1.4 First order linear PDEs

We begin our study of linear PDEs with the case of first order linear PDEs. To simplify the discussion ,we shall only consider equations in 2 dimensions, i.e., for $d = 2$; the case of three or more dimensions can be treated in a completely analogous fashion. We begin with an example.

Example 1.15 *We consider the PDE in $\mathbb{R}^2$:*

$$u_x + u_y = 0. \tag{1.4}$$

To find its general solution, we perform the following transformation of coordinates (also known as change of variables *in Calculus): we consider new variables $(\xi, \eta) \in \mathbb{R}^2$ defined by the transformation of coordinates*

$$(x, y) \to (\xi, \eta) \quad , \textit{where } \xi(x, y) = x + y \quad \textit{and} \quad \eta(x, y) = y - x.$$

We can also calculate the inverse transformation of coordinates

$$(\xi, \eta) \to (x, y),$$

by solving with respect to x and y, obtaining

$$x = \frac{1}{2}(\xi - \eta) \quad \textit{and} \quad y = \frac{1}{2}(\xi + \eta). \tag{1.5}$$

We write the PDE (1.4) in the new coordinates, using the chain rule from Calculus. Setting $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ we have, respectively:

$$u_x = v_\xi \xi_x + v_\eta \eta_x, \qquad u_y = v_\xi \xi_y + v_\eta \eta_y,$$

giving

$$u_x = v_\xi - v_\eta, \qquad u_y = v_\xi + v_\eta.$$

Putting these back to the PDE (1.4), we deduce

$$0 = u_x + u_y = v_\xi - v_\eta + v_\xi + v_\eta = 2v_\xi \quad \textit{or} \quad v_\xi = 0. \tag{1.6}$$

Integrating this equation with respect to ξ, we arrive to

$$v(\xi, \eta) = f(\eta),$$

for any differentiable function of one variable $f : \mathbb{R} \to \mathbb{R}$. Using now the inverse transformation of coordinates (1.5), we conclude that the general solution of the PDE (1.4) is given by:

$$u(x, y) = v(\xi(x, y), \eta(x, y)) = f(\eta(x, y)) = f(y - x).$$

We discuss some big ideas that are present in the previous example. The change of variables $(x, y) \to (\xi, \eta)$ is essentially a clockwise rotation of the axes by an angle $\frac{\pi}{4}$ (there is also stretching of size $\sqrt{2}$ taking place with this change of variables, but this is not really relevant to our discussion). Once the rotation is done, the PDE takes the simpler form (1.6), which can be interpreted geometrically as: v is constant with respect to the variable ξ, or in other words, the solution u is constant when $x + y = c$, for any constant $c \in \mathbb{R}$. This means that the solution remains constant as we move along straight lines of the form $y = -x + c$. Hence, if the value of the solution u at one point (x_0, y_0), say, on the plane is known, then the value of u along the straight line of slope $-\frac{\pi}{4}$ that passes through (x_0, y_0) is also known (i.e., it is the same value)! In other words, the straight lines of the form $y = -x + c$ "characterise" the solution of the PDE above; such curves are called *characteristic curves* of a PDE, as we shall see below.

Next, we shall incorporate these ideas into the case of the general first order linear PDE. The general form of a 1st order linear PDE in 2 dimensions can be written as:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = g(x, y), \quad \text{for} \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{1.7}$$

where a, b, c, g are functions of the independent variables x and y only. We also assume that a, b have continuous first partial derivatives, and that they do *not* vanish simultaneously at any point of the domain of definition Ω. Finally ,we assume that the solution u of the PDE (1.7) has continuous first partial derivatives.

Consider a transformation of coordinates of $\mathbb{R}^2$:

$$(x, y) \leftrightarrow (\xi, \eta),$$

with $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, which is assumed to be smooth (that is, the functions $\xi(x, y)$ and $\eta(x, y)$ have all derivatives with respect to x and y well-defined) and non-singular, i.e., its Jacobian

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} := \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0, \tag{1.8}$$

in Ω (this requirement ensures that the change of variables is meaningful, in the sense that it is one-to-one and onto). We also denote by $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ the inverse transformation, as it will be useful below.

We write the PDE (1.7) in the new coordinates, using the chain rule. Setting $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ we have, respectively:

$$u_x = v_\xi\xi_x + v_\eta\eta_x, \qquad u_y = v_\xi\xi_y + v_\eta\eta_y,$$

giving

$$(a\xi_x + b\xi_y)v_\xi + (a\eta_x + b\eta_y)v_\eta + cv = g(x(\xi, \eta), y(\xi, \eta)), \tag{1.9}$$

after substitution into (1.7). To simplify the above equation, we require that the function $\eta(x, y)$ is such that

$$a\eta_x + b\eta_y = 0; \tag{1.10}$$

if this is the case then (1.9) becomes an ordinary differential equation with respect to the independent variable ξ, whose solution can be found by standard separation of variables.

The equation (1.10) is a slightly simpler PDE of first order than the original PDE. To find the required η we are seek to construct curves such that $\eta(x, y) = const$ for any constant; these are called the *characteristic curves* of the PDE (compare this with the straight lines of the example above).

Differentiating this equation with respect to x, we get

$$0 = \frac{\mathrm{d}\, const}{\mathrm{d}x} = \frac{\mathrm{d}\eta(x,y)}{\mathrm{d}x} = \eta_x\frac{\mathrm{d}x}{\mathrm{d}x} + \eta_y\frac{\mathrm{d}y}{\mathrm{d}x} = \eta_x + \eta_y\frac{\mathrm{d}y}{\mathrm{d}x},$$

where in the penultimate equality we made use of the chain rule for functions of two variables; the above equality yields

$$\frac{\eta_x}{\eta_y} = -\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{1.11}$$

assuming, without loss of generality, that $\eta_y \neq 0$ (for otherwise, we argue as above with the rôles of the x and y variables interchanged, and we get necessarily $\eta_x \neq 0$ from hypothesis (1.8)).

Using (1.11) on (1.10), we deduce the *characteristic equation*:

$$-a\frac{\mathrm{d}y}{\mathrm{d}x} + b = 0, \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}, \tag{1.12}$$

assuming, without loss of generality that $a \neq 0$ near the point (x_0, y_0) (for otherwise, we have that necessarily $b \neq 0$ near the point (x_0, y_0), as a, b cannot vanish simultaneously at any point due to hypothesis, and we can apply the same argument as above with x and y interchanged). Equation (1.12) is an ordinary differential equation of first order that can be solved using standard separation of variables to give a solution $f(x, y) = const$, say. Setting $\eta = f(x, y)$ and ξ to be any function for which (1.8) holds, we can easily see that (1.10) holds also. Therefore, the PDE (1.7) can be written as

$$(a\xi_x + b\xi_y)v_\xi + cv = g(x(\xi, \eta), y(\xi, \eta)), \quad \text{or} \quad v_\xi + \frac{c}{(a\xi_x + b\xi_y)}v = \frac{g(x(\xi, \eta), y(\xi, \eta))}{(a\xi_x + b\xi_y)},$$

which is an ordinary differential equation of first order with respect to ξ and can be solved using the (standard) method of multipliers[1] to find $v(\xi, \eta)$. Using the inverse transformation of coordinates, we can now find the

[1]We recall the method of multipliers to solve this ordinary differential equation (ODE): we write the ODE in the form

$$v_\xi + Cv = G,$$

by dividing by $(a\xi_x + b\xi_y)$, and we multiply both sides by $\mathrm{e}^{\int C(s)\mathrm{d}s}$, which gives

$$\mathrm{e}^{\int^\xi C(s)\mathrm{d}s}v_\xi + C\mathrm{e}^{\int^\xi C(s)\mathrm{d}s}v = G\mathrm{e}^{\int^\xi C(s)\mathrm{d}s}, \quad \text{or} \quad (\mathrm{e}^{\int^\xi C(s)\mathrm{d}s}v)_\xi = G\mathrm{e}^{\int^\xi C(s)\mathrm{d}s}, \quad \text{or} \quad v = \mathrm{e}^{-\int^\xi C(s)\mathrm{d}s}\int^\xi G(\tau)\mathrm{e}^{\int^\tau C(s)\mathrm{d}s}\mathrm{d}\tau.$$

solution $u(x, y)$ from $v(\xi, \eta)$. This is the so-called *method of characteristics* in finding the solution to a first order PDE.

Example 1.16 *We use the method of characteristics described above to find the general solution to the PDE*

$$yu_x - xu_y + yu = xy.$$

We have $a = y$, $b = -x$, $c = y$ *and* $g = xy$. *To find the characteristic curves of this PDE, we solve the ordinary differential equation (1.12), which in this case becomes*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \quad \textit{or} \quad \int y\mathrm{d}y = -\int x\mathrm{d}x, \quad \textit{or} \quad \frac{y^2}{2} = -\frac{x^2}{2} + \textit{const}, \quad \textit{for } xy \neq 0,$$

using separation of variables. We set

$$\eta(x, y) := \frac{x^2 + y^2}{2};$$

then we have $\eta(x, y) = \textit{const}$ *as required by the method described above, i.e., the characteristic curves of this PDE are concentric circles centred at the origin. If we also set* $\xi(x, y) := x$, *say, we have*

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x\eta_y - \xi_y\eta_x = 1 \times y - 0 \times x = y \neq 0,$$

when $xy \neq 0$. *Hence the transformation of coordinates* $(x, y) \leftrightarrow (\xi, \eta)$ *is non-singular and smooth. The inverse transformation is given by*

$$x(\xi, \eta) = \xi \quad \textit{and} \quad y(\xi, \eta) = \begin{cases} \sqrt{\eta - \frac{1}{2}\xi^2}, & \textit{if } y \geq 0; \\ -\sqrt{\eta - \frac{1}{2}\xi^2}, & \textit{if } y < 0. \end{cases}$$

giving the transformed PDE

$$v_\xi + \frac{y(\xi, \eta)}{y(\xi, \eta) \times 1 + (-x(\xi, \eta)) \times 0} v = \frac{x(\xi, \eta)y(\xi, \eta)}{y(\xi, \eta) \times 1 + (-x(\xi, \eta)) \times 0}, \quad \textit{or} \quad v_\xi + v = x(\xi, \eta) = \xi.$$

Multiplying the last equation with the multiplier $\mathrm{e}^{\int^\xi 1\mathrm{d}s} = \mathrm{e}^\xi$, *we deduce*

$$\left(\mathrm{e}^\xi v\right)_\xi = \xi\mathrm{e}^\xi, \quad \textit{or} \quad v = \mathrm{e}^{-\xi}\int^\xi \tau\mathrm{e}^\tau\mathrm{d}\tau = \cdots = \xi - 1 + f(\eta),$$

for any differentiable function f *of one variable. Hence the general solution is given by*

$$u(x, y) = v(\xi(x, y), \eta(x, y)) = \xi(x, y) - 1 + f(\eta(x, y)) = x - 1 + f\big(\frac{x^2 + y^2}{2}\big).$$