2.2.1 Separation of variables

For simplicity of the presentation, let $\Omega = [0, L] \subset \mathbb{R}$. We seek the (unique) solution $u : [0, T] \times \Omega \to \mathbb{R}$ to the initial/boundary-value problem

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) \quad \text{in } (0,T] \times [0,L], \\ u(0,x) &= f(x), \quad \text{for} \quad 0 \le x \le L \\ u(t,0) = u(t,L) &= 0, \quad \text{for} \quad 0 < t \le T, \end{aligned} \tag{2.13}$$

where $f : [0, L] \to \mathbb{R}$ is a known function.

We begin by making the crucial assumption that the solution u of the problem (2.13) is of the form

$$u(t,x) = T(t)X(x),$$

for some twice differentiable functions of one variable T and X. (Indeed, if we find *one* solution to the problem (2.13), it has to be necessarily the only solution, due to the uniqueness of the solution property described above.) Then we have

$$u_t = T'(t)X(x) \quad \text{and} \quad u_{xx} = T(t)X''(x).$$

Inserting this into the PDE $u_t = u_{xx}$, we arrive to

$$T'(t)X(x) = T(t)X''(x), \quad \text{or} \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}, \tag{2.14}$$

after division by $T(t)X(x)$, which we can assume to be non-zero without loss of generality (for otherwise, the solution u is identically equal to zero which means that we found the solution if also $f = 0$, or that this is impossible if $f \neq 0$). We notice that the left-hand side of (2.14) depends only on the independent variable t and the right-hand side depends only on the independent variable x. Since t and x are independent variables, the only possibility for the relation (2.14) to hold is for both the left- and the right-hand sides to be constant, say equal to $\lambda \in \mathbb{R}$. From this we get the ordinary differential equations

$$T'(t) - \lambda T(t) = 0, \quad \text{and} \quad X''(x) - \lambda X(x) = 0.$$

Now from the the boundary conditions we get

$$T(t)X(0) = T(t)X(L) = 0 \quad \text{giving} \quad X(0) = X(L) = 0, \quad \text{and} \quad T(0)X(x) = f(x) \text{ for } x \in [0, L].$$

Now we separate 3 cases: whether λ is positive, negative or zero.

The case $\lambda > 0$:

If $\lambda > 0$, then the two-point boundary value problem

$$X''(x) - \lambda X(x) = 0, \; 0 < x < L, \quad \text{and} \quad X(0) = X(L) = 0,$$

has solution of the form

$$X(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x),$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(L) = 0$. We have

$$0 = X(0) = A\cosh(0) = A, \quad \text{and} \quad 0 = X(L) = B\sinh(\sqrt{\lambda}a),$$

which implies that also $B = 0$. This means that if $\lambda > 0$, we get $X(x) = 0$ and thus, the only solution is the trivial solution $u(t, x) = 0$, which is not acceptable as $u(0, x) \neq 0$, in general.

The case $\lambda = 0$:

If $\lambda = 0$, then the two-point boundary value problem becomes

$$X''(x) = 0, \; 0 < x < L, \quad \text{and} \quad X(0) = X(L) = 0;$$

it has solution of the form

$$X(x) = Ax + B,$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(L) = 0$. We have then

$$0 = X(0) = B, \quad \text{and} \quad 0 = X(L) = AL,$$

implying also that $A = 0$. Hence if $\lambda = 0$ we again arrive to the trivial solution $u = 0$ which is not acceptable.

The case $\lambda < 0$:

If $\lambda < 0$, then there exists $\kappa \in \mathbb{R}$ such that $\lambda = -\kappa^2$. The two-point boundary value problem

$$X''(x) + \kappa^2 X(x) = 0, \text{ for } 0 < x < L, \quad \text{and} \quad X(0) = X(L) = 0,$$

has solution of the form

$$X(x) = A\cos(\kappa x) + B\sin(\kappa x),$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(L) = 0$. We have

$$0 = X(0) = A\cos(0) = A, \quad \text{and} \quad 0 = X(L) = B\sin(\kappa L);$$

this implies $\sin(\kappa L) = 0$, which means $\kappa L = n\pi$ for any $n = 1, 2, \ldots$ integers. From this we find

$$\kappa = \frac{n\pi}{L},$$

and, thus, we obtain the solutions

$$X(x) = B_n \sin\big(\frac{n\pi x}{L}\big),$$

for all $n = 1, 2, \ldots$ integers and $B_n \in \mathbb{R}$. We now turn our attention to T, which satisfies the first order ODE:

$$T'(t) - \kappa^2 T(t) = 0, \text{ for } 0 < t < T, .$$

The solution of the ODE is of the form

$$T(t) = C\mathrm{e}^{-\kappa^2 t},$$

for some constant $C \in \mathbb{R}$; giving the family of solutions

$$T(t) = C_n \mathrm{e}^{-n^2\pi^2 t/L^2},$$

for all $n = 1, 2, \ldots$ and $C_n \in \mathbb{R}$. Hence, setting $D_n = B_n C_n$, we deduce that all the functions of the form

$$u_n(t, x) := D_n \mathrm{e}^{-n^2\pi^2 t/L^2} \sin\big(\frac{n\pi x}{L}\big),$$

are solutions to the problem (2.13). Since the heat equation is linear, it is not hard to see that if two functions are solutions to the heat equation, then any linear combination of these functions is a solution to the heat equation also. Hence, we can formally write the solution of the problem (2.13)

$$u(t, x) = \sum_{n=1}^{\infty} D_n \mathrm{e}^{-n^2\pi^2 t/L^2} \sin\big(\frac{n\pi x}{L}\big) \tag{2.15}$$

(at this point the above equality is only formal, as we do *not* know if the above series converges). Notice that the D_n's are still not determined; this is to be expected as we have not yet made use of the initial condition $u(0, x) = T(0)X(x) = f(x)$ which, in view of (2.15) can be written as

$$f(x) = u(0, x) = \sum_{n=1}^{\infty} D_n \sin\big(\frac{n\pi x}{L}\big). \tag{2.16}$$

Expanding the function $f(x)$ into a sine series, we have

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\big(\frac{n\pi x}{L}\big), \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\big(\frac{n\pi x}{L}\big)\mathrm{d}x,$$

as shown in (2.10). Hence, setting

$$D_n = a_n = \frac{2}{L}\int_0^L f(x)\sin\big(\frac{n\pi x}{L}\big)\mathrm{d}x,$$

we finally conclude that the solution to the problem (2.13) is given by

$$u(t, x) = \sum_{n=1}^{\infty} \Big(\frac{2}{L}\int_0^L f(x)\sin\big(\frac{n\pi x}{L}\big)\mathrm{d}x\Big)\, \mathrm{e}^{-n^2\pi^2 t/L^2} \sin\big(\frac{n\pi x}{L}\big).$$

Example 2.7 *We want to solve the problem (2.13), where $L = 1$ and $f : [0, 1] \to \mathbb{R}$ is defined by*

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le \frac{1}{2}; \\ 1 - x, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

To calculate the solution (2.15), we compute the Fourier coefficients:

$$\frac{2}{L}\int_0^L f(x)\sin\big(\frac{n\pi x}{L}\big)\mathrm{d}x = \cdots = \frac{4\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2},$$

(this was done in Example 2.6). The solution then is given by

$$u(t, x) = \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}\, \mathrm{e}^{-n^2\pi^2 t/L^2} \sin(n\pi x).$$

2.3 The wave equation and the initial/boundary value problem

The *wave equation* is a paradigm of equations of hyperbolic type as we saw in the previous chapter. As the name suggest, the solutions to the wave equation model wave propagation. In most applications, hyperbolic equations describe transport/propagation phenomena; in the following, t will be denoting the "time"-variable, ranging between time 0 and some final time $T > 0$. For (unknown) solution $u : [0, T] \times \Omega \to \mathbb{R}$, the wave equation reads

$$u_{tt} = \Delta u, \quad \text{for} \quad (x_1, x_2, \ldots, x_d) \in \Omega \subset \mathbb{R}^d, \quad \text{and} \quad t \in [0, T], \tag{2.17}$$

where Δ is the Laplace operator in d dimensions; in particular, in one space dimension the hear equation reads:

$$u_{tt} = u_{xx}, \quad \text{for} \quad x \in \Omega \subset \mathbb{R}, \quad \text{and} \quad t \in [0, T]. \tag{2.18}$$

As we saw in Chapter 1, the wave equation is of hyperbolic type. Therefore, it admits two families of characteristic curves. We also saw that, for the corresponding Cauchy problem to be well-defined, we should require two Cauchy initial conditions. Also, the PDE for each fixed time $t \in [0, T]$ takes the form of the Poisson problem. Hence, again at leat heuristically, we can see that Dirichlet and/or Neumann type boundary condition(s) are required on the boundary of Ω, for each time t, for the problem to be well posed.

Next, we shall be concerned with finding the solution to the initial/boundary value problem (**??**), using the separation of variables.

2.3.1 Separation of variables

For simplicity of the presentation, let $\Omega = [0, L] \subset \mathbb{R}$. We seek the (unique) solution $u : [0, T \times \Omega \to \mathbb{R}$ to the initial/boundary-value problem

$$\begin{aligned} u_{tt}(t, x) &= u_{xx}(t, x) \quad \text{in } (0, T] \times [0, L], \\ u(0, x) &= f(x), \quad \text{for} \quad 0 \le x \le L \\ u_t(0, x) &= g(x), \quad \text{for} \quad 0 \le x \le L \\ u(t, 0) = u(t, L) &= 0, \quad \text{for} \quad 0 < t \le T, \end{aligned} \tag{2.19}$$

where $f, g : [0, L] \to \mathbb{R}$ are known functions.

We begin by making the crucial assumption that the solution u of the problem (2.13) is of the form

$$u(t, x) = T(t)X(x),$$

for some twice differentiable functions of one variable T and X. (Indeed, if we find *one* solution to the problem (2.19), it has to be necessarily the only solution, due to the uniqueness of the solution property described above.) Then we have

$$u_{tt} = T''(t)X(x) \quad \text{and} \quad u_{xx} = T(t)X''(x).$$

Inserting this into the PDE $u_{tt} = u_{xx}$, we arrive to

$$T''(t)X(x) = T(t)X''(x), \quad \text{or} \quad \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}, \tag{2.20}$$

after division by $T(t)X(x)$, which we can assume to be non-zero without loss of generality (for otherwise, the solution u is identically equal to zero which means that we found the solution if also $f = 0$, or that this is impossible if $f \neq 0$). We notice that the left-hand side of (2.20) depends only on the independent variable t and the right-hand side depends only on the independent variable x; thus, the only possibility for the relation (2.20) to hold is for both the left- and the right-hand sides to be constant, say equal to $\lambda \in \mathbb{R}$. From this we get the ordinary differential equations

$$T''(t) - \lambda T(t) = 0, \quad \text{and} \quad X''(x) - \lambda X(x) = 0;$$

from the the boundary conditions we get

$$T(t)X(0) = T(t)X(L) = 0 \quad \text{giving} \quad X(0) = X(L) = 0.$$

Now we separate 3 cases: whether λ is positive, negative or zero. Completely analogously to the discussion in Section 2.2.1, we conclude that λ positive or zero yield only the trivial solution, which may not be admissible due to the non-zero initial conditions. So we are left with the case $\lambda < 0$, for which we set $\lambda = -\kappa^2$ for some $\kappa \in \mathbb{R}$. As in the case of the parabolic problem, the two-point boundary value problem

$$X''(x) + \kappa^2 X(x) = 0, \text{ for } 0 < x < L, \quad \text{and} \quad X(0) = X(L) = 0,$$

has solutions

$$X(x) = B_n \sin\big(\frac{n\pi x}{L}\big),$$

with $\kappa = \kappa_n = \frac{n\pi}{L}$, for all $n = 1, 2, \ldots$ integers and $B_n \in \mathbb{R}$. We now turn our attention to T, which satisfies the second order ODE:

$$T''(t) + \kappa_n^2 T(t) = 0, \text{ for } 0 < t < T,$$

which has the family of solutions

$$T(t) = C_n \cos\big(\frac{n\pi t}{L}\big) + D_n \sin\big(\frac{n\pi t}{L}\big),$$

for all $n = 1, 2, \ldots$ and $C_n \in \mathbb{R}$. Hence, setting $E_n = B_n C_n$ and $F_n = B_n D_n$, we deduce that all the functions of the form

$$u_n(t,x) := \Big(E_n \cos\big(\frac{n\pi t}{L}\big) + F_n \sin\big(\frac{n\pi t}{L}\big)\Big) \sin\big(\frac{n\pi x}{L}\big),$$

are solutions to the problem (2.19). The wave equation is linear; thus the principle of superposition yields that the solution of the problem (2.19) is (at least formally) of the form

$$u(t,x) = \sum_{n=1}^{\infty} \Big(E_n \cos\big(\frac{n\pi t}{L}\big) + F_n \sin\big(\frac{n\pi t}{L}\big)\Big) \sin\big(\frac{n\pi x}{L}\big) \tag{2.21}$$

(at this point the above equality is only formal, as we do *not* know if the above series converges). Notice that the E_n's and the F_n's are still not determined; this is to be expected as we have not yet made use of the initial conditions $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$. In view of (2.21), the initial condition $u(0,x) = f(x)$ can be written as

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} E_n \sin\big(\frac{n\pi x}{L}\big). \tag{2.22}$$

To apply the second initial condition $u_t(0,x) = g(x)$, we first calculate u_t from (2.21):

$$u_t(t,x) = \sum_{n=1}^{\infty} \Big(-E_n \frac{n\pi}{L} \sin\big(\frac{n\pi t}{L}\big) + F_n \frac{n\pi}{L} \cos\big(\frac{n\pi t}{L}\big)\Big) \sin\big(\frac{n\pi x}{L}\big);$$

and we then set $t = 0$, to deduce

$$g(x) = u_t(0,x) = \sum_{n=1}^{\infty} F_n \frac{n\pi}{L} \sin\big(\frac{n\pi x}{L}\big). \tag{2.23}$$

Expanding the functions f and g into sine series, we get

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\big(\frac{n\pi x}{L}\big), \quad \text{where} \quad a_n = \frac{2}{L}\int_0^L f(x) \sin\big(\frac{n\pi x}{L}\big)\mathrm{d}x,$$

$$g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{where} \quad c_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

as shown in (2.10). Hence, setting $E_n = a_n$ and $F_n \frac{n\pi}{L} = c_n$, or $F_n = \frac{Lc_n}{n\pi}$ we finally conclude that the solution to the problem (2.19) is given by (2.21).

Example 2.8 *We want to solve the problem (2.13), where $L = 1$ and $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are defined by*

$$f(x) = \sin(2\pi x), \quad \textit{and} \quad g(x) = 0.$$

To calculate the solution (2.21), we compute the Fourier coefficients:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 2\int_0^1 \sin(2\pi x)\sin(n\pi x)\mathrm{d}x = \begin{cases} 1, & \textit{if } n = 2; \\ 0, & \textit{othewise.} \end{cases},$$

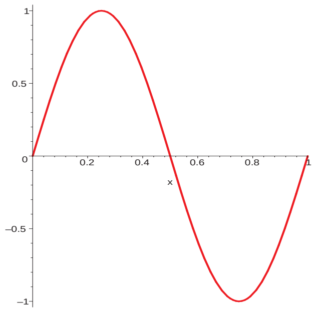
(the last equality is left as an exercise), and

$$c_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0,$$

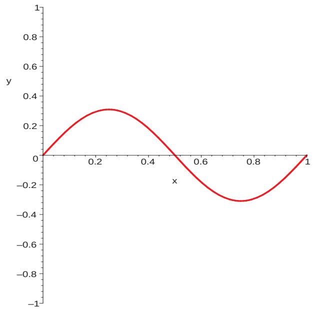
giving $E_2 = 2$, $E_n = 0$ for $n \neq 2$, and $F_n = 0$, for $n = 1, 2, \ldots$. The solution is then given by

$$u(t,x) = \cos(2\pi t)\sin(2\pi x),$$

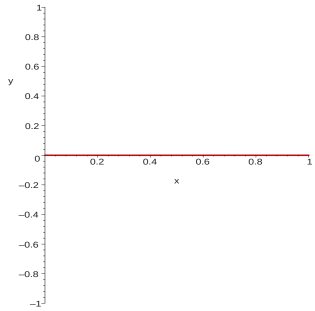
which is drawn in Figure 2.6.



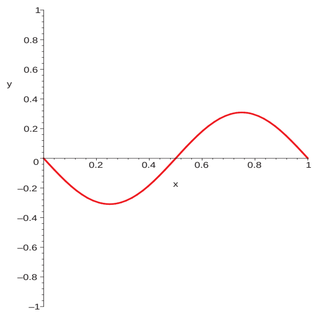
(a) Solution for $t = 0$



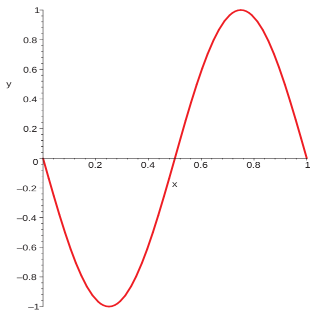
(b) Solution for $t = 0.2$



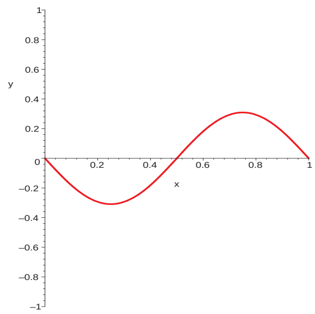
(c) Solution for $t = 0.25$



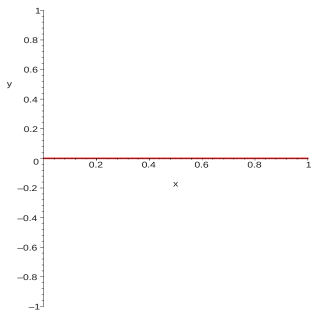
(d) Solution for $t = 0.3$



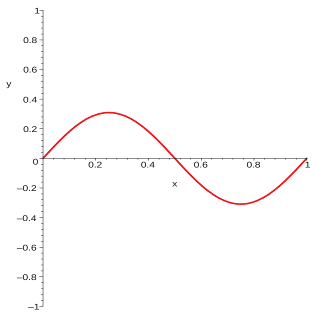
(e) Solution for $t = 0.5$



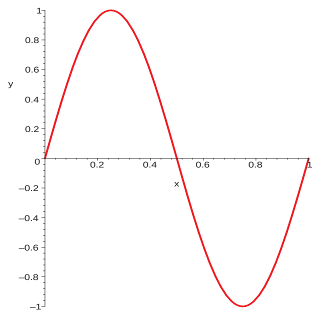
(f) Solution for $t = 0.7$



(g) Solution for $t = 0.75$



(h) Solution for $t = 0.8$



(i) Solution for $t = 1$

Figure 2.6: Example 2.8. The solution $u(t, x)$ for various t.