1.5 Second order linear PDEs

An important class of PDEs are the linear PDEs of 2nd order, which we shall be concerned in this section. For simplicity, we shall consider only equations in 2 dimensions, i.e., for $d = 2$. The general form of a 2nd order linear PDE in 2 dimensions can be written as:

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \quad \text{for} \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{1.13}$$

where a, b, c, d, e, f, g are functions of the independent variables x and y only. We also assume that a, b, c have continuous second partial derivatives, and that they do *not* vanish simultaneously at any point of the domain of definition Ω. Finally ,we assume that the solution u of the PDE (1.13) has continuous second partial derivatives. We shall classify PDEs of the form (1.13) in different types, depending on the sign of the *discriminant* defined by

$$\mathcal{D} := b^2 - ac,$$

at each point $(x_0, y_0) \in \Omega$. More specifically, we have the following definition.

Definition 1.17 *Let $\mathcal{D} = b^2 - ac$ be the discriminant of a second order PDE of the form (1.13) in $\Omega \subset \mathbb{R}^2$ and let a point $(x_0, y_0) \in \Omega$.*

- *If $\mathcal{D} > 0$ at the point (x_0, y_0), the PDE is said to be* hyperbolic *at (x_0, y_0).*
- *If $\mathcal{D} = 0$ at the point (x_0, y_0), the PDE is said to be* parabolic *at (x_0, y_0).*
- *If $\mathcal{D} < 0$ at the point (x_0, y_0), the PDE is said to be* elliptic *at (x_0, y_0).*

The equation is said to be hyperbolic, parabolic or elliptic in the domain Ω if it is, respectively, hyperbolic, parabolic or elliptic at all points of Ω.

We give some examples.

Example 1.18 *The so-called* wave equation

$$u_{tt} - u_{xx} = 0,$$

is hyperbolic in $\mathbb{R}^2$. Indeed, for this equation we have $a = 1$, $c = -1$, and $b = 0$, giving $\mathcal{D} = -1 < 0$ for all $(x, y) \in \mathbb{R}^2$.

Example 1.19 *The so-called* heat equation

$$u_t - u_{xx} = 0,$$

is parabolic in $\mathbb{R}^2$. Indeed, for this equation we have $c = -1$ and $a = b = 0$, giving $\mathcal{D} = 0$ for all $(x, y) \in \mathbb{R}^2$.

Example 1.20 *The so-called* Laplace equation

$$u_{xx} + u_{yy} = 0,$$

is elliptic in $\mathbb{R}^2$. Indeed, for this equation we have $a = c = 1$ and $b = 0$, giving $\mathcal{D} = -1 < 0$ for all $(x, y) \in \mathbb{R}^2$.

Example 1.21 *The equation*

$$u_{xx} + x^2 u_{yy} = 0,$$

is elliptic in the set $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ and parabolic in the set $\{(x, y) \in \mathbb{R}^2 : x = 0\}$. Indeed, for this equation we have $a = 1$, $c = x^2$ and $b = 0$, giving $\mathcal{D} = -x^2 < 0$ for all $(x, y) \in \mathbb{R}^2$ such that $x \neq 0$ and $\mathcal{D} = 0$ when $x = 0$. (This equation sometimes referred to in the literature as Grušin equation.)

Example 1.22 *The* Tricomi equation

$$yu_{xx} + u_{yy} = 0,$$

is elliptic in the set $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, parabolic in the set $\{(x, y) \in \mathbb{R}^2 : y = 0\}$, and hyperbolic in the set $\{(x, y) \in \mathbb{R}^2 : y < 0\}$. Indeed, for this equation we have $a = y$, $c = 1$ and $b = 0$, giving $\mathcal{D} = -y < 0$ for all $(x, y) \in \mathbb{R}^2$ such that $y > 0$, $\mathcal{D} = 0$ when $y = 0$ and $\mathcal{D} = -y > 0$ when $y < 0$.

The above classification of 2nd order linear PDEs can be very useful when studying the properties of such equations. For instance, the next result shows that the type of a 2nd order linear PDE at a point (i.e., if the PDE is hyperbolic, parabolic or elliptic), remains unchanged if we make a smooth non-singular transformation of coordinates (also known as *change of variables*) on $\mathbb{R}^2$. In particular, we have the following theorem.

Theorem 1.23 *The sign of the discriminant $\mathcal{D}$ of a second order PDE of the form (1.13) in $\Omega \subset \mathbb{R}^2$ is invariant under smooth non-singular transformations of coordinates.*

Proof. Consider a transformation of coordinates of $\mathbb{R}^2$:

$$(x, y) \leftrightarrow (\xi, \eta),$$

with $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, which is assumed to be smooth (that is, the functions $\xi(x, y)$ and $\eta(x, y)$ have all derivatives with respect to x and y well-defined) and non-singular, i.e., its Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} := \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0, \tag{1.14}$$

in Ω. We also denote by $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ the inverse transformation, as it will be useful below.

We write the PDE (1.13) in the new coordinates, using the chain rule. Setting $v(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$ we have, respectively:

$$u_x = v_\xi \xi_x + v_\eta \eta_x, \qquad u_y = v_\xi \xi_y + v_\eta \eta_y, \tag{1.15}$$

giving

$$\begin{aligned} u_{xx} &= v_{\xi\xi}\xi_x^2 + 2v_{\xi\eta}\xi_x\eta_x + v_{\eta\eta}\eta_x^2 + v_\xi \xi_{xx} + v_\eta \eta_{xx}, \\ u_{yy} &= v_{\xi\xi}\xi_y^2 + 2v_{\xi\eta}\xi_y\eta_y + v_{\eta\eta}\eta_y^2 + v_\xi \xi_{yy} + v_\eta \eta_{yy}, \\ u_{xy} &= v_{\xi\xi}\xi_x\xi_y + v_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + v_{\eta\eta}\eta_x\eta_y + v_\xi \xi_{xy} + v_\eta \eta_{xy}. \end{aligned} \tag{1.16}$$

Inserting (1.15) and (1.16) into (1.13), and factorising accordingly, we arrive to

$$Av_{\xi\xi} + 2Bv_{\xi\eta} + Cv_{\eta\eta} + Dv_\xi + Ev_\eta + fv = g, \tag{1.17}$$

where the new coefficients A, B, C, D and E are given by

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\ B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\ C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2, \\ D &= a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y, \\ E &= a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y. \end{aligned} \tag{1.18}$$

Thus the discriminant of the PDE in new variables (1.17), is given by

$$\begin{aligned} B^2 - AC &= (a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y)^2 - (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2) \\ &= \cdots = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2 = (b^2 - ac)\Big(\frac{\partial(\xi, \eta)}{\partial(x, y)}\Big)^2. \end{aligned} \tag{1.19}$$

This means that the discriminant $B^2 - AC$ of (1.17) has always the same sign as the discriminant $b^2 - ac$ of (1.13), as $\frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$ from the hypothesis and, therefore, $\big(\frac{\partial(\xi,\eta)}{\partial(x,y)}\big)^2 > 0$. Since the discriminant of the transformed PDE has always the same sign as the one of the original PDE, the type of the PDE remains invariant. □

Recalling now the methods of characteristics for the solution of first order linear PDEs presented in the previous section, we can see the relevance of the above theorem: it assures us that applying a change of variables will not alter the type of the PDE.

Let us now consider some special transformations for PDEs of each type. What we shall see is that, given certain transformation, it is possible to write (1.13) locally in much simpler form, the so-called *canonical form*. We begin with an example.

Example 1.24 *Consider the wave equation*

$$u_{xx} - y_{yy} = 0,$$

which as we saw before is hyperbolic in $\mathbb{R}^2$*. Let us also consider the transformation of coordinates of* $\mathbb{R}^2$*:*

$$(x, y) \leftrightarrow (\xi, \eta), \quad \textit{with} \quad \xi = x + y \quad \textit{and} \quad \eta = x - y.$$

It is, of course, smooth as $x + y$ *and* $x - y$ *are infinite times differentiable with respect to* x *and* y*, and it is non-singular, as*

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x \eta_y - \xi_y \eta_x = 1 \times (-1) - 1 \times 1 = -2 \neq 0, \quad \textit{for all} \quad (x, y) \in \mathbb{R}^2.$$

To calculate the transformed equation, we can use the formulas (1.18), with $a = 1$*,* $b = 0$*,* $c = -1$*,* $d = e = f = g = 0$*,* $\xi_x = 1$*,* $\xi_y = 1$*,* $\eta_x = 1$ *and* $\eta_y = -1$ *to calculate* $A = 1 - 0 - 1 = 0$*,* $B = 1 - 0 + 1 = 2$*,* $C = 1 - 0 - 1 = 0$*,* $D = E = 0$ *and, thus, the transformed equation is given by*

$$4v_{\xi\eta} = 0, \quad \textit{or} \quad v_{\xi\eta} = 0.$$

For this canonical form, we can in fact compute the general solution of the wave equation. Indeed, integrating with respect to η*, we arrive to*

$$v_\xi = h(\xi),$$

for an arbitrary continuously differentiable function h*. Integrating now the last equality with respect to* ξ*, we deduce*

$$v = \int^{\xi} h(s)\mathrm{d}s + G(\eta).$$

If we set $F(\xi) := \int^{\xi} h(s)\mathrm{d}s$*, to simplify the notation, we get*

$$v(\xi, \eta) = F(\xi) + G(\eta),$$

for an arbitrary twice continuously differentiable function G*, or equivalently*

$$u(x, y) = F(x + y) + G(x - y),$$

for all twice continuously differentiable functions F *and* G *of one variable.*

In the previous example, we found a general solution for the wave equation using a particular transformation of coordinates. Note that the general solution of the wave equation involves *unknown functions*! (Recall that the general solution of an ordinary differential equation involves unknown constants; this might help to draw an analogy. In the next sections we shall study some appropriate initial conditions and boundary conditions that will be sufficient to specify the unknown functions and arrive to unique solutions.)