therefore $C(2\pi) = 1$. Hence (vi) with $y = 2\pi$ gives

$$C(x + 2\pi) = C(x)C(2\pi) - S(x)S(2\pi) = C(x),$$

and

$$S(x + 2\pi) = S(x)C(2\pi) + C(x)S(2\pi) = S(x).$$

(xii) We note that $C\left(\frac{1}{2}\pi\right) = 0$, and it is an exercise to show that $S\left(\frac{1}{2}\pi\right) = 1$. If we employ these together with formulas (vi), the desired relations are obtained. Q.E.D.

Exercises for Section 8.4

1. Calculate cos(.2), sin(.2) and cos 1, sin 1 correct to four decimal places.
2. Show that $|\sin x| \le 1$ and $|\cos x| \le 1$ for all $x \in \mathbb{R}$.
3. Show that property (vii) of Theorem 8.4.8 does not hold if $x < 0$, but that we have $|\sin x| \le |x|$ for all $x \in \mathbb{R}$. Also show that $|\sin x - x| \le |x|^3/6$ for all $x \in \mathbb{R}$.
4. Show that if $x > 0$ then
$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \le \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$
Use this inequality to establish a lower bound for π.
5. Calculate π by approximating the smallest positive zero of sin. (Either bisect intervals or use Newton's Method of Section 6.4.)
6. Define the sequence (c_n) and (s_n) inductively by $c_1(x) := 1, s_1(x) := x$, and
$$s_n(x) := \int_0^x c_n(t)dt, \quad c_{n+1}(x) := 1 + \int_0^x s_n(t)dt$$
for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Reason as in the proof of Theorem 8.4.1 to conclude that there exist functions $c : \mathbb{R} \to \mathbb{R}$ and $s : \mathbb{R} \to \mathbb{R}$ such that (j) $c''(x) = c(x)$ and $s''(x) = s(x)$ for all $x \in \mathbb{R}$, and (jj) $c(0) = 1$, $c'(0) = 0$ and $s(0) = 0$, $s'(0) = 1$. Moreover, $c'(x) = s(x)$ and $s'(x) = c(x)$ for all $x \in \mathbb{R}$.

7. Show that the functions c, s in the preceding exercise have derivatives of all orders, and that they satisfy the identity $(c(x))^2 - (s(x))^2 = 1$ for all $x \in \mathbb{R}$. Moreover, they are the unique functions satisfying (j) and (jj). (The functions c, s are called the **hyperbolic cosine** and **hyperbolic sine functions**, respectively.)

8. If $f : \mathbb{R} \to \mathbb{R}$ is such that $f''(x) = f(x)$ for all $x \in \mathbb{R}$, show that there exist real numbers α, β such that $f(x) = \alpha c(x) + \beta s(x)$ for all $x \in \mathbb{R}$. Apply this to the functions $f_1(x) := e^x$ and $f_2(x) := e^{-x}$ for $x \in \mathbb{R}$. Show that $c(x) = \frac{1}{2}(e^x + e^{-x})$ and $s(x) = \frac{1}{2}(e^x - e^{-x})$ for $x \in \mathbb{R}$.

9. Show that the functions c, s in the preceding exercises are even and odd, respectively, and that

$$c(x + y) = c(x)c(y) + s(x)s(y), \quad s(x + y) = s(x)c(y) + c(x)s(y),$$

for all $x, y \in \mathbb{R}$.

10. Show that $c(x) \geq 1$ for all $x \in \mathbb{R}$, that both c and s are strictly increasing on $(0, \infty)$, and that $\lim_{x\to\infty} c(x) = \lim_{x\to\infty} s(x) = \infty$.