2.1.1 Separation of variables

For simplicity of the presentation, let $\Omega = [0,a] \times [0,b] \subset \mathbb{R}^2$ be the rectangular region with vertices the points $(0,0)$, $(a,0)$, (a,b), and $(0,b)$. We seek the (unique) solution $u : \Omega \to \mathbb{R}$ to the Laplace boundary-value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, && (2.4)\\ u(0,y) = u(a,y) = u(x,0) &= 0, \quad \text{for} \quad 0 \le x \le a,\ 0 \le y \le b, && (2.5)\\ u(x,b) &= f(x), \quad \text{for} \quad 0 \le x \le a, && (2.6) \end{aligned}$$

where $f : [0,a] \to \mathbb{R}$ is a known function.

We begin by making the crucial assumption that the solution u of the problem (2.4) is of the form

$$u(x,y) = X(x)Y(y),$$

for some twice differentiable functions of one variable X and Y. (Indeed, if we find *one* solution to the problem (2.4), it has to be necessarily the only solution, due to the uniqueness of the solution property described above.) Then we have

$$u_{xx} = X''(x)Y(y) \quad \text{and} \quad u_{yy} = X(x)Y''(y).$$

Inserting this into the PDE $\Delta u = 0$, we arrive to

$$X''(x)Y(y) + X(x)Y''(y) = 0, \quad \text{or} \quad \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0,$$

after division by $X(x)Y(y)$, which we can assume to be non-zero without loss of generality (for otherwise, the solution u is identically equal to zero which means that we found the solution if also $f = 0$, or that this is impossible if $f \neq 0$). This gives

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}, \qquad (2.7)$$

that is, the left-hand side depends only on the independent variable x and the right-hand side depends only on the independent variable y. Since x and y are independent variables, the only possibility for the relation (2.7) to hold is for both the left- and the right-hand sides to be constant, say equal to $\lambda \in \mathbb{R}$. From this we get the ordinary differential equations

$$X''(x) - \lambda X(x) = 0, \quad \text{and} \quad Y''(y) + \lambda Y(y) = 0.$$

Now from the the boundary conditions we get

$$X(0)Y(y) = X(a)Y(y) = 0 \quad \text{giving} \quad X(0) = X(a) = 0, \quad \text{and} \quad X(x)Y(0) = 0 \quad \text{giving} \quad Y(0) = 0.$$

Now we separate 3 cases: whether λ is positive, negative or zero.

The case $\lambda > 0$:

If $\lambda > 0$, then the two-point boundary value problem

$$X''(x) - \lambda X(x) = 0,\ 0 < x < a, \quad \text{and} \quad X(0) = X(a) = 0,$$

has solution of the form

$$X(x) = A\cosh(\sqrt{\lambda}x) + B\sinh(\sqrt{\lambda}x),$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(a) = 0$. We have

$$0 = X(0) = A\cosh(0) = A, \quad \text{and} \quad 0 = X(a) = B\sinh(\sqrt{\lambda}a),$$

which implies that also $B = 0$. This means that if $\lambda > 0$, we get $X(x) = 0$ and thus, the only solution is the trivial solution $u(x,y) = 0$, which is not acceptable as $u(x,y) \neq 0$ on the top boundary.

The case $\lambda = 0$:

If $\lambda = 0$, then the two-point boundary value problem becomes

$$X''(x) = 0, \ 0 < x < a, \quad \text{and} \quad X(0) = X(a) = 0;$$

it has solution of the form

$$X(x) = Ax + B,$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(a) = 0$. We have then

$$0 = X(0) = B, \quad \text{and} \quad 0 = X(a) = Aa,$$

implying also that $A = 0$. Hence if $\lambda = 0$ we again arrive to the trivial solution $u = 0$ which is not acceptable.

The case $\lambda < 0$:

If $\lambda < 0$, then there exists $\kappa \in \mathbb{R}$ such that $\lambda = -\kappa^2$. The two-point boundary value problem

$$X''(x) + \kappa^2 X(x) = 0, \ \text{for } 0 < x < a, \quad \text{and} \quad X(0) = X(a) = 0,$$

has solution of the form

$$X(x) = A\cos(\kappa x) + B\sin(\kappa x),$$

for some constants $A, B \in \mathbb{R}$, which can be determined using the boundary conditions $X(0) = X(a) = 0$. We have

$$0 = X(0) = A\cos(0) = A, \quad \text{and} \quad 0 = X(a) = B\sin(\kappa a);$$

this implies $\sin(\kappa a) = 0$, which means $\kappa a = n\pi$ for any $n = 1, 2, \dots$ integers. From this we find

$$\kappa = \frac{n\pi}{a},$$

and, thus, we obtain the solutions

$$X(x) = B_n \sin\big(\frac{n\pi x}{a}\big),$$

for all $n = 1, 2, \dots$ integers and $B_n \in \mathbb{R}$. We now turn our attention to Y, which satisfies the two-point boundary value problem

$$Y''(y) - \kappa^2 Y(y) = 0, \ \text{for } 0 < y < b, \quad \text{and} \quad Y(0) = 0, \ X(x)Y(b) = f(x).$$

As before, the solution of the ODE is of the form

$$Y(y) = C\cosh(\kappa y) + D\sinh(\kappa y),$$

for some constants $C, D \in \mathbb{R}$, which can be determined using the boundary conditions. From the "left" boundary condition, we get

$$0 = Y(0) = C\cosh(0) = C,$$

giving the family of solutions

$$Y(y) = D_n \sinh(\kappa y) = \sinh\big(\frac{n\pi y}{a}\big),$$

for all $n = 1, 2, \dots$ integers and $D_n \in \mathbb{R}$. Hence all the functions of the form

$$u_n(x, y) := E_n \sin\big(\frac{n\pi x}{a}\big)\sinh\big(\frac{n\pi y}{a}\big),$$

setting $E_n = B_n D_n$ for all $n = 1, 2, \dots,$ are solutions to the Laplace problem (2.4). Since the Laplace equation is linear, it is not hard to see that if two functions are solutions to Laplace equation, then any linear combination of these functions is a solution to the Laplace equation also[2]. Hence, we can formally write that the solution of the Laplace problem (2.4) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin\big(\frac{n\pi x}{a}\big)\sinh\big(\frac{n\pi y}{a}\big) \tag{2.8}$$

[2]This is the so-called *Principle of Superposition*, whereby the if two functions are solutions to a linear homogeneous PDE then their linear combination is also a solution.

(at this point the above equality is only formal, as we do *not* know if the above series converges). Notice that the E_n's are still not determined; this is to be expected as we have not yet made use of the remaining boundary condition $u(x,b) = X(x)Y(b) = f(x)$ which, in view of (2.8) can be written as

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = \sum_{n=1}^{\infty} \tilde{E}_n \sin\left(\frac{n\pi x}{a}\right), \tag{2.9}$$

where we have set $\tilde{E}_n := E_n \sinh\left(\frac{n\pi b}{a}\right)$.

To conclude the solution, we need to determine all $\tilde{E}_n$'s such that (2.9) is satisfied. Describing how to do that is the content of the next section.