2.1.2 Fourier series

In applied mathematics, it is often of great interest to be able to describe functions in terms of simpler functions. For instance, the classical theory of Power Series is concerned with representing functions as infinite sums of simpler polynomial functions, e.g., the exponential function is known to satisfy

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

i.e., the function $f(x) = \mathrm{e}^x$ can be written as an infinite sum of multiples of powers of x! Such representations can be very useful in various applications, such as in the solution of ordinary differential equations.

In this section, we investigate the question of describing functions in terms of simpler trigonometric functions; these are the celebrated *Fourier series expansions*[3]. In physical terms they can be interpreted as analysing a function into simple sine and cosine functions (waves) of different frequencies. This is a very general and powerful idea; indeed, Fourier series are at the heart of many applications such as signal processing and medical imaging.

Definition 2.1 *For $L > 0$ constant, let $f[-L, L] \to \mathbb{R}$ be a function with at most finite number of discontinuities in the interval $[-L, L]$. The* Fourier series expansion *associated with f is defined as*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where a_n and b_n, $n = 1, 2, \ldots$, are called the Fourier coefficients *and given by the formulas*

$$a_n := \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad and \quad b_n := \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

We investigate the above definition with an example.

Example 2.2 *We are seeking the Fourier series expansion of the function $f : [-1, 1] \to \mathbb{R}$, with $f(x) = |x|$. We have $L = 1$, and we calculate the Fourier coefficients:*

$$
\begin{aligned}
a_n &= \int_{-1}^{1} |x| \cos(n\pi x) \mathrm{d}x = \int_{-1}^{0} (-x) \cos(n\pi x) \mathrm{d}x + \int_{0}^{1} x \cos(n\pi x) \mathrm{d}x \\
&= \left[-x \frac{\sin(n\pi x)}{n\pi} \right]_{-1}^{0} + \int_{-1}^{0} \frac{\sin(n\pi x)}{n\pi} \mathrm{d}x + \left[x \frac{\sin(n\pi x)}{n\pi} \right]_{0}^{1} - \int_{0}^{1} \frac{\sin(n\pi x)}{n\pi} \mathrm{d}x \\
&= 0 + \left[-\frac{\cos(n\pi x)}{(n\pi)^2} \right]_{-1}^{0} + 0 - \left[-\frac{\cos(n\pi x)}{(n\pi)^2} \right]_{0}^{1} = -\frac{1 - (-1)^n}{(n\pi)^2} + \frac{(-1)^n - 1}{(n\pi)^2} = 2\frac{(-1)^n - 1}{(n\pi)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
b_n &= \int_{-1}^{1} |x| \sin(n\pi x) \mathrm{d}x = \int_{-1}^{0} (-x) \sin(n\pi x) \mathrm{d}x + \int_{0}^{1} x \sin(n\pi x) \mathrm{d}x \\
&= \left[x \frac{\cos(n\pi x)}{n\pi} \right]_{-1}^{0} - \int_{-1}^{0} \frac{\cos(n\pi x)}{n\pi} \mathrm{d}x + \left[-x \frac{\cos(n\pi x)}{n\pi} \right]_{0}^{1} + \int_{0}^{1} \frac{\cos(n\pi x)}{n\pi} \mathrm{d}x \\
&= \frac{(-1)^n}{n\pi} - \left[\frac{\sin(n\pi x)}{(n\pi)^2} \right]_{-1}^{0} - \frac{(-1)^n}{n\pi} + \left[\frac{\sin(n\pi x)}{(n\pi)^2} \right]_{0}^{1} = 0,
\end{aligned}
$$

for $n = 1, 2, \ldots$, and

$$a_0 = \int_{-1}^{1} |x| \mathrm{d}x = \int_{-1}^{0} (-x) \mathrm{d}x + \int_{0}^{1} x \mathrm{d}x = 1.$$

Hence the Fourier series expansion of $f(x) = |x|$, $x \in [-1, 1]$ reads

$$\frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n - 1}{(n\pi)^2} \cos(n\pi x).$$

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[3]The name refers to the great French mathematician and physicist Jean Baptiste Joseph Fourier (1768 - 1830) who first proposed solving PDEs using trigonometric series expansions.

To see how does the Fourier expansion compares to the original function, we consider the partial sums

$$\frac{1}{2} + \sum_{n=1}^{k} 2\frac{(-1)^n - 1}{(n\pi)^2} \cos(n\pi x);$$

then, for $k = 1$, we get

$$\frac{1}{2} - \frac{4}{\pi^2} \cos(\pi x),$$

for $k = 3$, we get

$$\frac{1}{2} - \frac{4}{\pi^2} \cos(\pi x) - \frac{4}{9\pi^2} \cos(3\pi x),$$

and so on (notice that the terms for n even are zero). In Figure 2.2, we plot the function $f(x) = |x|$ and the partial sums of its Fourier expansion for $k = 1$, $k = 3$ and $k = 7$.

(a) Graph of $f(x) = |x|$

(b) Fourier expansion for $k = 1$

(c) Fourier expansion for $k = 3$

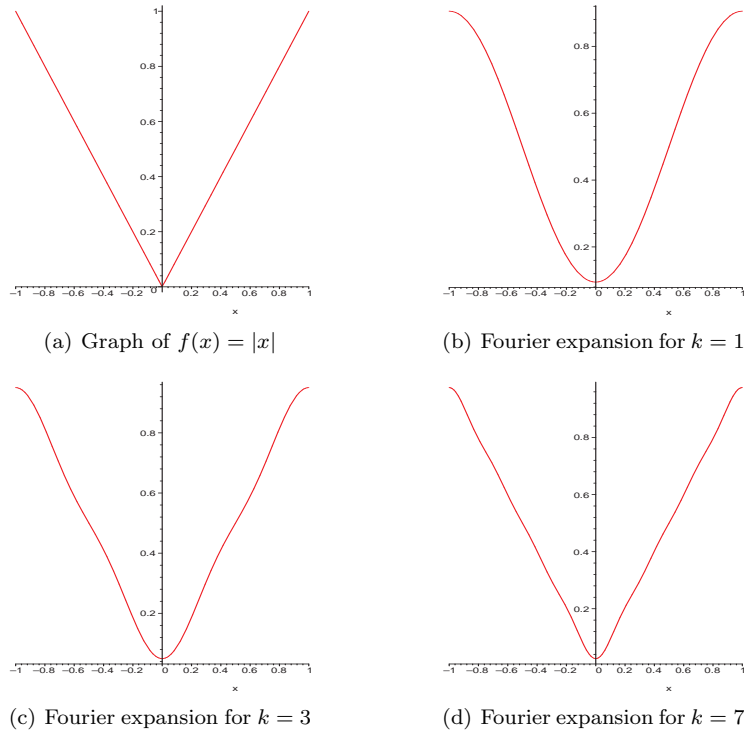(d) Fourier expansion for $k = 7$

Figure 2.2: Fourier synthesis...

The first question that may spring to mind is: does it hold

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \ ?$$

In the previous example, this seemed to be the case, as we take more and more terms in the series. The following theorem gives an answer to the above question.

Theorem 2.3 *Suppose that a function $f : [-L, L] \to \mathbb{R}$ and its derivative f' are bounded and continuous everywhere in $[-L, L]$ apart from a finite number of points. Then, at every point $x \in (-L, L)$ for which f is continuous, we have*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

at every point $x \in (-L, L)$ that f has a jump discontinuity, we have

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where $f(x_+)$ and $f(x_-)$ denote the limit values of f from the right and from the left of x, respectively [4]; finally, at the endpoints $x = -L$ and $x = L$, we have

$$\frac{1}{2}\big(f((-L)_+) + f(L_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n \cos\big(\frac{n\pi x}{L}\big) + b_n \sin\big(\frac{n\pi x}{L}\big)\Big).$$

The proof of this theorem is quite involved and somewhat outside the scope of these notes, so it is omitted.

Remark 2.4 *Note that when f is continuous at a point $x \in (0, L)$, then we have $f(x_+) = f(x_-) = f(x)$, and thus $\frac{1}{2}\big(f(x_+) + f(x_-)\big) = f(x)$, i.e., we go back to the first part of the theorem!*

Example 2.2 revisited. *We investigate the above theorem in conjunction with Example 2.2. The function $f(x) = |x|$ in Example 2.2 is continuous in $(-1, 1)$ and, therefore, Theorem 2.3 implies*

$$|x| = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x), \quad \text{for all} \quad x \in (-1, 1).$$

Moreover, at the endpoints $x = \pm 1$ of the interval $[-1, 1]$, we calculate $f((-1)_+) = 1$ and $f(1_-) = 1$. Hence, Theorem 2.3 implies

$$1 = \frac{1}{2}(f((-1)_+) + f(1_-)) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi(\pm 1)) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{1 - (-1)^n}{(n\pi)^2};$$

notice that in this case we have $f((-1)_+) = 1 = f(1_-)$.

Example 2.5 *We are seeking the Fourier series expansion of the function $f : [0, 2] \to \mathbb{R}$, with*

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1; \\ 0, & \text{if } 1 < x \le 2. \end{cases}.$$

We notice that the domain of the function f is not of the form $[-L, L]$ for some $L > 0$ and, therefore the theory above does not apply as is. One way of overcoming this difficulty is to extend the definition of the function f from $[0, 2]$ to $[-2, 2]$. Of course, there are infinite possible extensions of a function and we shall be seeking extensions that are "convenient" and "simple". Below, we consider two possible extensions.

*(a) **Extension to an even function.** We extend f to the even function[5] $\tilde{f} : [-2, 2] \to \mathbb{R}$, with*

$$\tilde{f}(x) = \begin{cases} 1, & \text{if } -1 \le x \le 1; \\ 0, & \text{if } -2 \le x < -1 \text{ or } 1 < x \le 2. \end{cases}.$$

We have $L = 2$, and we calculate the Fourier coefficients:

$$a_n = \frac{1}{2}\int_{-1}^{1}\cos\big(\frac{n\pi x}{2}\big)\mathrm{d}x = \Big[\frac{\sin\big(\frac{n\pi x}{2}\big)}{n\pi}\Big]_{-1}^{1} = \frac{\sin\big(\frac{n\pi}{2}\big)}{n\pi} - \frac{\sin\big(-\frac{n\pi}{2}\big)}{n\pi} = 2\frac{\sin\big(\frac{n\pi}{2}\big)}{n\pi},$$

and

$$b_n = \frac{1}{2}\int_{-1}^{1}\sin\big(\frac{n\pi x}{2}\big)\mathrm{d}x = \Big[-\frac{\cos\big(\frac{n\pi x}{2}\big)}{n\pi}\Big]_{-1}^{1} = -\frac{\cos\big(\frac{n\pi}{2}\big)}{n\pi} + \frac{\cos\big(-\frac{n\pi}{2}\big)}{n\pi} = 0,$$

for $n = 1, 2, \ldots$, and

$$a_0 = \frac{1}{2}\int_{-1}^{1}\mathrm{d}x = 1.$$

Hence the Fourier series expansion of $\tilde{f}$, $x \in [-2, 2]$ reads

$$\frac{1}{2} + \sum_{n=1}^{\infty} 2\frac{\sin\big(\frac{n\pi}{2}\big)}{n\pi}\cos\big(\frac{n\pi x}{2}\big).$$

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[4]We recall that the limit values of f from the right and from the left of x are, respectively, defined as

$$f(x_+) := \lim_{\substack{h \to 0 \\ h > 0}} f(x + h), \quad \text{and} \quad f(x_-) := \lim_{\substack{h \to 0 \\ h > 0}} f(x - h).$$

[5]We recall that an even function is one for which $f(-x) = f(x)$, i.e., it admits the y-axis as axis of symmetry.

To see how does the Fourier expansion compares to the original function, we consider the partial sums

$$\frac{1}{2} + \sum_{n=1}^{k} 2\frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi} \cos\left(\frac{n\pi x}{2}\right).$$

In Figure 2.3, we plot the function $\tilde{f}$ and the partial sums of its Fourier expansion for $k = 1$, $k = 3$ and $k = 9$ and $k = 39$.
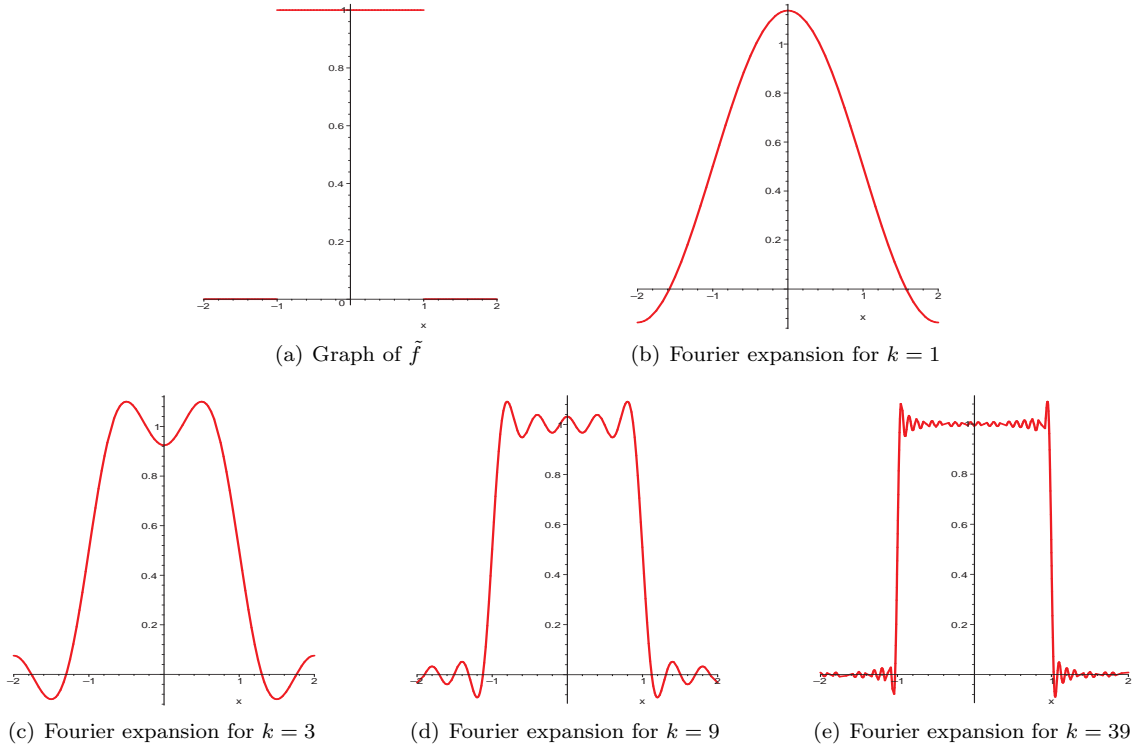
(a) Graph of $\tilde{f}$

(b) Fourier expansion for $k = 1$

(c) Fourier expansion for $k = 3$

(d) Fourier expansion for $k = 9$

(e) Fourier expansion for $k = 39$

Figure 2.3: Fourier synthesis...

(b) Extension to an odd function. *We extend f to the odd function[6] $\hat{f} : [-2, 2] \to \mathbb{R}$, with*

$$\hat{f}(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1; \\ -1, & \text{if } -1 \le x < 0; \\ 0, & \text{if } -2 \le x < -1 \text{ or } 1 < x \le 2. \end{cases}.$$

We have $L = 2$, and we calculate the Fourier coefficients:

$$a_n = -\frac{1}{2}\int_{-1}^{0} \cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{1} \cos\left(\frac{n\pi x}{2}\right)dx = -\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_{0}^{1} = 0,$$

and

$$b_n = -\frac{1}{2}\int_{-1}^{0} \sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{1} \sin\left(\frac{n\pi x}{2}\right)dx = -\left[-\frac{\cos\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_{-1}^{0} + \left[-\frac{\cos\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_{0}^{1} = 2\frac{1 - \cos\left(\frac{n\pi}{2}\right)}{n\pi},$$

for $n = 1, 2, \ldots,$ and

$$a_0 = -\frac{1}{2}\int_{-1}^{0} dx + \frac{1}{2}\int_{0}^{1} dx = 0.$$

Hence the Fourier series expansion of $\hat{f}$, $x \in [-2, 2]$ reads

$$\sum_{n=1}^{\infty} 2\frac{1 - \cos\left(\frac{n\pi}{2}\right)}{n\pi} \sin\left(\frac{n\pi x}{2}\right).$$

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[6]We recall that an odd function is one for which $f(-x) = -f(x)$, i.e., it admits the origin as the centre of symmetry.

To see how does the Fourier expansion compare to the original function, we consider the partial sums

$$\sum_{n=1}^{k} 2 \frac{1 - \cos\left(\frac{n\pi}{2}\right)}{n\pi} \sin\left(\frac{n\pi x}{2}\right).$$

In Figure 2.4, we plot the function $\hat{f}$ and the partial sums of its Fourier expansion for $k = 1$, $k = 3$ and $k = 10$ and $k = 40$.
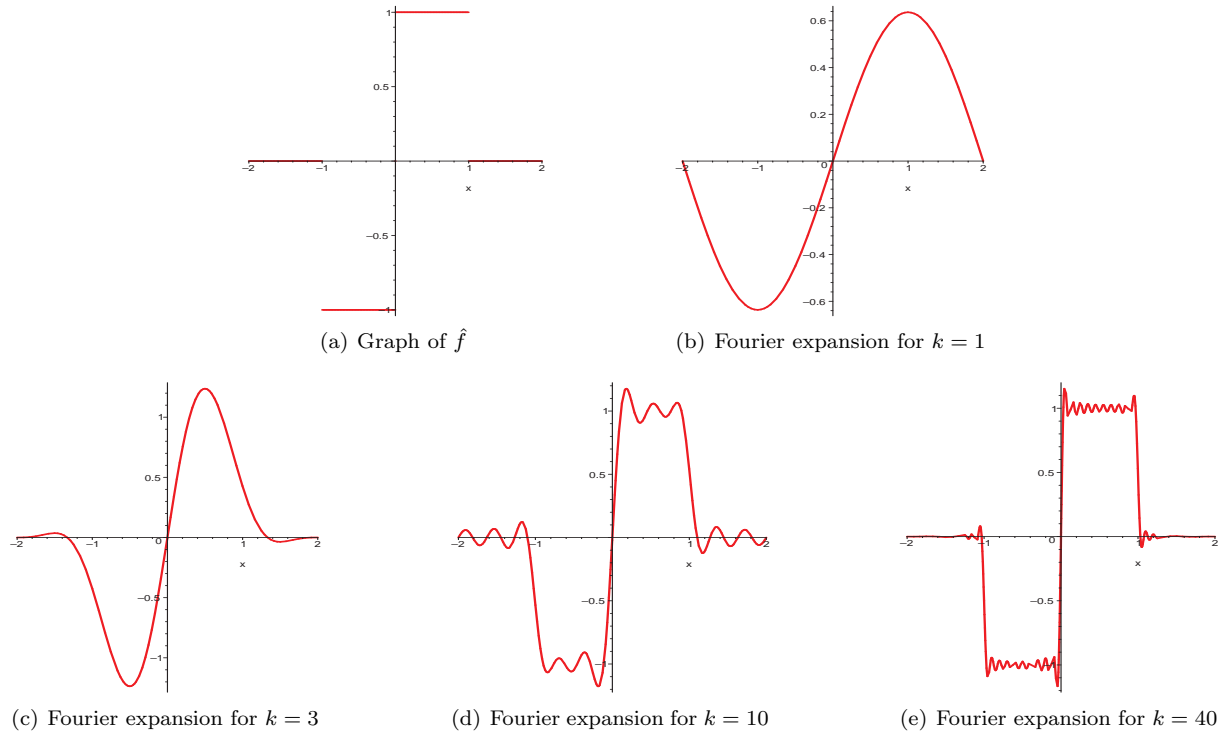

(a) Graph of $\hat{f}$

(b) Fourier expansion for $k = 1$

(c) Fourier expansion for $k = 3$

(d) Fourier expansion for $k = 10$

(e) Fourier expansion for $k = 40$

Figure 2.4: Fourier synthesis...

When we extended the function $f : [0, 2] \to \mathbb{R}$ to an even function $\tilde{f} : [-2, 2] \to \mathbb{R}$, the Fourier coefficients b_n vanished, yielding a Fourier series expansion consisting only of cosine functions – this is called a *cosine series expansion*. On the contrary when we extended the function $f : [0, 2] \to \mathbb{R}$ to an even function $\hat{f} : [-2, 2] \to \mathbb{R}$, the Fourier coefficients a_n vanished, yielding a Fourier series expansion consisting only of sine functions – this is called a *sine series expansion*. In fact, this is a general observation: any function $f : [0, L] \to \mathbb{R}$ can be expressed in terms of either a cosine series or a sine series; this will be particularly useful when solving PDEs as presented below.

We observe that the partial sums of Fourier series expansions develop oscillations near discontinuities. This phenomenon is common whenever one tries to approximate discontinuous functions by continuous ones and it is usually referred to as *Gibbs phenomenon*.

It is possible to see from Figures 2.3 and 2.4 that at points where the functions admit jump discontinuities then the Fourier series expansions appear to converge to the average of the limit values at either side of each discontinuity; this is in accordance with Theorem 2.3!