2.1.3 Back to the Laplace problem

In the previous section we saw how to represent functions in terms of Fourier series expansions; this discussion was motivated by the formula (2.9), which was resulted from solving the Laplace problem on a rectangular domain using separation of variables.

Going back to (2.9), we are now in position to determine the $\tilde{E}_n$'s as the Fourier coefficients of the sine series expansion of the function $f : [0, a] \to \mathbb{R}$ (which is the Dirichlet boundary condition on the top part of the rectangular boundary). More specifically, we extend f to an odd function in $\hat{f} : [-a, a] \to \mathbb{R}$, i.e., we set

$$\hat{f}(x) := \begin{cases} f(x), & \text{if } 0 \le x \le a; \\ -f(-x), & \text{if } -a \le x < 0; \end{cases}$$

then, we have

$$\begin{aligned} \tilde{E}_n &= \frac{1}{a}\int_{-a}^{a} \hat{f}(x) \sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x = -\frac{1}{a}\int_{-a}^{0} f(-x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x + \frac{1}{a}\int_{0}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x \\ &= \frac{1}{a}\int_{a}^{0} f(y)\sin\left(-\frac{n\pi y}{a}\right)\mathrm{d}y + \frac{1}{a}\int_{0}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x \\ &= \frac{1}{a}\int_{0}^{a} f(y)\sin\left(\frac{n\pi y}{a}\right)\mathrm{d}y + \frac{1}{a}\int_{0}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x = \frac{2}{a}\int_{0}^{a} f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x, \end{aligned} \tag{2.10}$$

using the change of variables $y = -x$ in the penultimate equality, which is now computable provided we know $f(x)$!

Example 2.6 *We want to solve the Laplace problem (2.4), where $a = 1$ and $f : [0, 1] \to \mathbb{R}$ is defined by*

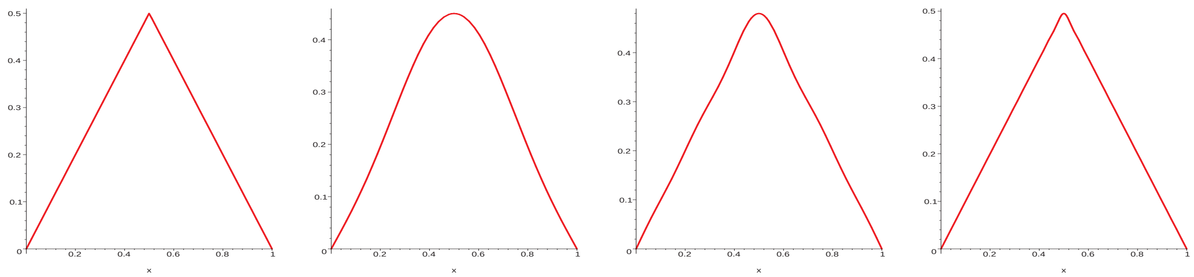
$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le \frac{1}{2}; \\ 1 - x, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

The method of solution of the Laplace problem using separation of variables is presented in Section 2.3.1, where we arrived to the solution (2.8) with unknown coefficients E_n, $n = 1, 2, \ldots$, which can be determined from the boundary condition (2.9) after expanding $f(x)$ into a sine series.

Using (2.10) with $a = 1$, we calculate

$$\tilde{E}_n = 2\int_{0}^{\frac{1}{2}} x\sin(n\pi x)\mathrm{d}x + 2\int_{\frac{1}{2}}^{1}(1 - x)\sin(n\pi x)\mathrm{d}x = \cdots = \frac{4\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2}.$$

In Figure 2.5, we plot the function f and the partial sums with 3, 9 and 39 first terms,respectively. Recalling



(a) Graph of $\hat{f}$ (b) 3 terms (c) 9 terms (d) 39 terms

Figure 2.5: Fourier synthesis...

now the definition of $\tilde{E}_n := E_n \sinh\left(\frac{n\pi b}{a}\right)$, we conclude that the solution of (2.4) for $a = 1$ is given by the series

$$u(x, y) = \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{n\pi}{2}\right)}{(n\pi)^2\sinh(n\pi b)}\sin(n\pi x)\sinh(n\pi y).$$

2.2 The heat equation and the initial/boundary value problem

The *heat equation* (also known as the diffusion equation) is a paradigm of equations of parabolic type as we saw in the previous chapter. In most applications the parabolic equations describe phenomena that evolve in time; in the following, t will be denoting the "time"-variable, ranging between time 0 and some final time $T > 0$. For (unknown) solution $u : [0, T] \times \Omega \to \mathbb{R}$, the heat equation reads

$$u_t = \Delta u, \quad \text{for} \quad (x_1, x_2, \ldots, x_d) \in \Omega \subset \mathbb{R}^d, \quad \text{and} \quad t \in [0, T], \tag{2.11}$$

where Δ is the Laplace operator in d dimensions; in particular, in one space dimension the hear equation reads:

$$u_t = u_{xx}, \quad \text{for} \quad x \in \Omega \subset \mathbb{R}, \quad \text{and} \quad t \in [0, T]. \tag{2.12}$$

As we also saw in Problem Sheet 1, the heat equation can viewed as one canonical form of the *Black-Scholes* equation of mathematical finance.

As we saw in Chapter 1, the heat equation is of parabolic type. Therefore, it admits one family of characteristic curves and, at least intuitively, we can see that it requires Cauchy initial condition. Also, the PDE for each fixed time $t \in [0, T]$ takes the form of the Poisson problem. Hence, again at leat heuristically, we can see that Dirichlet and/or Neumann type boundary condition(s) are required on the boundary of Ω, for each time t, for the problem to be well posed. We shall refer to the heat equation together with the Cauchy initial condition and the boundary conditions, as the *initial/boundary value problem.*

Next, we shall be concerned with finding the solution to the initial/boundary value problem, using the separation of variables.