We now investigate the following question: is it always possible to find transformations of coordinates that make the general PDE (1.13) "simpler"? In Example (1.24) we saw that for the case of the wave equation it is indeed possible to reduce the wave equation in the simpler PDE $v_{\xi\eta} = 0$.

For the general PDE, we employ a geometric argument. We seek functions $\xi(x, y)$ and $\eta(x, y)$ for which we have

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \quad \text{and} \quad a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0; \tag{1.20}$$

i.e., $A = C = 0$ for the coefficients of the transformed PDE (1.17). The equations (1.20) are PDEs of first order, for which we are now seeking to construct curves such that $\xi(x, y) = const$ for any constant. When (x, y) are points on a curve, i.e, they are such that $\xi(x, y) = const$, they are dependent. Hence, differentiating this equation with respect to x, we get

$$0 = \frac{\mathrm{d}\, const}{\mathrm{d}x} = \frac{\mathrm{d}\xi(x, y)}{\mathrm{d}x} = \xi_x \frac{\mathrm{d}x}{\mathrm{d}x} + \xi_y \frac{\mathrm{d}y}{\mathrm{d}x} = \xi_x + \xi_y \frac{\mathrm{d}y}{\mathrm{d}x},$$

where in the penultimate equality we made use of the chain rule for functions of two variables; the above equality yields

$$\frac{\xi_x}{\xi_y} = -\frac{\mathrm{d}y}{\mathrm{d}x}, \tag{1.21}$$

assuming, without loss of generality, that $\xi_y \neq 0$ (for otherwise, we argue as above with the rôles of the x and y variables interchanged, and we get necessarily $\xi_x \neq 0$ from hypothesis (1.14)). Now, we go back to the desired equations (1.20), and we divide the first equation by ξ_y^2 to obtain

$$a\Big(\frac{\xi_x}{\xi_y}\Big)^2 + 2b\frac{\xi_x}{\xi_y} + c = 0,$$

and, using (1.21), we arrive to

$$a\Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2 - 2b\frac{\mathrm{d}y}{\mathrm{d}x} + c = 0, \tag{1.22}$$

which is called the *characteristic equation* for the PDE (1.13). This is a quadratic equation for $\frac{\mathrm{d}y}{\mathrm{d}x}$, with discriminant $\mathcal{D} = b^2 - ac$! The roots of the characteristic equation are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{\mathcal{D}}}{a}. \tag{1.23}$$

Each of the equations above is a first order ordinary differential equation that can be solved using standard separation of variables to give (families of) solutions $f_1(x, y) = const$ and $f_2(x, y) = const$, say. The curves defined by the equations $f_1(x, y) = const$ and $f_2(x, y) = const$ are called the *characteristic curves* of the second order PDE.

Therefore, if the original PDE (1.13) is hyperbolic, i.e., if $\mathcal{D} > 0$, the characteristic equation has two real distinct roots, giving two real distinct characteristics curves for the PDE. If the original PDE (1.13) is parabolic, thereby $\mathcal{D} = 0$, the characteristic equation has one double root, giving one real characteristic curve for the PDE. Finally, if the original PDE (1.13) is elliptic, thereby $\mathcal{D} < 0$, the characteristic equation has no real roots, and therefore the PDE has **no** real characteristic curves, but as we shall see below it has complex characteristic curves. The characteristic curves can be thought as the "natural directions" in which the PDE "communicates information" to different points in its domain of definition Ω. With this statement in mind, it is possible to see that each type of PDE models different phenomena and also admits different properties, rendering the above classification into hyperbolic, parabolic and elliptic PDEs of great importance.

The case of hyperbolic PDE

Now we go back to the question of the possibility of simplification of the original PDE (1.13), assuming that (1.13) is hyperbolic. Therefore, as we have seen above it will have two real distinct characteristics for which the equation (1.22), and thus (1.20) holds too (by observing that all the steps followed above are in fact equivalences). This means that for every $(x_0, y_0) \in \Omega$ there exists a local transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ with $\xi = f_1(x, y)$ and $\eta = f_2(x, y)$ such that $A = C = 0$ in (1.17) (i.e., we can use one characteristic curve for each new variable, since both functions satisfy (1.22), and thus (1.20). Finally, we check if this transformation of coordinates has non-zero Jacobian:

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x\eta_y - \xi_y\eta_x = \xi_y\eta_y\Big(\frac{\xi_x}{\xi_y} - \frac{\eta_x}{\eta_y}\Big) == -\xi_y\eta_y\Big(\frac{b + \sqrt{\mathcal{D}}}{a} - \frac{b - \sqrt{\mathcal{D}}}{a}\Big) = -\xi_y\eta_y\frac{2\sqrt{\mathcal{D}}}{a} \neq 0,$$

whereby the penultimate equality follows from (1.21). Thus, we have essentially proven the following theorem.

Theorem 1.25 *Let (1.13) be a hyperbolic PDE. Then, for every* $(x_0, y_0) \in \Omega$ *there exists a transformation of coordinates* $(x, y) \leftrightarrow (\xi, \eta)$ *in the neighbourhood of* (x_0, y_0)*, such that (1.13) can be written as*

$$v_{\xi\eta} + \cdots = g, \tag{1.24}$$

where "…" are used to signify the terms involving u*,* u_x*, or* u_y*. This is called the* canonical form *of a hyperbolic PDE.*

Proof. The proof essentially follows from the above discussion: since we are able to show that for every $(x_0, y_0) \in \Omega$ there exists a local transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ for which we have $A = C = 0$ in (1.17), then (1.17) becomes

$$2Bv_{\xi\eta} + Dv_\xi + Ev_\eta + fv = g.$$

Dividing now the above equation by $2B$ (which is not zero, as it can be seen from (1.19)), (1.24) follows. □

The above theorem shows that each second order linear hyperbolic PDE can be written in the (simpler) canonical form (1.24).

Example 1.26 *We shall calculate the characteristic curves of the wave equation (Example (1.18). In this case we have* $a = 1$*,* $b = 0$*, and* $c = -1$*. Thus the characteristic equation reads:*

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 1 = 0, \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \pm 1,$$

from which we get two solutions $y - x = C_1$ *and* $y + x = C_2$*, for* $C_1, C_2 \in \mathbb{R}$ *arbitrary constants. This yields the transformation of coordinates* $\xi = y - x$ *and* $\eta = y + x$*. Comparing this to (Example (1.18), we can see that we have arrived to the same transformation of coordinates!*

Example 1.27 *We shall calculate the characteristic curves and the canonical form of the Tricomi equation*

$$yu_{xx} + u_{yy} = 0$$

In this case we have $a = y$*,* $b = 0$*, and* $c = 1$*. As we saw in Example (1.22), this equation is hyperbolic for* $y < 0$*, parabolic for* $y = 0$ *and elliptic for* $y > 0$*. We first consider the case* $y < 0$*. Then the characteristic equation reads:*

$$y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 = 0, \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{-y}},$$

from which we get two solutions in implicit form[2] $\frac{2}{3}(-y)^{3/2} + x = C_1$ *and* $\frac{2}{3}(-y)^{3/2} - x = C_2$*, for* $C_1, C_2 \in \mathbb{R}$ *arbitrary constants. This yields the transformation of coordinates* $\xi = \frac{2}{3}(-y)^{3/2} + x$ *and* $\eta = \frac{2}{3}(-y)^{3/2} - x$*. Notice that for* $y = 0$ *(i.e., when the PDE is parabolic), we have* $\xi = \eta$*, i.e., the two characteristic curves meet, i.e., we only have one characteristic direction! Moreover,* ξ *and* η *are not well defined for* $y > 0$*, which is again consistent with the theory developed above, as when* $y > 0$ *the PDE is elliptic and, therefore, it has no real characteristic curves! (More details about the last two cases can be found in the discussion below.)*

Also, it is a simple (but worthwhile) exercise to verify that, with the above change of variables, the Tricomi equation can be written in the canonical form (1.24) when $y < 0$*.*

The case of parabolic PDE

We now assume that (1.13) is parabolic, i.e., $\mathcal{D} = b^2 - ac = 0$. Therefore, the equation (1.22) has one double root given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}. \tag{1.25}$$

which yields *one* family of characteristic curves, say $f_2(x, y) = \mathit{const}$ for which (1.25), and thus (1.20) holds. We set $\eta = f_2(x, y)$ as before. As far as ξ is concerned, we now have flexibility in its choice: the only

[2] We remind the reader how an ordinary differential equation is solved using separation of variables: we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{1}{\sqrt{-y}}, \quad \text{or} \quad \sqrt{-y}\mathrm{d}y = \pm\mathrm{d}x, \quad \text{or} \quad \int\sqrt{-y}\mathrm{d}y = \int\pm\mathrm{d}x, \quad \text{or} \quad -\frac{2}{3}(-y)^{3/2} = \pm x + \mathit{const}.$$

requirement is that the Jacobian of the resulting transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ is non-zero, i.e.,

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x \eta_y - \xi_y \eta_x \neq 0. \tag{1.26}$$

If the above are true, we have $C = 0$ (from the choice of ξ) in (1.17). Moreover, in this case we necessarily have that $B = 0$, too. This is because the PDE is parabolic, i.e., $\mathcal{D} = 0$, which from (1.19) and (1.26) implies that $B^2 - AC = 0$. But $C = 0$, giving finally $B = 0$ also. Thus, we have essentially proven the following theorem.

Theorem 1.28 *Let (1.13) be parabolic PDE. Then, for every $(x_0, y_0) \in \Omega$ there exists a transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ in the neighbourhood of (x_0, y_0), such that (1.13) can be written as*

$$v_{\xi\xi} + \cdots = g, \tag{1.27}$$

where "…" are used to signify the terms involving u, u_x, or u_y. This is called the canonical form *of a parablic PDE.*

Proof. The proof essentially follows from the above discussion: since we are able to show that for every $(x_0, y_0) \in \Omega$ there exists a local transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ for which we have $B = C = 0$ in (1.17), then (1.17) becomes

$$Av_{\xi\xi} + Dv_\xi + Ev_\eta + fv = g.$$

Dividing now the above equation by A (which is necessarily non-zero from assumption (1.26)), the result follows. □

The above theorem shows that each second order linear parabolic PDE can be written in the (simpler) canonical form (1.27).

The case of elliptic PDE

We now assume that (1.13) is elliptic, i.e., $\mathcal{D} < 0$. Therefore, the equation (1.22) has no real roots and, therefore, if (1.13) is elliptic then it has **no** real characteristic curves. Since complex variables (and the theory of analytic functions) are beyond the scope of these notes, we shall only state the main result for elliptic problems, without proof.

Theorem 1.29 *Let (1.13) be an elliptic PDE. Then, for every $(x_0, y_0) \in \Omega$ there exists a transformation of coordinates $(x, y) \leftrightarrow (\xi, \eta)$ in the neighbourhood of (x_0, y_0), such that (1.13) can be written as*

$$v_{\xi\xi} + v_{\eta\eta} + \cdots = g, \tag{1.28}$$

where "…" are used to signify the terms involving u, u_x, or u_y. This is called the canonical form *of an elliptic PDE.*

The above theorem shows that each second order linear parabolic PDE can be written in the (simpler) canonical form (1.28).

Remark 1.30 *Notice that the whole discussion in this section about linear second order PDEs will still be valid for the case of semilinear second order PDEs too! Indeed, since in second order semilinear PDEs the non-linearities are not present in the coefficients of the second order derivatives, the calculations and the theorems above will still be valid (as all the calculations above are done to control the coefficients of the second order derivatives).*