7

SEQUENCES AND SERIES OF FUNCTIONS

In the present chapter we confine our attention to complex-valued functions (including the real-valued ones, of course), although many of the theorems and proofs which follow extend without difficulty to vector-valued functions, and even to mappings into general metric spaces. We choose to stay within this simple framework in order to focus attention on the most important aspects of the problems that arise when limit processes are interchanged.

DISCUSSION OF MAIN PROBLEM

7.1 Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$(1) \qquad\qquad f(x) = \lim_{n \to \infty} f_n(x) \qquad (x \in E).$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f *pointwise* on E" if (1) holds. Similarly, if $\Sigma f_n(x)$ converges for every $x \in E$, and if we define

$$(2) \qquad f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in E),$$

the function f is called the *sum* of the series Σf_n.

The main problem which arises is to determine whether important properties of functions are preserved under the limit operations (1) and (2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f'_n and f', say, or between the integrals of f_n and that of f?

To say that f is continuous at a limit point x means

$$\lim_{t \to x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$(3) \qquad \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t),$$

i.e., whether the order in which limit processes are carried out is immaterial. On the left side of (3), we first let $n \to \infty$, then $t \to x$; on the right side, $t \to x$ first, then $n \to \infty$.

We shall now show by means of several examples that limit processes cannot in general be interchanged without affecting the result. Afterward, we shall prove that under certain conditions the order in which limit operations are carried out is immaterial.

Our first example, and the simplest one, concerns a "double sequence."

7.2 Example For $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$, let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n,

$$\lim_{m \to \infty} s_{m,n} = 1,$$

so that

$$(4) \qquad \lim_{n \to \infty} \lim_{m \to \infty} s_{m,n} = 1.$$

On the other hand, for every fixed m,

$$\lim_{n \to \infty} s_{m,n} = 0,$$

so that

(5)
$$\lim_{m \to \infty} \lim_{n \to \infty} s_{m,n} = 0.$$

7.3 Example Let

$$f_n(x) = \frac{x^2}{(1 + x^2)^n} \qquad (x \text{ real}; n = 0, 1, 2, \ldots),$$

and consider

(6)
$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n}.$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series in (6) is a convergent geometric series with sum $1 + x^2$ (Theorem 3.26). Hence

(7)
$$f(x) = \begin{cases} 0 & (x = 0), \\ 1 + x^2 & (x \neq 0), \end{cases}$$

so that a convergent series of continuous functions may have a discontinuous sum.

7.4 Example For $m = 1, 2, 3, \ldots$, put

$$f_m(x) = \lim_{n \to \infty} (\cos m!\pi x)^{2n}.$$

When $m!x$ is an integer, $f_m(x) = 1$. For all other values of x, $f_m(x) = 0$. Now let

$$f(x) = \lim_{m \to \infty} f_m(x).$$

For irrational x, $f_m(x) = 0$ for every m; hence $f(x) = 0$. For rational x, say $x = p/q$, where p and q are integers, we see that $m!x$ is an integer if $m \geq q$, so that $f(x) = 1$. Hence

(8)
$$\lim_{m \to \infty} \lim_{n \to \infty} (\cos m!\pi x)^{2n} = \begin{cases} 0 & (x \text{ irrational}), \\ 1 & (x \text{ rational}). \end{cases}$$

We have thus obtained an everywhere discontinuous limit function, which is not Riemann-integrable (Exercise 4, Chap. 6).

7.5 Example Let

$$(9) \qquad f_n(x) = \frac{\sin nx}{\sqrt{n}} \qquad (x \text{ real}, \ n = 1, 2, 3, \ldots),$$

and

$$f(x) = \lim_{n \to \infty} f_n(x) = 0.$$

Then $f'(x) = 0$, and

$$f_n'(x) = \sqrt{n} \cos nx,$$

so that $\{f_n'\}$ does not converge to f'. For instance,

$$f_n'(0) = \sqrt{n} \to +\infty$$

as $n \to \infty$, whereas $f'(0) = 0$.

7.6 Example Let

$$(10) \qquad f_n(x) = n^2 x(1 - x^2)^n \qquad (0 \leq x \leq 1, n = 1, 2, 3, \ldots).$$

For $0 < x \leq 1$, we have

$$\lim_{n \to \infty} f_n(x) = 0,$$

by Theorem 3.20(d). Since $f_n(0) = 0$, we see that

$$(11) \qquad \lim_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1).$$

A simple calculation shows that

$$\int_0^1 x(1 - x^2)^n \, dx = \frac{1}{2n + 2}.$$

Thus, in spite of (11),

$$\int_0^1 f_n(x) \, dx = \frac{n^2}{2n + 2} \to +\infty$$

as $n \to \infty$.

If, in (10), we replace n^2 by n, (11) still holds, but we now have

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \lim_{n \to \infty} \frac{n}{2n + 2} = \frac{1}{2},$$

whereas

$$\int_0^1 \left[\lim_{n \to \infty} f_n(x) \right] dx = 0.$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

After these examples, which show what can go wrong if limit processes are interchanged carelessly, we now define a new mode of convergence, stronger than pointwise convergence as defined in Definition 7.1, which will enable us to arrive at positive results.

UNIFORM CONVERGENCE

7.7 Definition We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, \ldots,$ converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

(12) $$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If $\{f_n\}$ converges pointwise on E, then there exists a function f such that, for every $\varepsilon > 0$, and for every $x \in E$, there is an integer N, depending on ε *and* on x, such that (12) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E, it is possible, for each $\varepsilon > 0$, to find *one* integer N which will do for *all* $x \in E$.

We say that the series $\Sigma f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(x) = s_n(x)$$

converges uniformly on E.

The Cauchy criterion for uniform convergence is as follows.

7.8 Theorem *The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

(13) $$|f_n(x) - f_m(x)| \leq \varepsilon.$$

Proof Suppose $\{f_n\}$ converges uniformly on E, and let f be the limit function. Then there is an integer N such that $n \geq N$, $x \in E$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2},$$

so that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

if $n \geq N, m \geq N, x \in E$.

Conversely, suppose the Cauchy condition holds. By Theorem 3.11, the sequence $\{f_n(x)\}$ converges, for every x, to a limit which we may call $f(x)$. Thus the sequence $\{f_n\}$ converges on E, to f. We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n, and let $m \to \infty$ in (13). Since $f_m(x) \to f(x)$ as $m \to \infty$, this gives

$$(14) \qquad\qquad |f_n(x) - f(x)| \leq \varepsilon$$

for every $n \geq N$ and every $x \in E$, which completes the proof.

The following criterion is sometimes useful.

7.9 Theorem *Suppose*

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Since this is an immediate consequence of Definition 7.7, we omit the details of the proof.

For series, there is a very convenient test for uniform convergence, due to Weierstrass.

7.10 Theorem *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \leq M_n \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Then Σf_n converges uniformly on E if ΣM_n converges.

Note that the converse is not asserted (and is, in fact, not true).

Proof If ΣM_n converges, then, for arbitrary $\varepsilon > 0$,

$$\left| \sum_{i=n}^{m} f_i(x) \right| \leq \sum_{i=n}^{m} M_i \leq \varepsilon \qquad (x \in E),$$

provided m and n are large enough. Uniform convergence now follows from Theorem 7.8.

UNIFORM CONVERGENCE AND CONTINUITY

7.11 Theorem *Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that*

$$(15) \qquad \lim_{t \to x} f_n(t) = A_n \qquad (n = 1, 2, 3, \ldots).$$

Then $\{A_n\}$ converges, and

$$(16) \qquad \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

In other words, the conclusion is that

$$(17) \qquad \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

Proof Let $\varepsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, there exists N such that $n \geq N$, $m \geq N$, $t \in E$ imply

$$(18) \qquad |f_n(t) - f_m(t)| \leq \varepsilon.$$

Letting $t \to x$ in (18), we obtain

$$|A_n - A_m| \leq \varepsilon$$

for $n \geq N, m \geq N$, so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to A.

Next,

$$(19) \qquad |f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

We first choose n such that

$$(20) \qquad |f(t) - f_n(t)| \leq \frac{\varepsilon}{3}$$

for all $t \in E$ (this is possible by the uniform convergence), and such that

$$(21) \qquad |A_n - A| \leq \frac{\varepsilon}{3}.$$

Then, for this n, we choose a neighborhood V of x such that

$$(22) \qquad |f_n(t) - A_n| \leq \frac{\varepsilon}{3}$$

if $t \in V \cap E$, $t \neq x$.

Substituting the inequalities (20) to (22) into (19), we see that

$$|f(t) - A| \leq \varepsilon,$$

provided $t \in V \cap E$, $t \neq x$. This is equivalent to (16).

7.12 Theorem *If* $\{f_n\}$ *is a sequence of continuous functions on* E, *and if* $f_n \to f$ *uniformly on* E, *then* f *is continuous on* E.

This very important result is an immediate corollary of Theorem 7.11.

The converse is not true; that is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform. Example 7.6 is of this kind (to see this, apply Theorem 7.9). But there is a case in which we can assert the converse.

7.13 Theorem *Suppose* K *is compact, and*

(a) $\{f_n\}$ *is a sequence of continuous functions on* K,

(b) $\{f_n\}$ *converges pointwise to a continuous function* f *on* K,

(c) $f_n(x) \geq f_{n+1}(x)$ *for all* $x \in K$, $n = 1, 2, 3, \ldots$.

Then $f_n \to f$ *uniformly on* K.

Proof Put $g_n = f_n - f$. Then g_n is continuous, $g_n \to 0$ pointwise, and $g_n \geq g_{n+1}$. We have to prove that $g_n \to 0$ uniformly on K.

Let $\varepsilon > 0$ be given. Let K_n be the set of all $x \in K$ with $g_n(x) \geq \varepsilon$. Since g_n is continuous, K_n is closed (Theorem 4.8), hence compact (Theorem 2.35). Since $g_n \geq g_{n+1}$, we have $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \to 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N (Theorem 2.36). It follows that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \geq N$. This proves the theorem.

Let us note that compactness is really needed here. For instance, if

$$f_n(x) = \frac{1}{nx + 1} \qquad (0 < x < 1; n = 1, 2, 3, \ldots)$$

then $f_n(x) \to 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

7.14 Definition If X is a metric space, $\mathscr{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

[Note that boundedness is redundant if X is compact (Theorem 4.15). Thus $\mathscr{C}(X)$ consists of all complex continuous functions on X if X is compact.] We associate with each $f \in \mathscr{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$. It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

If we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, it follows that Axioms 2.15 for a metric are satisfied.

We have thus made $\mathscr{C}(X)$ into a metric space.

Theorem 7.9 can be rephrased as follows:

A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Accordingly, closed subsets of $\mathscr{C}(X)$ are sometimes called *uniformly closed*, the closure of a set $\mathscr{A} \subset \mathscr{C}(X)$ is called its *uniform closure*, and so on.

7.15 Theorem *The above metric makes $\mathscr{C}(X)$ into a complete metric space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. This means that to each $\varepsilon > 0$ corresponds an N such that $\|f_n - f_m\| < \varepsilon$ if $n \geq N$ and $m \geq N$. It follows (by Theorem 7.8) that there is a function f with domain X to which $\{f_n\}$ converges uniformly. By Theorem 7.12, f is continuous. Moreover, f is bounded, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, and f_n is bounded.

Thus $f \in \mathscr{C}(X)$, and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$.

UNIFORM CONVERGENCE AND INTEGRATION

7.16 Theorem *Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \ldots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and*

$$(23) \qquad \int_a^b f \, d\alpha = \lim_{n \to \infty} \int_a^b f_n \, d\alpha.$$

(The existence of the limit is part of the conclusion.)

Proof It suffices to prove this for real f_n. Put

$$(24) \qquad \varepsilon_n = \sup |f_n(x) - f(x)|,$$

the supremum being taken over $a \leq x \leq b$. Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n,$$

so that the upper and lower integrals of f (see Definition 6.2) satisfy

$$(25) \qquad \int_a^b (f_n - \varepsilon_n) \, d\alpha \leq \underline{\int} f \, d\alpha \leq \overline{\int} f \, d\alpha \leq \int_a^b (f_n + \varepsilon_n) \, d\alpha.$$

Hence

$$0 \leq \overline{\int} f \, d\alpha - \underline{\int} f \, d\alpha \leq 2\varepsilon_n[\alpha(b) - \alpha(a)].$$

Since $\varepsilon_n \to 0$ as $n \to \infty$ (Theorem 7.9), the upper and lower integrals of f are equal.

Thus $f \in \mathscr{R}(\alpha)$. Another application of (25) now yields

(26)
$$\left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| \leq \varepsilon_n[\alpha(b) - \alpha(a)].$$

This implies (23).

Corollary If $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (a \leq x \leq b),$$

the series converging uniformly on $[a, b]$, *then*

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

In other words, the series may be integrated term by term.

UNIFORM CONVERGENCE AND DIFFERENTIATION

We have already seen, in Example 7.5, that uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$. Thus stronger hypotheses are required for the assertion that $f_n' \to f'$ if $f_n \to f$.

7.17 Theorem *Suppose* $\{f_n\}$ *is a sequence of functions, differentiable on* $[a, b]$ *and such that* $\{f_n(x_0)\}$ *converges for some point* x_0 *on* $[a, b]$. *If* $\{f_n'\}$ *converges uniformly on* $[a, b]$, *then* $\{f_n\}$ *converges uniformly on* $[a, b]$, *to a function* f, *and*

(27)
$$f'(x) = \lim_{n \to \infty} f_n'(x) \qquad (a \leq x \leq b).$$

Proof Let $\varepsilon > 0$ be given. Choose N such that $n \geq N$, $m \geq N$, implies

(28)
$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

and

(29)
$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b - a)} \qquad (a \leq t \leq b).$$

If we apply the mean value theorem 5.19 to the function $f_n - f_m$, (29) shows that

$$(30) \qquad |f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x - t|\varepsilon}{2(b - a)} \leq \frac{\varepsilon}{2}$$

for any x and t on $[a, b]$, if $n \geq N$, $m \geq N$. The inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

implies, by (28) and (30), that

$$|f_n(x) - f_m(x)| < \varepsilon \qquad (a \leq x \leq b, n \geq N, m \geq N),$$

so that $\{f_n\}$ converges uniformly on $[a, b]$. Let

$$f(x) = \lim_{n \to \infty} f_n(x) \qquad (a \leq x \leq b).$$

Let us now fix a point x on $[a, b]$ and define

$$(31) \qquad \phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

for $a \leq t \leq b$, $t \neq x$. Then

$$(32) \qquad \lim_{t \to x} \phi_n(t) = f_n'(x) \qquad (n = 1, 2, 3, \ldots).$$

The first inequality in (30) shows that

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b - a)} \qquad (n \geq N, m \geq N),$$

so that $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges to f, we conclude from (31) that

$$(33) \qquad \lim_{n \to \infty} \phi_n(t) = \phi(t)$$

uniformly for $a \leq t \leq b$, $t \neq x$.

If we now apply Theorem 7.11 to $\{\phi_n\}$, (32) and (33) show that

$$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x);$$

and this is (27), by the definition of $\phi(t)$.

Remark: If the continuity of the functions f_n' is assumed in addition to the above hypotheses, then a much shorter proof of (27) can be based on Theorem 7.16 and the fundamental theorem of calculus.

7.18 Theorem *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof Define

(34)
$$\varphi(x) = |x| \qquad (-1 \le x \le 1)$$

and extend the definition of $\varphi(x)$ to all real x by requiring that

(35)
$$\varphi(x + 2) = \varphi(x).$$

Then, for all s and t,

(36)
$$|\varphi(s) - \varphi(t)| \le |s - t|.$$

In particular, φ is continuous on R^1. Define

(37)
$$f(x) = \sum_{n=0}^{\infty} (\tfrac{3}{4})^n \varphi(4^n x).$$

Since $0 \le \varphi \le 1$, Theorem 7.10 shows that the series (37) converges uniformly on R^1. By Theorem 7.12, f is continuous on R^1.

Now fix a real number x and a positive integer m. Put

(38)
$$\delta_m = \pm \tfrac{1}{2} \cdot 4^{-m}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done, since $4^m |\delta_m| = \tfrac{1}{2}$. Define

(39)
$$\gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}.$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. When $0 \le n \le m$, (36) implies that $|\gamma_n| \le 4^n$.

Since $|\gamma_m| = 4^m$, we conclude that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right|$$
$$\ge 3^m - \sum_{n=0}^{m-1} 3^n$$
$$= \tfrac{1}{2}(3^m + 1).$$

As $m \to \infty$, $\delta_m \to 0$. It follows that f is not differentiable at x.

EQUICONTINUOUS FAMILIES OF FUNCTIONS

In Theorem 3.6 we saw that every bounded sequence of complex numbers contains a convergent subsequence, and the question arises whether something similar is true for sequences of functions. To make the question more precise, we shall define two kinds of boundedness.

7.19 Definition Let $\{f_n\}$ be a sequence of functions defined on a set E.

We say that $\{f_n\}$ is *pointwise bounded* on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \qquad (x \in E, n = 1, 2, 3, \ldots).$$

We say that $\{f_n\}$ is *uniformly bounded* on E if there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Now if $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E, it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$. This can be done by the diagonal process which is used in the proof of Theorem 7.23.

However, even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E. In the following example, this would be quite trouble-some to prove with the equipment which we have at hand so far, but the proof is quite simple if we appeal to a theorem from Chap. 11.

7.20 Example Let

$$f_n(x) = \sin nx \qquad (0 \le x \le 2\pi, n = 1, 2, 3, \ldots).$$

Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$. In that case we must have

$$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad (0 \le x \le 2\pi);$$

hence

(40) $$\lim_{k \to \infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad (0 \le x \le 2\pi).$$

By Lebesgue's theorem concerning integration of boundedly convergent sequences (Theorem 11.32), (40) implies

(41) $$\lim_{k \to \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx = 0.$$

But a simple calculation shows that

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 \, dx = 2\pi,$$

which contradicts (41).

Another question is whether every convergent sequence contains a uniformly convergent subsequence. Our next example will show that this need not be so, even if the sequence is uniformly bounded on a compact set. (Example 7.6 shows that a sequence of bounded functions may converge without being uniformly bounded; but it is trivial to see that uniform convergence of a sequence of bounded functions implies uniform boundedness.)

7.21 Example Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \qquad (0 \le x \le 1, n = 1, 2, 3, \ldots).$$

Then $|f_n(x)| \le 1$, so that $\{f_n\}$ is uniformly bounded on $[0, 1]$. Also

$$\lim_{n \to \infty} f_n(x) = 0 \qquad (0 \le x \le 1),$$

but

$$f_n\!\left(\frac{1}{n}\right) = 1 \qquad (n = 1, 2, 3, \ldots),$$

so that no subsequence can converge uniformly on $[0, 1]$.

The concept which is needed in this connection is that of equicontinuity; it is given in the following definition.

7.22 Definition A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be *equicontinuous* on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$. Here d denotes the metric of X.

It is clear that every member of an equicontinuous family is uniformly continuous.

The sequence of Example 7.21 is not equicontinuous.

Theorems 7.24 and 7.25 will show that there is a very close relation between equicontinuity, on the one hand, and uniform convergence of sequences of continuous functions, on the other. But first we describe a selection process which has nothing to do with continuity.

7.23 Theorem *If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.*

Proof Let $\{x_i\}$, $i = 1, 2, 3, \ldots$, be the points of E, arranged in a sequence. Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence, which we shall denote by $\{f_{1,k}\}$, such that $\{f_{1,k}(x_1)\}$ converges as $k \to \infty$.

Let us now consider sequences $S_1, S_2, S_3, \ldots$, which we represent by the array

$$
\begin{array}{llllll}
S_1\colon & f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\
S_2\colon & f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\
S_3\colon & f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\
& \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}
$$

and which have the following properties:

(a) S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, \ldots$.

(b) $\{f_{n,k}(x_n)\}$ converges, as $k \to \infty$ (the boundedness of $\{f_n(x_n)\}$ makes it possible to choose S_n in this way);

(c) The order in which the functions appear is the same in each sequence; i.e., if one function precedes another in S_1, they are in the same relation in every S_n, until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence

$$
S\colon \quad f_{1,1} \quad f_{2,2} \quad f_{3,3} \quad f_{4,4} \cdots.
$$

By (c), the sequence S (except possibly its first $n - 1$ terms) is a subsequence of S_n, for $n = 1, 2, 3, \ldots$. Hence (b) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \to \infty$, for every $x_i \in E$.

7.24 Theorem *If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.*

Proof Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly, there is an integer N such that

(42) $$\|f_n - f_N\| < \varepsilon \qquad (n > N).$$

(See Definition 7.14.) Since continuous functions are uniformly continuous on compact sets, there is a $\delta > 0$ such that

(43) $$|f_i(x) - f_i(y)| < \varepsilon$$

if $1 \le i \le N$ and $d(x, y) < \delta$.

If $n > N$ and $d(x, y) < \delta$, it follows that

$$|f_n(x) - f_n(y)| \le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon.$$

In conjunction with (43), this proves the theorem.

7.25 Theorem *If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*

(a) *$\{f_n\}$ is uniformly bounded on K,*

(b) *$\{f_n\}$ contains a uniformly convergent subsequence.*

Proof

(a) Let $\varepsilon > 0$ be given and choose $\delta > 0$, in accordance with Definition 7.22, so that

$$(44) \qquad |f_n(x) - f_n(y)| < \varepsilon$$

for all n, provided that $d(x, y) < \delta$.

Since K is compact, there are finitely many points $p_1, \ldots, p_r$ in K such that to every $x \in K$ corresponds at least one p_i with $d(x, p_i) < \delta$. Since $\{f_n\}$ is pointwise bounded, there exist $M_i < \infty$ such that $|f_n(p_i)| < M_i$ for all n. If $M = \max(M_1, \ldots, M_r)$, then $|f_n(x)| < M + \varepsilon$ for every $x \in K$. This proves (a).

(b) Let E be a countable dense subset of K. (For the existence of such a set E, see Exercise 25, Chap. 2.) Theorem 7.23 shows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}(x)\}$ converges for every $x \in E$.

Put $f_{n_i} = g_i$, to simplify the notation. We shall prove that $\{g_i\}$ converges uniformly on K.

Let $\varepsilon > 0$, and pick $\delta > 0$ as in the beginning of this proof. Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$. Since E is dense in K, and K is compact, there are finitely many points $x_1, \ldots, x_m$ in E such that

$$(45) \qquad K \subset V(x_1, \delta) \cup \cdots \cup V(x_m, \delta).$$

Since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer N such that

$$(46) \qquad |g_i(x_s) - g_j(x_s)| < \varepsilon$$

whenever $i \geq N, j \geq N, 1 \leq s \leq m$.

If $x \in K$, (45) shows that $x \in V(x_s, \delta)$ for some s, so that

$$|g_i(x) - g_i(x_s)| < \varepsilon$$

for every i. If $i \geq N$ and $j \geq N$, it follows from (46) that

$$|g_i(x) - g_j(x)| \leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)|$$
$$< 3\varepsilon.$$

This completes the proof.

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on* $[a, b]$, *there exists a sequence of polynomials* P_n *such that*

$$\lim_{n \to \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. *If f is real, the* P_n *may be taken real.*

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \qquad (0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f, since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

(47) $$Q_n(x) = c_n(1 - x^2)^n \qquad (n = 1, 2, 3, \ldots),$$

where c_n is chosen so that

(48) $$\int_{-1}^{1} Q_n(x)\, dx = 1 \qquad (n = 1, 2, 3, \ldots).$$

We need some information about the order of magnitude of c_n. Since

$$\int_{-1}^{1} (1 - x^2)^n\, dx = 2 \int_{0}^{1} (1 - x^2)^n\, dx \ge 2 \int_{0}^{1/\sqrt{n}} (1 - x^2)^n\, dx$$

$$\ge 2 \int_{0}^{1/\sqrt{n}} (1 - nx^2)\, dx$$

$$= \frac{4}{3\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}},$$

it follows from (48) that

(49) $$c_n < \sqrt{n}.$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

(50) $$Q_n(x) \leq \sqrt{n}\,(1 - \delta^2)^n \qquad (\delta \leq |x| \leq 1),$$

so that $Q_n \to 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

(51) $$P_n(x) = \int_{-1}^{1} f(x + t)Q_n(t)\,dt \qquad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x + t)Q_n(t)\,dt = \int_{0}^{1} f(t)Q_n(t - x)\,dt,$$

and the last integral is clearly a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} [f(x + t) - f(x)]Q_n(t)\,dt \right|$$

$$\leq \int_{-1}^{1} |f(x + t) - f(x)|\,Q_n(t)\,dt$$

$$\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt + 2M \int_{\delta}^{1} Q_n(t)\,dt$$

$$\leq 4M\sqrt{n}\,(1 - \delta^2)^n + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

for all large enough n, which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary *For every interval* $[-a, a]$ *there is a sequence of real polynomials* P_n *such that* $P_n(0) = 0$ *and such that*

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

> **Proof** By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \to 0$ as $n \to \infty$. The polynomials
>
> $$P_n(x) = P_n^*(x) - P_n^*(0) \qquad (n = 1, 2, 3, \ldots)$$
>
> have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathscr{A}$, (ii) $fg \in \mathscr{A}$, and (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$ and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ ($n = 1, 2, 3, \ldots$) and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly closed*.

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$. (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem *Let* $\mathscr{B}$ *be the uniform closure of an algebra* $\mathscr{A}$ *of bounded functions. Then* $\mathscr{B}$ *is a uniformly closed algebra.*

> **Proof** If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \to f$, $g_n \to g$ and $f_n \in \mathscr{A}$, $g_n \in \mathscr{A}$. Since we are dealing with bounded functions, it is easy to show that
>
> $$f_n + g_n \to f + g, \qquad f_n g_n \to fg, \qquad cf_n \to cf,$$
>
> where c is any constant, the convergence being uniform in each case.
> Hence $f + g \in \mathscr{B}$, $fg \in \mathscr{B}$, and $cf \in \mathscr{B}$, so that $\mathscr{B}$ is an algebra.
> By Theorem 2.27, $\mathscr{B}$ is (uniformly) closed.

7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes at no point of* E.

The algebra of all polynomials in one variable clearly has these properties on R^1. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

The following theorem will illustrate these concepts further.

7.31 Theorem *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof The assumptions show that $\mathscr{A}$ contains functions g, h, and k such that

$$g(x_1) \neq g(x_2), \qquad h(x_1) \neq 0, \qquad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathscr{A}$, $v \in \mathscr{A}$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

We shall divide the proof into four steps.

STEP 1 *If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.*

Proof Let

(52) $$a = \sup |f(x)| \qquad (x \in K)$$

and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers $c_1, \ldots, c_n$ such that

(53)
$$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \qquad (-a \le y \le a).$$

Since $\mathscr{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathscr{B}$. By (52) and (53), we have

$$\left| g(x) - |f(x)| \right| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this shows that $|f| \in \mathscr{B}$.

STEP 2 *If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.*

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \ge g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2},$$

$$\min(f, g) = \frac{f + g}{2} - \frac{|f - g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathscr{B}$, then $\max(f_1, \ldots, f_n) \in \mathscr{B}$, and

$$\min(f_1, \ldots, f_n) \in \mathscr{B}.$$

STEP 3 *Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and*

(54)
$$g_x(t) > f(t) - \varepsilon \qquad (t \in K).$$

Proof Since $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ satisfies the hypotheses of Theorem 7.31 so does $\mathscr{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathscr{B}$ such that

(55)
$$h_y(x) = f(x), \qquad h_y(y) = f(y).$$

By the continuity of h_y there exists an open set J_y, containing y, such that

(56) $$h_y(t) > f(t) - \varepsilon \qquad (t \in J_y).$$

Since K is compact, there is a finite set of points $y_1, \ldots, y_n$ such that

(57) $$K \subset J_{y_1} \cup \cdots \cup J_{y_n}.$$

Put

$$g_x = \max(h_{y_1}, \ldots, h_{y_n}).$$

By step 2, $g_x \in \mathscr{B}$, and the relations (55) to (57) show that g_x has the other required properties.

STEP 4 *Given a real function f, continuous on K, and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that*

(58) $$|h(x) - f(x)| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof Let us consider the functions g_x, for each $x \in K$, constructed in step 3. By the continuity of g_x, there exist open sets V_x containing x, such that

(59) $$g_x(t) < f(t) + \varepsilon \qquad (t \in V_x).$$

Since K is compact, there exists a finite set of points $x_1, \ldots, x_m$ such that

(60) $$K \subset V_{x_1} \cup \cdots \cup V_{x_m}.$$

Put

$$h = \min(g_{x_1}, \ldots, g_{x_m}).$$

By step 2, $h \in \mathscr{B}$, and (54) implies

(61) $$h(t) > f(t) - \varepsilon \qquad (t \in K),$$

whereas (59) and (60) imply

(62) $$h(t) < f(t) + \varepsilon \qquad (t \in K).$$

Finally, (58) follows from (61) and (62).

Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on $\mathscr{A}$, namely, that $\mathscr{A}$ be *self-adjoint*. This means that for every $f \in \mathscr{A}$ its complex conjugate $\bar{f}$ must also belong to $\mathscr{A}$; $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem *Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is dense $\mathscr{C}(K)$.*

Proof Let $\mathscr{A}_R$ be the set of all real functions on K which belong to $\mathscr{A}$.

If $f \in \mathscr{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since $\mathscr{A}$ is self-adjoint, we see that $u \in \mathscr{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathscr{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that $\mathscr{A}_R$ separates points on K. If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathscr{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence $\mathscr{A}_R$ vanishes at no point of K.

Thus $\mathscr{A}_R$ satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of $\mathscr{A}_R$, hence lies in $\mathscr{B}$. If f is a complex continuous function on K, $f = u + iv$, then $u \in \mathscr{B}$, $v \in \mathscr{B}$, hence $f \in \mathscr{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

5. Let

$$
f_n(x) = \begin{cases}
0 & \left(x < \dfrac{1}{n+1}\right), \\[2mm]
\sin^2 \dfrac{\pi}{x} & \left(\dfrac{1}{n+1} \le x \le \dfrac{1}{n}\right), \\[2mm]
0 & \left(\dfrac{1}{n} < x\right).
\end{cases}
$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series Σf_n to show that absolute convergence, even for all x, does not imply uniform convergence.

6. Prove that the series

$$
\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}
$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

7. For $n = 1, 2, 3, \ldots, x$ real, put

$$
f_n(x) = \frac{x}{1 + nx^2}.
$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$
f'(x) = \lim_{n \to \infty} f_n'(x)
$$

is correct if $x \ne 0$, but false if $x = 0$.

8. If

$$
I(x) = \begin{cases}
0 & (x \le 0), \\
1 & (x > 0),
\end{cases}
$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\Sigma |c_n|$ converges, prove that the series

$$
f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \qquad (a \le x \le b)
$$

converges uniformly, and that f is continuous for every $x \ne x_n$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$
\lim_{n \to \infty} f_n(x_n) = f(x)
$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

10. Letting (x) denote the fractional part of the real number x (see Exercise 16, Chap. 4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \qquad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

11. Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

(a) Σf_n has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\Sigma f_n g_n$ converges uniformly on E. *Hint*: Compare with Theorem 3.42.

12. Suppose g and $f_n(n = 1, 2, 3, \ldots)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g, f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^{\infty} g(x)\, dx < \infty.$$

Prove that

$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)\, dx = \int_0^{\infty} f(x)\, dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathscr{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

for every $x \in R^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$, for any $x \in R^1$, to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

14. Let f be a continuous real function on R^1 with the following properties: $0 \leq f(t) \leq 1, f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t), \qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n} t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0, 1]$ *onto* the unit square $I^2 \subset R^2$. If fact, show that Φ maps the Cantor set onto I^2.

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1} (2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.

(This simple example of a so-called "space-filling curve" is due to I. J. Schoenberg, *Bull. A.M.S.*, vol. 44, 1938, pp. 519.)

15. Suppose f is a real continuous function on R^1, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

17. Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any R^k.

18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t) \, dt \qquad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

19. Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$. *Hint:* The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x) \, dx = 0$.

21. Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta} \qquad (\theta \text{ real}).$$

Then $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$. *Hint:* For every $f \in \mathscr{A}$

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta} \, d\theta = 0,$$

and this is also true for every f in the closure of $\mathscr{A}$.

22. Assume $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n \to \infty} \int_a^b |f - P_n|^2 \, d\alpha = 0.$$

(Compare with Exercise 12, Chap. 6.)

23. Put $P_0 = 0$, and define, for $n = 0, 1, 2, \ldots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n \to \infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

(This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.)

Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right]$$

to prove that $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ if $|x| \le 1$, and that

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \le 1$.

24. Let X be a metric space, with metric d. Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x, p) - d(x, a) \qquad (x \in X).$$

Prove that $|f_p(x)| \leq d(a, p)$ for all $x \in X$, and that therefore $f_p \in \mathscr{C}(X)$.
Prove that

$$\|f_p - f_q\| = d(p, q)$$

for all $p, q \in X$.

If $\Phi(p) = f_p$ it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X) \subset \mathscr{C}(X)$.

Let Y be the closure of $\Phi(X)$ in $\mathscr{C}(X)$. Show that Y is complete.

Conclusion: X is isometric to a dense subset of a complete metric space Y.
(Exercise 24, Chap. 3 contains a different proof of this.)

25. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \qquad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 27, Chap. 5.)

Hint: Fix n. For $i = 0, \ldots, n$, put $x_i = i/n$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$,

$$f_n'(t) = \phi(x_i, f_n(x_i)) \qquad \text{if } x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f_n'(t) - \phi(t, f_n(t)),$$

except at the points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] \, dt.$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

(a) $|f_n'| \leq M$, $|\Delta_n| \leq 2M$, $\Delta_n \in \mathscr{R}$, and $|f_n| \leq |c| + M = M_1$, say, on $[0, 1]$, for all n.

(b) $\{f_n\}$ is equicontinuous on $[0, 1]$, since $|f_n'| \leq M$.

(c) Some $\{f_{n_k}\}$ converges to some f, uniformly on $[0, 1]$.

(d) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$,

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0, 1]$.

(e) $\Delta_n(t) \to 0$ uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}).

(*f*) Hence

$$f(x) = c + \int_0^x \phi(t, f(t)) \; dt.$$

This *f* is a solution of the given problem.

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, $\mathbf{y} \in R^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \le x \le 1$, $\mathbf{y} \in R^k$ into R^k. (Compare Exercise 28, Chap. 5.) *Hint:* Use the vector-valued version of Theorem 7.25.