1.6 The Cauchy problem and well-posedness of PDEs

In the previous sections, we studied the method of characteristics for the solution of first and second order linear PDEs. We found that, normally, the general solutions of these PDEs contain unknown functions. We also gave some heuristic arguments on the importance of characteristic curves in describing the properties and the solution of PDEs. In particular, we mentioned that information "travels along characteristic curves", whenever these exist, i.e., the solution of the PDE has "preferred direction(s)" to relate its values from one point in space to another.

In the theory of ordinary differential equations, we have seen that the general solution of an ODE involves unknown constants, which can be determined when we equip the ODE with some "initial condition", e.g., the ODE

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = 3u(t),$$

has general solution given by $u(t) = A\mathrm{e}^{3t}$, for all constants $A \in \mathbb{R}$. If we add the requirement that the solution of the above ODE must satisfy also the initial condition

$$u(0) = 5,$$

we find that necessarily $A = 5$, giving the solution $u(t) = 5\mathrm{e}^{3t}$.

In this section, we shall study some appropriate corresponding conditions for PDEs, that will be sufficient to specify the unknown functions and arrive to unique solutions.

Definition 1.31 *Consider a PDE of the form (1.1), of order k in Ω in d dimensions and let S be a (given) smooth surface on $\mathbb{R}^d$. Let also $n = n(x)$ denote the unit normal vector to the surface S at a point $\mathbf{x} = (x_1, x_2, \dots, x_d) \in S$. Suppose that on any point $\mathbf{x}$ of the surface S the values of the solution u and of all its directional derivatives up to order $k-1$ in the direction of n are given, i.e., we are given functions $f_0, f_1, \dots, f_{k-1} : S \to \mathbb{R}$ such that*

$$u(\mathbf{x}) = f_0(\mathbf{x}), \quad \textit{and} \quad \frac{\partial u}{\partial n}(\mathbf{x}) = f_1(\mathbf{x}), \quad \textit{and} \quad \frac{\partial^2 u}{\partial n^2}(\mathbf{x}) = f_2(\mathbf{x})\, dots, \quad \textit{and} \quad \frac{\partial^{k-1} u}{\partial n^{k-1}}(\mathbf{x}) = f_{k-1}(\mathbf{x}). \quad (1.29)$$

The Cauchy problem *consists of finding the unknown function(s) u that satisfy simultaneously the PDE and the conditions (1.29). The conditions (1.29) are called the* initial conditions *and the given functions $f_0, f_1, \dots, f_{k-1}$, will be referred to as the* initial data.

The degenerate case of $d = 1$ and $k = 1$, i.e., the case of and ODE of first order with the corresponding initial condition is given above. We now consider some less trivial examples.

Example 1.32 *We want to find a solution to the Cauchy problem consisting of the PDE*

$$u_x + u_y = 0, \quad (1.30)$$

together with the initial condition

$$u(0, y) = \sin y.$$

(Here the surface S in Definition 1.31 is implicitly given by the initial condition: we have $S = \{(x, y) \in \mathbb{R}^2 : x = 0\}$, i.e., the surface S (which is now just a curve as we are in $\mathbb{R}^2$) is the y-axis on the Cartesian plane.)

In Example 1.15, we used the method of characteristics to deduce that the general solution to the PDE (1.30) is

$$u(x, y) = f(y - x), \quad \textit{for all } (x, y) \in \mathbb{R}^2.$$

If we set $x = 0$ we get, using the initial condition:

$$\sin y = u(0, y) = f(y).$$

Hence a solution to the Cauchy problem is given by

$$u(x, y) = \sin(y - x).$$

In Figure 1.1 we sketch S for this problem, along with some characteristic curves (which are of the form $y = x + c$). Notice that S intersects all *characteristic curves.*

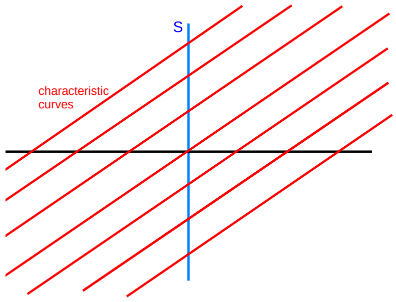

Figure 1.1: Example 1.32. Sketch of the Cauchy problem.

Example 1.33 *We want to find a solution to the Cauchy problem consisting of the wave equation*

$$u_{xx} - u_{yy} = 0, \tag{1.31}$$

together with the initial conditions

$$u(x,0) = \sin x, \quad \textit{and} \quad u_y(x,0) = 0.$$

(Again, here the surface S in Definition 1.31 is implicitly given by the initial condition: we have $S = \{(x,y) \in \mathbb{R}^2 : y = 0\}$, i.e., the surface S (which is now just a curve as we are in $\mathbb{R}^2$) is the x-axis on the Cartesian plane.) In Example 1.24, we used the method of characteristics to deduce that the general solution to the PDE (1.31) is

$$u(x,y) = F(x+y) + G(x-y), \quad \textit{for all } (x,y) \in \mathbb{R}^2,$$

for some functions F, G; thus, if we set $y = 0$ we get, using the first initial condition:

$$\sin x = u(x,0) = F(x) + G(x). \tag{1.32}$$

Differentiating the general solution with respect to y, we get

$$u_y(x,y) = F'(x+y)(x+y)_y + G'(x-y)(x-y)_y = F'(x+y) - G'(x-y);$$

setting $y = 0$ and using the first initial condition, we arrive to

$$0 = u_y(x,0) = F'(x) - G'(x), \quad \textit{or} \quad F(x) - G(x) = c, \tag{1.33}$$

for some constant $c \in \mathbb{R}$. Solving the system (1.32) and (1.33) with respect to $F(x)$ and $G(x)$ (two equations with two unknowns!), we get

$$F(x) = \frac{1}{2}(\sin x + c), \quad \textit{and} \quad G(x) = \frac{1}{2}(\sin x - c).$$

Now that we have specified F and G, we can write the solution to the above Cauchy problem

$$u(x,y) = F(x+y) + G(x-y) = \frac{1}{2}(\sin(x+y) + c) + \frac{1}{2}(\sin(x-y) - c) = \frac{1}{2}(\sin(x+y) + \sin(x-y)).$$

Notice that S intersects all *characteristic curves.*

One question that arises is whether the solutions to the Cauchy problems in the previous examples are unique. A partial answer to this question is given by the celebrated Cauchy-Kovalevskaya Theorem.

Theorem 1.34 (The Cauchy-Kovalevskaya Theorem) *Consider the Cauchy problem from Definition (1.31) for the case of a linear PDE of the form (1.3). Let $\mathbf{x}_0$ be a point of the initial surface S, which is assumed to be analytic*[3]*. Suppose that S is* not *a characteristic surface at the point $\mathbf{x}_0$. Assume that all the coefficients of the PDE (1.3), the right-hand side f, and all the initial data $f_0, f_1, \ldots, f_{k-1}$ are analytic functions on a neighbourhood of the point $\mathbf{x}_0$. Then the Cauchy problem has a solution u, defined in the neighbourhood of $\mathbf{x}_0$. Moreover, the solution u is analytic in a neighbourhood of $\mathbf{x}_0$ and it is unique in the class of analytic functions.*

The proof of the above Theorem is out of the scope of these notes; it can be found in any standard PDE theory textbook.

[3] An analytic surface which can be described by a function $g(\mathbf{x}) = const$ for g analytic function. An analytic function is a function that can be written as a (absolutely convergent) power series.