Chapter 1

Introduction to Partial Differential Equations

1.1 Introduction

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables with some of its partial derivatives. PDEs are of fundamental importance in applied mathematics and physics, and have recently shown to be useful in as varied disciplines as financial modelling and modelling of biological systems. More specifically, we have the following definition.

Definition 1.1 *Let $\Omega \subset \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$ (called the* domain of definition*), for $d > 1$ a positive integer (called the* dimension*), and denote by $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ a vector in Ω. Let (unknown) function $u : \Omega \to \mathbb{R}$ whose partial derivatives up to order k (for k positive integer)*

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial u^2}{\partial x_1^2}, \frac{\partial u^2}{\partial x_2^2}, \ldots, \frac{\partial u^2}{\partial x_d^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_d},$$

$$\ldots, \frac{\partial u^k}{\partial x_1^k}, \frac{\partial u^k}{\partial x_2^k}, \ldots, \frac{\partial u^k}{\partial x_d^k}, \frac{\partial u^k}{\partial x_1^{k-1} \partial x_2}, \ldots, \frac{\partial u^k}{\partial x_{d-1} \partial x_d^{k-1}}$$

exist. A partial differential equation of order k in Ω in d dimensions *is an equation of the form:*

$$F(x, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_d}, \ldots, \frac{\partial u^k}{\partial x_{d-1} \partial x_d^{k-1}}) = 0, \tag{1.1}$$

where F is a given function.

Example 1.2 *Let $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and a function $u : \mathbb{R}^2 \to \mathbb{R}$. The equation*

$$\frac{\partial u}{\partial x_1} = 0,$$

is a PDE of 1st order on $\mathbb{R}^2$ in 2 dimensions.

Example 1.3 *Let $u : [0, 1]^3 \to \mathbb{R}$. The equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3x^3,$$

is a PDE of 2nd order on $[0, 1]^3$ in 3 dimensions. (This is an instance of the so-called Poisson equation*.)*

Solution. Indeed this is in accordance with Definition 1.1 with $d = 3$, $x_1 = x$, $x_2 = y$, $x_3 = z$ and $\Omega = \mathbb{R}^3$.
□

Example 1.4 *Let* $u : \mathbb{R}^3 \to \mathbb{R}$*. The equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

is a PDE of 2*nd order on* $\mathbb{R}^3$ *in* 3 *dimensions. (This is an instance of the so-called* heat equation.*)*

Example 1.5 *Let* $u : \mathbb{R}^2 \to \mathbb{R}$*, with* $u = u(t, x)$*. The equation*

$$\frac{\partial u}{\partial t} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0,$$

is a PDE of 2*nd order on* $\mathbb{R}^2$ *in* 2 *dimensions. (This is the so-called* Black-Scholes equation.*)*

Example 1.6 *Let* $u : \mathbb{R}^2 \to \mathbb{R}$*. The equation*

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \Big(\frac{\partial^2 u}{\partial x \partial y}\Big)^2 = 0,$$

is a PDE of 2*nd order on* $\mathbb{R}^2$ *in* 2 *dimensions. (This is the so-called* Monge-Ampère equation.*)*

We shall be mostly interested in PDEs in two and three dimensions (as these are the ones most often appearing in practical applications), and we shall confine the notation to these cases using (x, y) and (t, x) or (x, y, z) and (t, x, y) to describe two- and three-dimensional vectors respectively (when the notation t is used for an independent variable, this variable should almost always describing "time"). Nevertheless, many properties and ideas described below apply also to the general case of d-dimensions for $d > 3$.

Also, to simplify the notation, we shall often resort to the more compact notation u_x, u_y, u_{xx}, u_{xy}, etc., to signify partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, etc., respectively.

1.2 Solution of PDEs

Studying and solving PDEs has been one of the central areas of research in applied mathematics during the last 2 centuries.

Definition 1.7 *Consider the notation of Definition (1.1). We call the* general solution *of the PDE (1.1), the family of functions* $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ *that has continuous partial derivatives up to (and including) order* k *and that satisfies (1.1).*

Example 1.8 *We want to find the general solution of the PDE in* $\mathbb{R}^2$*:*

$$\frac{\partial u}{\partial x_1} = x_1^3.$$

Integrating with respect to x_1*, we get*

$$u(x_1, x_2) = \frac{x_1^4}{4} + f(x_2), \tag{1.2}$$

for any differentiable function $f : \mathbb{R} \to \mathbb{R}$*. Indeed, if we differentiate this solution with respect to* x_1*, we get back the PDE. It is also not hard to see that if* u *is a solution of the PDE then it has to be of the form (1.2) as this follows from the Fundamental Theorem of Calculus (Why?).*

Solving PDEs is often a far more tricky pursuit than the previous example seems to indicate. Let's try to see why. Consider, for example, the PDE

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0.$$

It is not too hard to guess that any constant function u satisfies this PDE. However, there are more functions that satisfy this PDE that just the constant ones. It is clear that an integration will not be of any help here and more elaborate methods need to be introduced. Moreover, as we shall see below, there is *no* method of solving PDEs that works in general. Instead, different methods work for different families of PDEs. Therefore, it is important to identify such families of PDEs that can be solved in an similar fashion and, subsequently to describe particular methods of solving PDEs from each such family. This will be the content of the rest of this chapter, where we shall classify PDEs in various families and present some of their properties.

1.3 Classification of PDEs

To study PDEs it is often useful to classify them into various families, since PDEs belonging to particular families can be characterised by similar behaviour and properties. There are many and varied classifications for PDEs. Perhaps the most widely accepted and generally useful classification is the distinction between linear and non-linear PDEs. In particular, we have the following definition.

Definition 1.9 *If the PDE (1.1) can be written in the form*

$$a(\mathbf{x})u+b_1(\mathbf{x})u_{x_1}+b_2(\mathbf{x})u_{x_2}+\cdots+b_d(\mathbf{x})u_{x_d}+c_1(\mathbf{x})u_{x_1x_1}+\cdots+c_2(\mathbf{x})u_{x_1x_2}+\cdots+c_{d^2}(\mathbf{x})u_{x_dx_d}+\cdots=f(\mathbf{x}), \tag{1.3}$$

i.e., if the coefficients of the unknown function u and of all its derivatives depend only *on the independent variables $\mathbf{x}=(x_1,x_2,\ldots,x_d)$, then it is called a* linear PDE. *If it is not possible to write (1.1) in the form (1.3), then it is called a* nonlinear PDE.

Example 1.10 *The PDEs in Examples 1.2, 1.3, 1.4, and 1.5 are linear PDEs.*

Indeed, the PDE in Example 1.2, can be written in the form (1.3) with $a(x)=0$, $b_1(x)=1$, $f(x)=0$ and all the other coefficients of the derivatives equal to zero.

Similarly, for Example 1.3, we have $f(x)=3x^3$, the coefficients of the second derivatives u_{xx}, u_{yy} and u_{zz} are equal to 1 *and all the other coefficients are zero.*

Also, for Example 1.4, $f(x)=0$, the coefficients of u_t, u_{xx} and u_{yy} are equal to 1 and all the other coefficients are zero.

Finally, for Example 1.5, $f(x)=0$, the coefficients of u_t, u_{xx}, u_x, and u depend only on the independent varibale x and do not *depend of u.*

Example 1.11 *The PDE in Example 1.6 a nonlinear PDE. This is clear, since the coefficient of u_{xx} is equal to u_{yy} (or to put it differently: the coefficient of u_{yy} is equal to u_{xx}) and the coefficient of u_{xy} is equal to u_{xy}, i.e., the coefficients of at least one of the partial derivatives contain u or its derivatives.*

Example 1.12 *The* inviscid Burgers' equation

$$u_t+uu_x=0,$$

for an unknown function $u=u(t,x)$ is a nonlinear PDE.

The family of nonlinear PDEs can be further subdivided into smaller families of PDEs. In particular we have the following definition.

Definition 1.13 *Consider a nonlinear PDE of order k with unknown solution u.*

- *If the coefficients of the k order partial derivatives of u are functions of the independent variables* $\mathbf{x}=(x_1,x_2,\ldots,x_d)$ only*, then this is called a* semilinear PDE.
- *If the coefficients of the k order partial derivatives of u are functions of the independent variables* $\mathbf{x}=(x_1,x_2,\ldots,x_d)$ *and/or of partial derivatives of u of order at most $k-1$ (including u itself) , then this is called a* quasilinear PDE.
- *If a (nonlinear) PDE is not quasilinear, then it is called* fully nonlinear.

Clearly a semilinear PDE is also a quasilinear PDE.

Example 1.14 *We give some examples of nonlinear PDEs along with their classifications.*

- *The* reaction-diffusion equation
$$u_t=u_{xx}+u^2,$$
is a semilinear PDE.
- *The* inviscid Burgers' equation
$$u_t+uu_x=0,$$
is a quasilinear PDE and it is NOT a semilinear PDE.

- *The* Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0,$$

is a semilinear PDE.

- *The* Monge-Ampère equation

$$u_{xx}u_{yy} - (u_{xy})^2 = 0,$$

is a fully nonlinear PDE.

The above classification of PDEs into linear, semilinear, quasilinear, and fully nonlinear is, roughly speaking, a classification of "increasing difficulty" in terms of studying and solving PDEs. Indeed, the mathematical theory of linear PDEs is now well understood. On the other hand, less is known about semilinear PDEs and quasilinear PDEs, and even less about fully nonlinear PDEs.