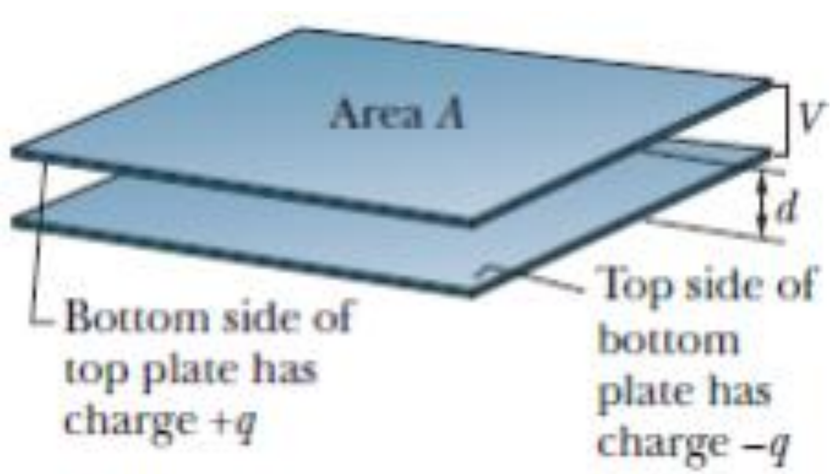
Capacitors and Capacitance: Parallel Plate; Cylindrical and Spherical capacitors; Capacitors in Series and Parallel; Energy Stored in an Electric Field; Dielectrics and Gauss' Law

Capacitor:

A capacitor is a passive electronic component that stores energy in the form of an electrostatic field. In its simplest form, a capacitor consists of two conducting plates separated by an insulating material called the dielectric.

The capacitance is directly proportional to the surface areas of the plates, and is inversely proportional to the separation between the plates. Capacitance also depends on the dielectric constant of the substance separating the plates.



This conventional arrangement, called a parallel-plate capacitor, consisting of two parallel conducting plates of area A separated by a distance d.

The symbol we use to represent a capacitor (⊣⊢) is based on the structure of a parallel-plate capacitor but is used for capacitors of all geometries.

We assume for the time being that no material medium (such as glass or plastic) is present in the region between the plates.

When a capacitor is charged, its plates have charges of equal magnitudes but opposite signs: +q and -q. However, we refer to the charge of a capacitor as being q, the absolute value of these charges on the plates. (Note that q is not the net charge on the capacitor, which is zero.)

Because the plates are conductors, they are equipotential surfaces; all points on a plate are at the same electric potential. Moreover, there is a potential difference between the two plates. For historical reasons, we represent the absolute value of this potential difference with V rather than with the ΔV we used in previous notation.

The charge q and the potential difference V for a capacitor are proportional to each other; that is,

$$q = CV.$$

The proportionality constant C is called the capacitance of the capacitor. Its value depends only on the geometry of the plates and not on their charge or potential difference. The capacitance is a measure of how much charge must be put on the plates to produce a certain potential difference between them: The greater the capacitance, the more charge is required.

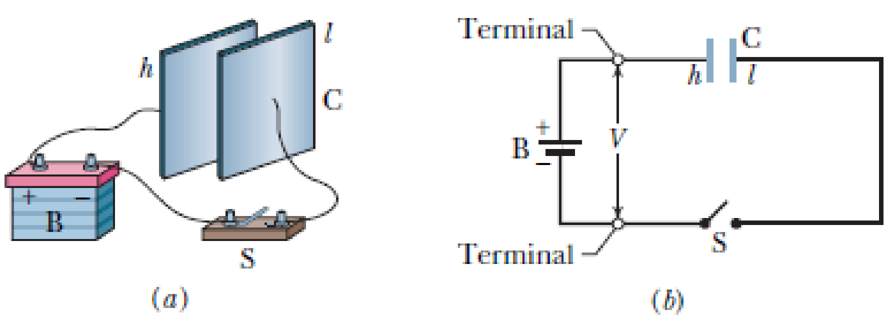
The SI unit, capacitance is the coulomb per volt. This unit occurs so often that it is given a special name, the farad (F):

$$1 \text{ farad} = 1 \text{ F} = 1 \text{ coulomb per volt} = 1 \text{ C/V}.$$

As you will see, the farad is a very large unit. Submultiples of the farad, such as the microfarad ($1\ \mu F = 10^{-6}$ F) and the picofarad ($1\ pF = 10^{-12}$ F), are more convenient units in practice

Charging a Capacitor

One way to charge a capacitor is to place it in an electric circuit with a battery. An electric circuit is a path through which charge can flow. A battery is a device that maintains a certain potential difference between its terminals.



In Fig.a, a battery B, a switch S, an uncharged capacitor C, and interconnecting wires form a circuit.

The same circuit is shown in the schematic diagram of Fig. b, in which the symbols for a battery, a switch, and a capacitor represent those devices. The battery maintains potential difference V between its terminals. The terminal of higher potential is labeled + and is often called the positive terminal; the terminal of lower potential is labeled - and is often called the negative terminal.

The circuit shown in Figs. *a* and *b* is said to be incomplete because switch S is open; that is, the switch does not electrically connect the wires attached to it. When the switch is closed, electrically connecting those wires, the circuit is complete and charge can then flow through the switch and the wires.

As we discussed, the charge that can flow through a conductor, such as a wire, is that of electrons.

When the circuit of Fig. (a,b) is completed, electrons are driven through the wires by an electric field that the battery sets up in the wires. The field drives electrons from capacitor plate *h* to the positive terminal of the battery; thus, plate *h*, losing electrons, becomes positively charged.

The field drives just as many electrons from the negative terminal of the battery to capacitor plate *l*; thus, plate *l*, gaining electrons, becomes negatively charged.

Initially, when the plates are uncharged, the potential difference between them is zero. As the plates become oppositely charged, that potential difference increases until it equals the potential difference V between the terminals of the battery.

Then plate h and the positive terminal of the battery are at the same potential, and there is no longer an electric field in the wire between them.

Similarly, plate *l* and the negative terminal reach the same potential, and there is then no electric field in the wire between them.

Thus, with the field zero, there is no further drive of electrons. The capacitor is then said to be fully charged, with a potential difference V and charge q that are related by Eq.

$$Q = C\,V$$

Calculating the Capacitance

To calculate the capacitance of a capacitor once we know its geometry. Because we shall consider a number of different geometries, it seems wise to develop a general plan to simplify the work.

In brief our plan is as follows:

(1) Assume a charge q on the plates

(2) Calculate the electric field between the plates in terms of this charge, using Gauss' law

$$\Phi = \oint \vec{E} \cdot d\vec{A}$$

(3) Calculate the potential difference V between the plates from Eq. ($V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}.$)

(4) Calculate C from Eq.(q = CV).

Calculating the Electric Field

To relate the electric field between the plates of a capacitor to the charge q on either plate, we shall use Gauss' law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q. \qquad (1)$$

Here q is the charge enclosed by a Gaussian surface and $\oint \vec{E} \cdot d\vec{A} = q.$ is the net electric flux through that surface. In all cases that we shall consider, the Gaussian surface will be such that whenever there is an electric flux through it, $\vec{E}$ will have a uniform magnitude E and the vectors $\vec{E}$ and $d\vec{A}$ will be parallel. The above equation, then reduces to

$$q = \varepsilon_0 E A \qquad (2)$$

Calculating the Potential Difference

The potential difference between the plates of a capacitor is related to the field by

$$V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}. \qquad (3)$$

in which the integral is to be evaluated along any path that starts on one plate and ends on the other.

We shall always choose a path that follows an electric field line, from the negative plate to the positive plate.

For this path, the vectors $\vec{E}$ and $d\vec{s}$ will have opposite directions; so the dot product will be equal to $\vec{E} \cdot d\vec{s}$ .Thus, the right side of above Eq. will then be positive. Letting V represent the difference V_f-V_i, we can then recast Eq. as

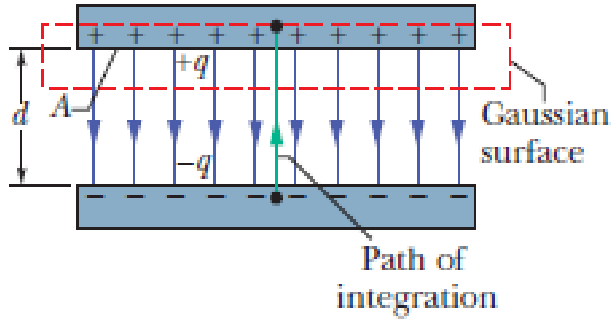
$$V = \int_-^+ E\, ds \qquad (4)$$

in which the - and + remind us that our path of integration starts on the negative plate and ends on the positive plate.

A Parallel-Plate Capacitor

We assume, as Fig. 25-5 suggests, that the plates of our parallel-plate capacitor are so large and so close together that we can neglect the fringing of the electric field at the edges of the plates, taking $\vec{E}$ to be constant throughout the region between the plates.

q.



We draw a Gaussian surface that encloses just the charge q on the positive plate, as in above Fig.. From Eq. 2 we can then write

$$q = \varepsilon_0 EA \tag{5}$$

Equation 4 yield

$$V = \int_{-}^{+} E\, ds = E \int_0^d ds = Ed. \tag{6}$$

In Eq. 6, E can be placed outside the integral because it is a constant; the second integral then is simply the plate separation d.

put q and V into the relation q = CV , we get

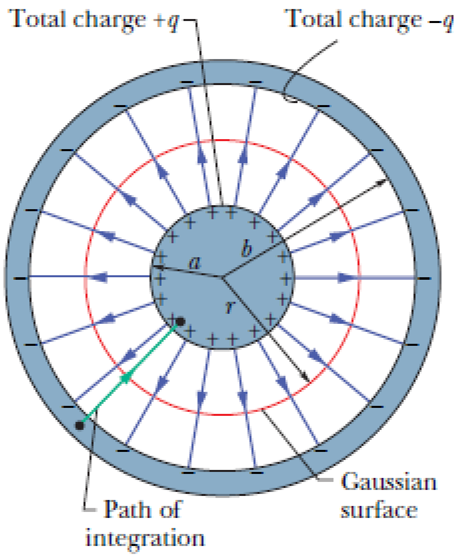
$$C = \frac{\varepsilon_0 A}{d} \tag{7}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{F/m} = 8.85\ \text{pF/m}.$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \text{C}^2/\text{N} \cdot \text{m}^2$$

A Cylindrical Capacitor

Figure shows, in cross section, a cylindrical capacitor of length L formed by two coaxial cylinders of radii a and b. We assume that L>>(b so that we can neglect the fringing of the electric field that occurs at the ends of the cylinders. Each plate contains a charge of magnitude q.



As a Gaussian surface, we choose a cylinder of length L and radius r, closed by end caps and placed as is shown in Fig. It is coaxial with the cylinders and encloses the central cylinder and thus also the charge q on that cylinder. Equation 2 then relates that charge and the field magnitude E as

$$q = \varepsilon_0 EA = \varepsilon_0 E(2\pi rL)$$

In which 2πrL is the area of the curved part of the Gaussian surface. There is no flux through the end caps. Solving for E yields

$$E = \frac{q}{2\pi\varepsilon_0 Lr}. \quad (8)$$

Put these values in eq 4

$$V = \int_{-}^{+} E\, ds = -\frac{q}{2\pi\varepsilon_0 L}\int_b^a \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right) \quad (9)$$

where we have used the fact that here ds = -dr (we integrated radially inward).

From the relation C = q/V, we then have

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \tag{10}$$

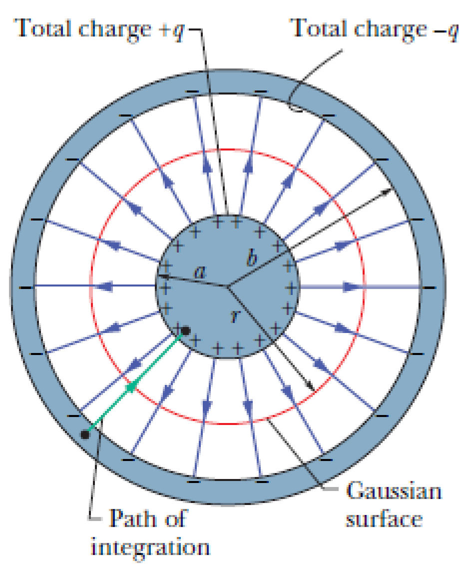
We see that the capacitance of a cylindrical capacitor, like that of a parallel-plate capacitor, depends only on geometrical factors, in this case the length L and the two radii b and a.

A Spherical Capacitor

This also Figure can also serve as a central cross section of a capacitor that consists of two concentric spherical shells, of radii a and b. As a Gaussian surface we draw a sphere of radius r concentric with the two shells; then Eq. 2 yields

$$q = \varepsilon_0 EA = \varepsilon_0 E(4\pi r^2),$$

in which $4\pi r^2$ is the area of the spherical Gaussian surface. We solve this equation for E, obtaining



$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \tag{11}$$

Which we recognize as the expression for the electric field due to a uniform spherical charge distribution (Eq. 11).

If we substitute this expression into Eq. 4, we find

$$V = \int_{-}^{+} E\,ds = -\frac{q}{4\pi\varepsilon_0}\int_{b}^{a}\frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{q}{4\pi\varepsilon_0}\frac{b-a}{ab} \qquad (12)$$

where again we have substituted -dr for ds. If we now substitute Eq. 12 into
q = CV, we find

$$C = 4\pi\varepsilon_0 \frac{ab}{b-a} \qquad (13)$$

Capacitors in Parallel

Figure. (a) Shows an electric circuit in which three capacitors are connected in parallel to battery B.

Each capacitor has the same potential difference V, which produces charge on the capacitor. (In Fig. a, the applied potential V is maintained by the battery.) In general, When we analyze a circuit of capacitors in parallel, we can simplify it with this mental replacement:

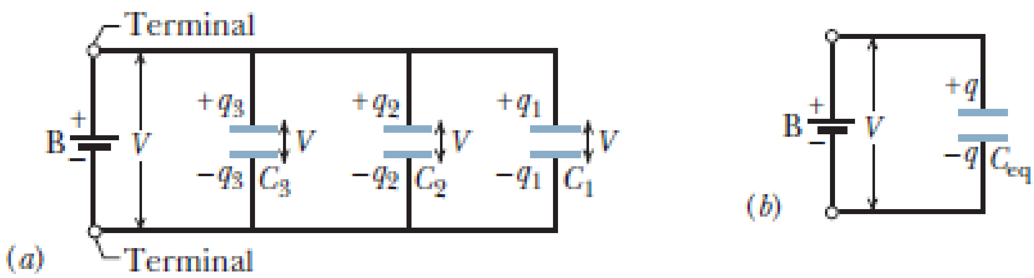


Figure b shows the equivalent capacitor (with equivalent capacitance C_{eq}) that has replaced the three capacitors (with actual capacitances C_1,C_2, and C_3) of Fig. a.
To derive an expression for Ceq in Fig. b, we first use Eq. q = CV to find the charge on each actual capacitor:

$$q_1 = C_1V, \quad q_2 = C_2V, \quad \text{and} \quad q_3 = C_3V.$$

The total charge on the parallel combination

$$q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V$$

The equivalent capacitance, with the same total charge *q* and applied potential difference *V* as the combination, is then

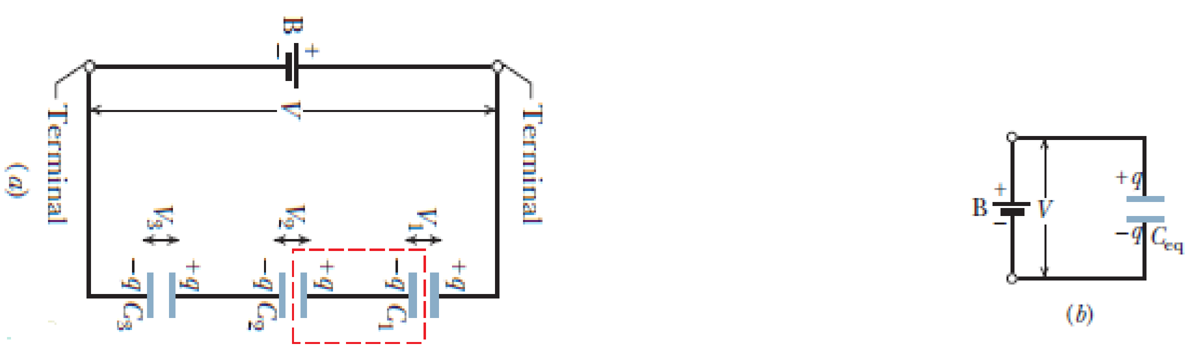
$$C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3,$$

For *n* number of capacitors,

$$C_{eq} = \sum_{j=1}^{n} C_j$$

Capacitors in Series

When the battery is first connected to the series of capacitors, it produces charge -q on the bottom plate of capacitor 3. That charge then repels negative charge from the top plate of capacitor 3 (leaving it with charge +q). The repelled negative charge moves to the bottom plate of capacitor 2 (giving it charge - q). That charge on the bottom plate of capacitor 2 then repels negative charge from the top plate of capacitor 2 (leaving it with charge +q) to the bottom plate of capacitor 1 (giving it charge - q). Finally, the charge on the bottom plate of capacitor 1 helps move negative charge from the top plate of capacitor 1 to the battery, leaving that top plate with charge +q.



To derive an expression for *C*eq in Fig. 25-9*b*, we first use q = CV to find the potential difference of each actual capacitor:

$$V_1 = \frac{q}{C_1}, \quad V_2 = \frac{q}{C_2}, \quad \text{and} \quad V_3 = \frac{q}{C_3}.$$

The total potential difference V due to the battery is the sum of these three potential differences. Thus,

$$V = V_1 + V_2 + V_3 = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right).$$

$$C_{eq} = \frac{q}{V} = \frac{1}{1/C_1 + 1/C_2 + 1/C_3},$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

For *n* number *of* capacitors as

$$\frac{1}{C_{eq}} = \sum_{j=1}^{n} \frac{1}{C_j}$$

The equivalent capacitance of a series of capacitances is always less than the least capacitance in the series

DI ELECTRICS AND GAUSS'S LAW

In our discussion of Gauss' law, we assumed that the charges existed in a vacuum. Here we shall see how to modify and generalize that law if dielectric materials. Figure 1 shows a parallel-plate capacitor of plate area A, both with and without a dielectric. We assume that the charge q on the plates is the same in both situations. For the situation, without a dielectric, we can find the electric field between the plates as we did in Gauss' law (electric field of oppositely charged parallel plates).

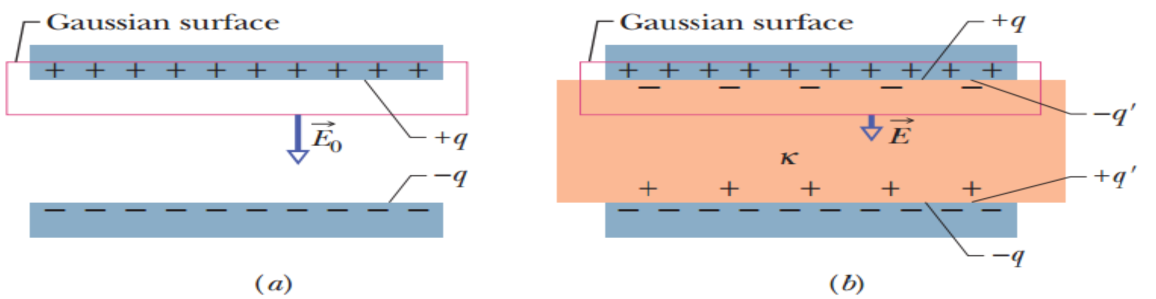


Fig 1. A parallel-plate capacitor (*a*) without and (*b*) with a dielectric slab inserted. The charge *q* on the plates is assumed to be the same in both cases.

We enclose the charge +q on the top plate with a Gaussian surface and then apply Gauss' law. Letting E_0 represent the magnitude of the field, we find

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = q,$$

or

$$E_0 = \frac{q}{\varepsilon_0 A}.$$

In Fig. 1*b*, with the dielectric in place, we can find the electric field between the plates (and within the dielectric) by using the same Gaussian surface. However, now the surface encloses two types of charge. It still encloses charge +*q* on the top plate, but it now also encloses the induced charge -*q*, on the top face of the dielectric. The charge on the conducting plate is said to be *free charge* because it can move if we change the electric potential of the plate; the induced charge on the surface of the dielectric is not free charge because it cannot move from that surface. The net charge enclosed by the Gaussian surface in **Fig 1*b*** is

$\boldsymbol{q - q^{/}}$ so Gauss' law now gives

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E A = q - q', \longrightarrow \boxed{1}$$

or

$$E = \frac{q - q'}{\varepsilon_0 A}. \longrightarrow \boxed{2}$$

The effect of the dielectric is to weaken the original field E_0 by a factor of k; so we may write

$$E = \frac{E_0}{\kappa} = \frac{q}{\kappa \varepsilon_0 A}. \longrightarrow \boxed{3}$$

Comparison of Eqs. 2 and 3 shows that

$$q - q' = \frac{q}{\kappa}. \longrightarrow \boxed{4}$$

Equation 4 shows correctly that the magnitude q, of the induced surface charge is less than that of the free charge q and is zero if no dielectric is present (because then k =1 in Eq. 4).

By substituting for q - q, from Eq. 4 in Eq. 1 , we can write Gauss'law in the form

$$\varepsilon_0 \oint \kappa \vec{E} \cdot d\vec{A} = q \qquad \text{(Gauss' law with dielectric).} \qquad (5)$$

This equation, although derived for a parallel-plate capacitor, is true generally and is the most general form in which Gauss' law can be written.

Note

1. The flux integral now involves $k\vec{E}$, not just $\vec{E}$. (The vector $\varepsilon_0 k\vec{E}$ is sometimes called the electric displacement $\vec{D}$, so that Eq. 5 can be written in the form $\oint \vec{D} \cdot d\vec{A} = q$.)

2. The charge q enclosed by the Gaussian surface is now taken to be the *free charge only*. The induced surface charge is deliberately ignored on the right side of Eq. 5, having been taken fully into account by introducing the dielectric constant k on the left side.

3. Equation 5 differs from eq ,i.e., ($\varepsilon_0 \oint \vec{E} \cdot dA = q_{enc}$), our original statement of Gauss' law, only in that $\boldsymbol{\varepsilon_0}$ in the latter equation has been replaced by $k\varepsilon_0$. We keep k inside the integral of Eq. 5 to allow for cases in which k is not constant over the entire Gaussian surface.

ENERGY STORED IN AN ELECTRIC FIELD

Work must be done by an external agent to charge a capacitor. Starting with an uncharged capacitor, for example, imagine that—using "magic tweezers"—you remove electrons from one plate and transfer them one at a time to the other plate. The electric field that builds up in the space between the plates has a direction that tends to oppose further transfer. Thus, as charge accumulates on the capacitor plates, you have to do increasingly larger amounts of work to transfer additional electrons. In practice, this work is done not by "magic tweezers" but by a battery, at the expense of its store of chemical energy.

We visualize the work required to charge a capacitor as being stored in the form of electric potential energy **U** in the electric field between the plates. You can recover this energy at will, by discharging the capacitor in a circuit, just as you can recover the potential energy stored in a stretched bow by releasing the bowstring to transfer the energy to the kinetic energy of an arrow. Suppose that, at a given instant, a charge $\mathbf{q}'$, has been transferred from one plate of a capacitor to the other. The potential difference $\mathbf{V}'$ **,** between the plates at that instant will be $\mathbf{q}'/\mathbf{C}$. If an extra increment of charge $\mathbf{dq}'$ is then transferred, the increment of work required will be,

$$dW = V'\,dq' = \frac{q'}{C}\,dq'.$$

The work required to bring the total capacitor charge up to a final value *q* is

$$W = \int dW = \frac{1}{C}\int_0^q q'\,dq' = \frac{q^2}{2C}.$$

This work is stored as potential energy U in the capacitor, so that

$$U = \frac{q^2}{2C} \qquad \text{(potential energy).} \rightarrow \boxed{1}$$

From eq, ($q = cv$), we can also write this as

$$U = \tfrac{1}{2}CV^2 \qquad \text{(potential energy).} \rightarrow \boxed{2}$$

Equations **1** and **2** hold no matter what the geometry of the capacitor is.

To gain some physical insight into energy storage, consider two parallel-plate capacitors that are identical except that capacitor 1 has twice the plate separation of capacitor 2. Then capacitor 1 has twice the volume between its plates and also, **from Eq,** ($\boldsymbol{C = \frac{A\varepsilon_0}{d}}$)**,** half the capacitance of capacitor 2. **Eq (**$\boldsymbol{q = \varepsilon_0 EA}$) tells us that if both capacitors have the same charge q, the electric fields between their plates are identical. And **Eq. 1** tells us that capacitor 1 has twice the stored potential energy of capacitor 2. Thus, of two otherwise identical capacitors with the same charge and same electric field, the one with twice the volume between its plates has twice the stored potential energy. Arguments like this tend to verify our earlier assumption:

'*The potential energy of a charged capacitor may be viewed as being stored in the electric field between its plates'*