Therefore, according to the Cauchy-Kovalevskaya Theorem (under the analyticity assumptions), the Cauchy problem has a solution which is unique in the space of analytic functions. Showing existence and uniqueness of solutions to PDE problems (i.e., PDEs together with some initial or boundary conditions) is, undoubtedly, a task of paramount importance in the theory of PDEs. Indeed, once a PDE model is studied, it is extremely useful to know if that model has a solution (for otherwise, we are wasting our efforts trying to solve it). If it does have a solution, then it is very important to be able to show that the solution is also unique (for otherwise, the same PDE problem will produce many different solutions, and this is not usually natural in mathematical modelling).

Even if a PDE problem has a unique solution, this does not necessarily mean that the PDE problem is "well behaved". By well-behaved here we understand if the PDE problem changes "slightly" (e.g., by altering "slightly" some coefficient), then also its solution should change only "slightly" also. In other words, "well behaved" is to be understood as follows: "small" changes in the initial data or the PDE itself should *not* result to arbitrarily "large" changes in the behaviour of the solution to the PDE problem.

Definition 1.35 *A PDE problem is* well-posed *if the following 3 properties hold:*

- *the PDE problem has a solution*
- *the solution is unique*
- *the solution depends continuously on the PDE coefficients and the problem data.*

If a PDE problem is not *well-posed, then we say that it is* ill-posed*.*

The concept of well-posedness is due to Hadamard[4].

Example 1.36 *The Cauchy problem consisting of the wave equation*

$$u_{xx} - u_{yy} = 0,$$

together with the initial conditions

$$u(x,0) = f(x), \quad u_y(x,0) = 0,$$

for some known *initial datum f, is an example of a well posed problem. Indeed, working completely analogously to Example (1.33), we can see that a solution to the above problem is given by*

$$u(x,y) = \frac{1}{2}\big(f(x-y) + f(x+y)\big).$$

The proof of uniqueness of solution is more involved and will be omitted (it is based on the so-called energy property of the wave equation).

Finally, to show the continuity of the solution to the initial data, we consider also the Cauchy problem

$$\tilde{u}_{xx} - \tilde{u}_{yy} = 0, \quad \text{together with the initial conditions} \quad \tilde{u}(x,0) = \tilde{f}(x), \quad \tilde{u}_y(x,0) = 0,$$

i.e., we consider a different initial condition $\tilde{f}$ for the Cauchy problem, giving a new solution $\tilde{u}$. Working as above, we can immediately see that the solution to this Cauchy problem is given by

$$\tilde{u}(x,y) = \frac{1}{2}\big(\tilde{f}(x-y) + \tilde{f}(x+y)\big).$$

No, we look at the difference of the solutions of the two Cauchy problems above. We have

$$u(x,y)-\tilde{u}(x,y) = \frac{1}{2}\big(f(x-y)+f(x+y)\big)-\frac{1}{2}\big(\tilde{f}(x-y)+\tilde{f}(x+y)\big) = \frac{1}{2}\Big(\big(f(x-y)-\tilde{f}(x-y)\big)+\big(f(x+y)-\tilde{f}(x+y)\big)\Big).$$

Hence if the difference $f(z) - \tilde{f}(z)$ is small for all $z \in \mathbb{R}$, then the difference $u - \tilde{u}$ will also be small! That is the solution depends continuously on the PDE coefficients and the problem data.

We now consider an example of an ill-posed problem, which is also due to Hadamard.

[4] Jacques Salomon Hadamard (1865 - 1963), French mathematician

Example 1.37 *The Cauchy problem consisting of the Laplace equation*

$$u_{xx} + u_{yy} = 0, \quad \textit{for } -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \textit{and} \quad y > 0,$$

(i.e., $\Omega = (-\pi/2, \pi/2) \times (0, +\infty)$*), together with the initial conditions*

$$u(x,0) = 0, \quad u_y(x,0) = \mathrm{e}^{-\sqrt{n}} \cos(nx), \quad \textit{for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

for every $n = 1, 3, 5, \ldots,$ *and*

$$u(-\pi/2, y) = 0 = u(\pi/2, y), \quad \textit{for } y \geq 0.$$

As we shall see in Chapter 2, it is possible to calculate (using the method of separation of variables) that a solution to the above problem is given by

$$u(x,y) = \frac{\mathrm{e}^{-\sqrt{n}}}{n} \cos(nx) \sinh(ny).$$

(It is easy to check that this is a solution to the Cauchy problem just by differentiating back and verifying that it indeed satisfies the PDE and the initial conditions.)

Now we study what happens as we vary the odd number n *appearing in the initial conditions. We can see that*

$$|u_y(x,0)| = |\mathrm{e}^{-\sqrt{n}} \cos(nx)| \leq \mathrm{e}^{-\sqrt{n}},$$

i.e., as we increase n*, the initial condition* $u_y(x,0)$ *changes at an* exponentially small *manner. Also, recalling the definition of the hyperbolic sine*[5]*, we have*

$$u(x,y) = \frac{\mathrm{e}^{-\sqrt{n}}}{n} \cos(nx) \sinh(ny) = \frac{\mathrm{e}^{-\sqrt{n}+ny} - \mathrm{e}^{-\sqrt{n}-ny}}{2n} \cos(nx).$$

Notice that when $y \neq 0$*, the exponent of the first exponential is positive and thus, a change in* n *results to an* exponentially large *change in* $u(x,y)$*. Hence, a small change in the initial data (realised when changing the constant* n*), result to exponentially large change in the solution* u *for* $y \neq 0$*! Hence the problem is ill-posed.*

In Chapter 2, we shall consider appropriate conditions for each type of linear second order equations (elliptic, parabolic, hyperbolic), that result to well-posed problems.

[5] We recall that the hyperbolic sine and the hyperbolic cosine are defined as

$$\sinh x := \frac{1}{2}(\mathrm{e}^{x} - \mathrm{e}^{-x}), \quad \text{and} \quad \cosh x := \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x})$$

Chapter 2

Problems of Mathematical Physics

In this chapter we shall be concerned with the classical equations of mathematical physics, together with appropriate initial (and boundary) conditions.

2.1 The Laplace equation

We begin the discussion with Laplace equation:

$$\Delta u = 0, \quad \text{for} \quad (x_1, x_2, \ldots, x_d) \in \Omega \subset \mathbb{R}^d, \tag{2.1}$$

where $\Delta := (\cdot)_{x_1 x_1} + (\cdot)_{x_2 x_2} + \cdots + (\cdot)_{x_d x_d}$ denotes the so-called *Laplace operator* in d dimensions; in particular, in two dimensions Laplace equation reads:

$$\Delta u = u_{xx} + u_{yy} = 0, \quad \text{for} \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{2.2}$$

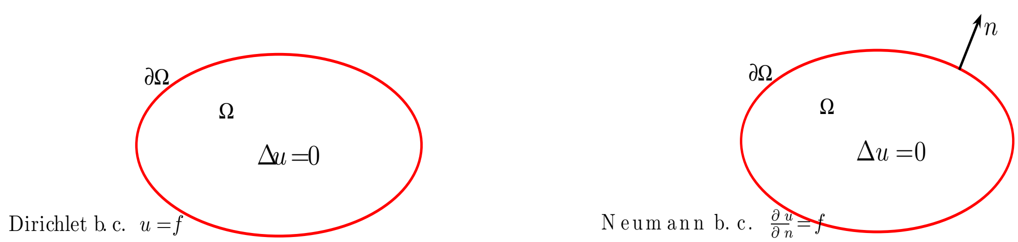
where $\Delta := (\cdot)_{xx} + (\cdot)_{yy}$. The non-homogeneous version of the Laplace equation, namely

$$\Delta u = f \quad \text{in } \Omega \tag{2.3}$$

for some known function $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, is known as the *Poisson equation*.

Laplace and Poisson equations model predominately phenomena that do *not* evolve in time, typically properties of materials (elasticity, electric or gravitational charge), probability densities of random variables, etc.

As we saw in Chapter 1, Laplace (and therefore, Poisson) equation is of elliptic type. in fact, Laplace equation is the archetypical equation of elliptic type (see also Theorem 1.29 for the canonical form of PDEs of elliptic type).



(a) Dirichlet boundary value problem. (b) Neumann boundary value problem.

Figure 2.1: Dirichlet and Neumann boundary value problems.

For the problem to be well posed, we equip the Laplace equation with conditions along the whole of the boundary $\partial\Omega$ of the domain Ω. We shall call these *boundary conditions*[1]. We shall consider two types of

[1]In the previous chapter, we talked about the Cauchy problem consisting of a PDE, together with initial conditions. The term "initial conditions" is used for PDEs that model evolution phenomena (i.e., PDEs for which one variable is "time"), for which the Cauchy problem is well posed. For elliptic PDEs, however, which model phenomena that do *not* evolve in time, it is conventional to use the term "boundary conditions" instead.

boundary conditions, namely the *Dirichlet boundary condition*:

$$u(x,y) = f(x,y), \quad \text{for}(x,y) \in \partial\Omega,$$

where $f : \partial\Omega \to \mathbb{R}$ is a known function, and the *Neumann boundary condition*:

$$\frac{\partial u}{\partial n}(x,y) = f(x,y), \quad \text{for}(x,y) \in \partial\Omega,$$

where $\frac{\partial u}{\partial n}(x,y)$ is the directional derivative of u in the direction of the unit outward normal vector n at the point (x,y) of the boundary $\partial\Omega$. We shall refer to the Laplace equation together with the Dirichlet boundary condition as the *Dirichlet boundary value problem* and to the Laplace equation together with the Neumann boundary condition as the *Neumann boundary value problem* (see Figure 2.1 for an illustration).

Next, we shall be concerned with finding the solution to the Laplace equation with the above boundary conditions.